AS 4022: Cosmology

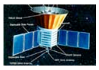
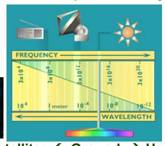
HS Zhao and K Horne

Online notes:
star-www.st-and.ac.uk/~hz4/cos/cos.html

Handouts in Library
Summary sheet of key results (from John Peacock)
take your own notes (including blackboard lectures)

Observable Space-Time and Bands

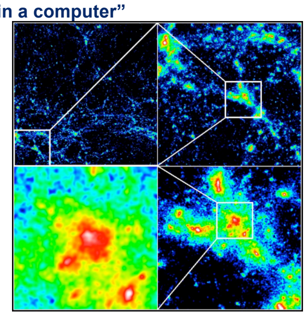
- **See What is out there? In all Energy bands**
  - Pupil → Galileo's Lens → 8m telescopes → square km arrays
  - Radio, Infrared ← optical → X-ray, Gamma-Ray (spectrum)

  - COBE satellites ← Ground → Underground DM detector
- **Know How were we created? XYZ & T ?**
  - Us, CNO in Life, Sun, Milky Way, … further and further
  - → first galaxy → first star → first Helium → first quark
  - Now → Billion years ago → first second → quantum origin

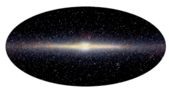
The Visible Cosmos: a hierarchy of structure and motion

- **"Cosmos in a computer"**



Observe A Hierarchical Universe

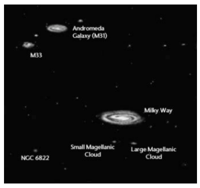
- **Planets**
  - moving around stars;
- **Stars grouped together,**
  - moving in a slow dance around the center of galaxies.



- **Galaxies themselves**
  - some 100 billion of them in the observable universe—
  - form galaxy clusters bound by gravity as they journey through the void.
- **But the largest structures of all are superclusters,**
  - each containing thousands of galaxies
  - and stretching many hundreds of millions of light years.
  - are arranged in filament or sheet-like structures,
  - between which are gigantic voids of seemingly empty space.

Cosmic Village

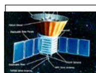
- **The Milky Way and Andromeda galaxies,**
  - along with about fifteen or sixteen smaller galaxies,
  - form what's known as the Local Group of galaxies.
- **The Local Group**
  - sits near the outer edge of a supercluster, the Virgo cluster.
  - the Milky Way and Andromeda are moving toward each other,
  - the Local Group is falling into the middle of the Virgo cluster, and



- **the entire Virgo cluster itself,**
  - is speeding toward a mass
  - known only as "The Great Attractor."

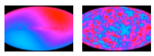
Introducing Gravity and DM (Key players)

- **These structures and their movements**
  - can't be explained purely by the expansion of the universe
- **must be guided by the gravitational pull of matter.**
- **Visible matter is not enough**
- **one more player into our hierarchical scenario:**
- **dark matter.**



Cosmologists hope to answer these questions:

- **How old is the universe? H_0**
- **Why was it so smooth? P(k), inflation**



- **How did structures emerge from smooth? N-body**
- **How did galaxies form? Hydro**

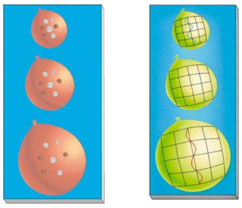
- **Will the universe expand forever? Omega, Lamda**
- **Or will it collapse upon itself like a bubble?**

1^{st} main concept in cosmology

- Cosmological Redshift

Stretch of photon wavelength in expanding space

- Emitted with intrinsic wavelength λ_0 from Galaxy A at time $t<t_{now}$ in smaller universe $R(t) < R_{now}$



- → Received at Galaxy B now (t_{now}) with λ
- $\lambda / \lambda_0 = R_{now} / R(t) = 1+z(t) > 1$

1^{st} main concept: Cosmological Redshift

- The space/universe is expanding,
  - Galaxies (pegs on grid points) are receding from each other
- As a photon travels through space, its wavelength becomes stretched gradually with time.
  - Photons wave-packets are like links between grid points
- This redshift is defined by:

$$z \equiv \frac{\lambda - \lambda_o}{\lambda_o}$$

$$\frac{\lambda}{\lambda_o} = 1 + z$$

- E.g. Consider a quasar with redshift z=2. Since the time the light left the quasar the universe has expanded by a factor of 1+z=3. At the epoch when the light left the quasar,
  - What was the distance between us and Virgo (presently 15Mpc)?
  - What was the CMB temperature then (presently 3K)?

$$1 + z = \frac{\lambda_{now}}{\lambda(t)} \quad \text{(wavelength)}$$

$$= \frac{R_{now}}{R(t)} \quad \text{(expansion factor)}$$

$$= \frac{T(t)}{T_{now}} \quad (\text{Photon Blackbody T } \propto 1/\lambda, \textit{why}?)$$

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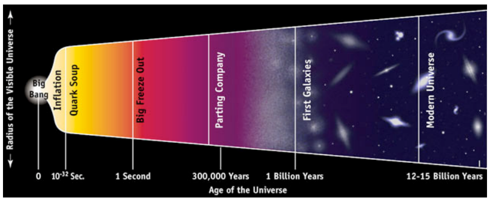
Cosmic Timeline

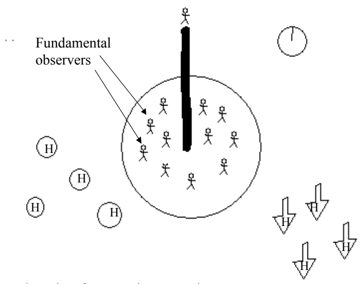
- **Past → Now**



Trafalgar Square

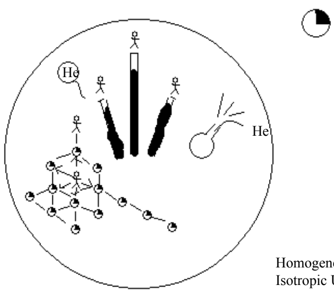
London Jan 1

Set your watches 0h:0m:0s



A comic explanation for cosmic expansion …

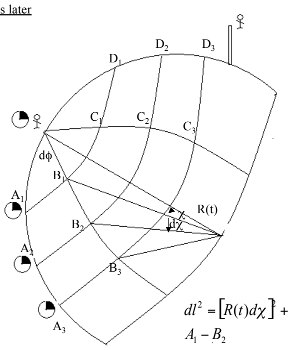
3 mins later



Homogeneous
Isotropic Universe

Walking ↔ *Elevating* ↔ *Earth Radius Stretching* $R(t)$

Feb 14 t=45 days later



$$dl^2 = [R(t)d\chi]^2 + [R(t)\sin\chi d\phi]^2$$

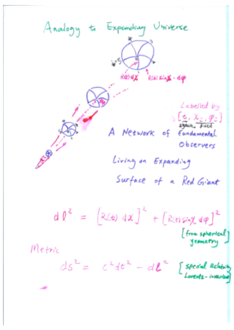
$$A_1 - B_2$$

Four Pillars of Hot Big Bang

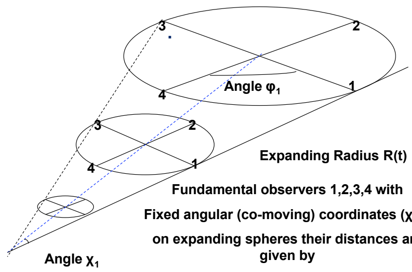
- **Galaxies moving apart from each other**
  - Redshift or receding from each other
  - Universe was smaller
- **Helium production outside stars**
  - Universe was hot, at least 10^9K to fuse 4H → He, to overcome a potential barrier of 1MeV.
- **Nearly Uniform Radiation 3K Background (CMB)**
  - Universe has cooled, hence expanded by at least a factor 10^9
- **Missing mass in galaxies and clusters (Cold Dark Matter: CDM)**
  - Cluster potential well is deeper than the potential due to baryons
  - CMB temperature fluctuations: photons climbed out of random potentials of DM

2nd Concept: metric of 1+2D universe

- **Analogy of a network of civilization living on an expanding star (red giant).**
  - What is fixed (angular coordinates of the grid points)
  - what is changing (distance).



Analogy: a network on a expanding sphere



Fundamental observers 1,2,3,4 with
Fixed angular (co-moving) coordinates (χ,φ)
on expanding spheres their distances are given by
Metric at cosmic time t $ds^2 = c^2 dt^2 - dl^2$,
$dl^2 = R^2(t) (d\chi^2 + \sin^2\chi\, d\varphi^2)$

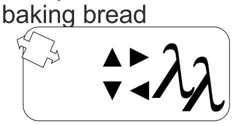
3rd Concept: The Energy density of Universe

- The Universe is made up of three things:
  - VACUUM
  - MATTER
  - PHOTONS (radiation fields)
- The total energy density of the universe is made up of the sum of the energy density of these three components.

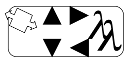
$$\varepsilon(t) = \varepsilon_{vac} + \varepsilon_{matter} + \varepsilon_{rad}$$

- From t=0 to t=10^9 years the universe has expanded by R(t).

Eq. of State for Expansion & analogy of baking bread



- Vacuum~air holes in bread
- Matter ~nuts in bread
- Photons ~words painted



- Verify expansion doesn't change N_{hole}, N_{proton}, N_{photon}
  - No Change with rest energy of a proton, changes energy of a photon

$$\varepsilon(t) = \rho_{eff}(t)c^2$$

$$\frac{\varepsilon(t)}{c^2} = \rho_{eff}(t)$$

- VACUUM ENERGY: $\rho = \text{constant} \quad \Rightarrow \text{E}_{vac} \propto R^3$
- MATTER:

$$\rho R^3 = \text{constant}, \quad \Rightarrow m \approx \text{constant}$$

- RADIATION:number of photons N_{ph} = constant

$$\Rightarrow n_{ph} \approx \frac{N_{ph}}{R^3} \qquad \text{Wavelength stretches} : \lambda \sim R$$

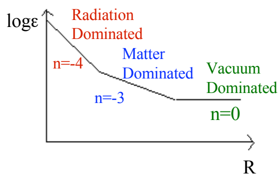
$$\text{Photons:E} = \text{h}\upsilon = \frac{\text{hc}}{\lambda} \sim \frac{1}{R}$$

$$\Rightarrow \varepsilon_{ph} \sim n_{ph} \times \frac{hc}{\lambda} \sim \frac{1}{R^4}$$

- The total energy density is given by:

$$\varepsilon \propto \varepsilon_{vac} + \varepsilon_{matter} + \varepsilon_{ph}$$

$\propto R^0$ $\qquad$ $\propto R^{-3}$ $\qquad$ $\propto R^{-4}$



Key Points

- **Scaling Relation among**
  - Redshift: z,
  - expansion factor: R
    - Distance between galaxies
  - Temperature of CMB: T
    - Wavelength of CMB photons: lambda
- **Metric of an expanding 2D+time universe**
  - Fundamental observers
    - Galaxies on grid points with fixed angular coordinates
- **Energy density in**
  - vacuum, matter, photon
  - How they evolve with R or z
- **If confused, recall the analogies of**
  - balloon, bread, a network on red giant star, microwave oven

Topics
Theoretical and Observational

- **Universe of uniform density**
  - Metrics ds, Scale R(t) and Redshift
  - EoS for mix of vacuum, photon, matter
- **Thermal history**
  - Nucleosynthesis
  - He/D/H
- **Structure formation**
  - Growth of linear perturbation
  - Origin of perturbations
  - Relation to CMB

Hongsheng.Zhao (hz4)

- **Quest of H0 (obs.)**
  - Applications of expansion models
  - Distances Ladders
    - (GL, SZ)
- **Quest for Omega (obs.)**
  - Galaxy/SNe surveys
  - Luminosity/Correlation Functions
- **Cosmic Background**
  - COBE/MAP/PLANCK etc.
  - Parameters of cosmos

Keith D. Horne (kdh1)

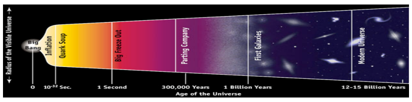
Lec 3

Acronyms in Cosmology

- **Cosmic Background Radiation (CBR)**
  - Or CMB (microwave because of present temperature 3K)
    - Argue about 10^5 photons fit in a 10cmx10cmx10cm microwave oven. [Hint: 3kT = h c / λ]
- **CDM/WIMPs: Cold Dark Matter, weakly-interact massive particles**
  - At time DM decoupled from photons, T ~ 10^{14}K, kT ~ 0.1 mc^2
    - Argue that dark particles were
      - non-relativistic (v/c << 1), hence "cold".
      - Massive (m >> m_{proton} =1 GeV)

Brief History of Universe

- **Inflation**
  - Quantum fluctuations of a tiny region
  - Expanded exponentially
- **Radiation cools with expansion $T \sim 1/R \sim t^{-2/n}$**
  - He and D are produced (lower energy than H)
  - Ionized H turns neutral (recombination)
  - Photon decouple (path no longer scattered by electrons)
- **Dark Matter Era**
  - Slight overdensity in Matter can collapse/cool.
  - Neutral transparent gas
- **Lighthouses (Galaxies and Quasars) form**
  - UV photons re-ionize H
  - Larger Scale (Clusters of galaxies) form



Acronyms and Physics Behind

- **DL: Distance Ladder**
  - Estimate the distance of a galaxy of size 1 kpc and angular size 1 arcsec? [About 0.6 10^9 light years]
- **GL: Gravitational Lensing**
  - Show that a light ray grazing a spherical galaxy of 10^{10} Msun at typical b=1 kpc scale will be bent ~$4GM/bc^2$ radian ~1 arcsec
  - It is a distance ladder
- **SZ: Sunyaev-Zeldovich effect**
  - A cloud of 1kev thermal electrons scattering a 3K microwave photon generally boost the latter's energy by 1kev/500kev=0.2%
  - This skews the blackbody CMB, moving low-energy photons to high-energy; effect is proportional to electron column density.

- **the energy density of universe now consists roughly**
  - Equal amount of vacuum and matter,
  - 1/10 of the matter is ordinary protons, rest in dark matter particles of 10Gev
    - Argue dark-particle-to-proton ratio ~ 1
  - Photons (3K ~10^{-4}ev) make up only 10^{-4} part of total energy density of universe (which is ~ proton rest mass energy density)
    - Argue photon-to-proton ratio ~ 10^{-4} GeV/(10^{-4}ev) ~ 10^9

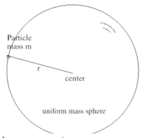
What have we learned?

- **Concepts of Thermal history of universe**
  - Decoupling
  - Last scattering
  - Dark Matter era
  - Compton scattering
  - Gravitational lensing
  - Distance Ladder
- **Photon-to-baryon ratio >>1**
- **If confused, recall the analogy of**
  - Crystalization from comic soup,
  - Last scattering photons escape from the photosphere of the sun

The rate of expansion of Universe

- Consider a sphere of radius $r=R(t)$ χ,
- If energy density inside is ρc^2

➔ Total effective mass inside is
$M = 4 \pi \rho r^3 /3$



- Consider a test mass m on this expanding sphere,
- For Test mass its
Kin.Energy + Pot.E. = const E

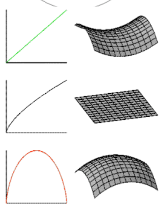
➔ $m (dr/dt)^2/2 - G m M/r = cst$

➔ $(dR/dt)^2/2 - 4 \pi G \rho R^2/3 = cst$

cst>0, cst=0, cst<0

$(dR/dt)^2/2 = 4 \pi G (\rho + \rho_{cur}) R^2/3$

where cst is absorbed by $\rho_{cur} \sim R^{(-2)}$



Typical solutions of expansion rate

$H^2=(dR/dt)^2/R^2=8\pi G (\rho_{cur}+ \rho_m + \rho_r + \rho_v)/3$

Assume domination by a component $\rho \sim R^{-n}$

Show Typical Solutions Are

$$\rho \propto R^{-n} \propto t^{-2}$$
$$n = 2(\text{curvature constant dominate})$$
$$n = 3(\textit{matter} \text{ dominate})$$
$$n = 4(\textit{radiation} \text{ dominate})$$
$$n \sim 0(\text{vaccum dominate}) : \ln(R) \sim t$$

- Argue also $H = (2/n)\, t^{-1} \sim t^{-1}$. Important thing is scaling!

Lec 4 Feb 22

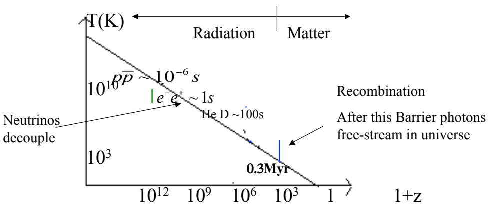
Where are we heading?

Next few lectures will cover a few chapters of
- Malcolm S. Longair's "Galaxy Formation" [Library Short Loan]

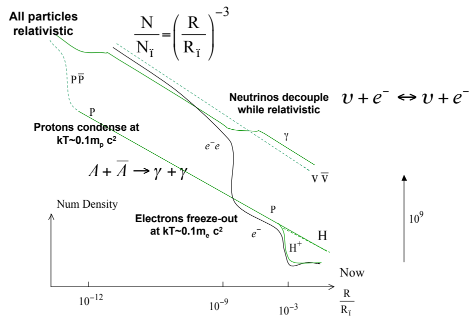
- Chpt 1: Introduction
- Chpt 2: Metrics, Energy density and Expansion
- Chpt 9-10: Thermal History

Thermal Schedule of Universe [chpt 9-10]

- At very early times, photons are typically energetic enough that they interact strongly with matter so the whole universe sits at a temperature dictated by the radiation.
- The energy state of matter changes as a function of its temperature and so a number of key events in the history of the universe happen according to a schedule dictated by the temperature-time relation.
- Crudely $(1+z) \sim 1/R \sim (T/3) \sim 10^9 (t/100s)^{(-2/n)} \sim 1000 (t/0.3Myr)^{-2/n}$, $H \sim 1/t$
- n~4 during radiation domination



A summary: Evolution of Number Densities of γ, P, e, υ



A busy schedule for the universe

- Universe crystalizes with a sophisticated schedule, much more confusing than simple expansion!
  - Because of many bosonic/fermionic players changing balance
  - Various phase transitions, numbers NOT conserved unless the chain of reaction is broken!
  - $p + p^- \leftrightarrow \gamma + \gamma$ (baryongenesis)
  - $e + e^+ \leftrightarrow \gamma + \gamma$, $v + e \leftrightarrow v + e$ (neutrino decouple)
  - $n \leftrightarrow p + e^- + v$, $p + n \leftrightarrow D + \gamma$ (BBN)
  - $H^+ + e^- \leftrightarrow H + \gamma$, $\gamma + e \leftrightarrow \gamma + e$ (recombination)
- Here we will try to single out some rules of thumb.
  - We will caution where the formulae are not valid, exceptions.
  - You are not required to reproduce many details, but might be asked for general ideas.

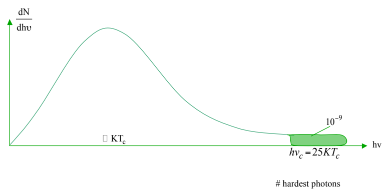
What is meant Particle-Freeze-Out?

- Freeze-out of equilibrium means NO LONGER in thermal equilibrium, means insulation.
- Freeze-out temperature means a species of particles have the SAME TEMPERATURE as radiation up to this point, then they bifurcate.
- Decouple = switch off = the chain is broken = Freeze-out

A general history of a massive particle

- **Initially mass doesn't matter in hot universe**
- **relativistic, dense (comparable to photon number density $\sim T^3 \sim R^{-3}$),**
  - **frequent collisions with other species to be in thermal equilibrium and cools with photon bath.**
  - **Photon numbers (approximately) conserved, so is the number of relativistic massive particles**

energy distribution in the photon bath



Initially zero chemical potential (~ Chain is on, equilibrium with photon)

- **The number density of photon or massive particles is :**

$$n = \frac{g}{h^3}\int_0^\infty \frac{d\left(\frac{4\pi}{3}p^3\right)}{\exp(E/kT)\pm 1} \qquad \begin{array}{l} + \text{ for Fermions} \\ - \text{ for Bosons} \end{array}$$

- **Where we count the number of particles occupied in momentum space and g is the degeneracy factor. Assuming zero cost to annihilate/decay/recreate.**

$$E = \sqrt{c^2p^2 + (mc^2)^2} \approx cp \quad \text{relativistic cp >> mc}^2$$

$$\approx m\,c^2 + \frac{1}{2}\frac{p^2}{m} \quad \text{non relativistic cp < mc}^2$$

- **As kT cools, particles go from**
- **From <u>Ultrarelativistic</u> limit. (kT>>mc²)**
  particles behave as if they were massless→

$$n = \left(\frac{kT}{c}\right)^3 \frac{4\pi g}{(2\pi\hbar)^3}\int_0^\infty \frac{y^2 dy}{e^y \pm 1} \Rightarrow n \sim T^3$$

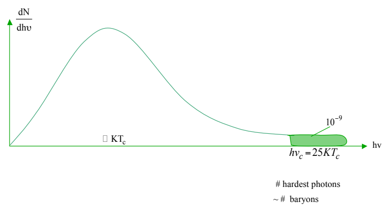
- **To <u>Non relativistic</u> limit (θ=mc²/kT > 10 , i.e., kT<< 0.1mc²) Here we can neglect the ±1 in the occupancy number→**

$$n = e^{-\frac{mc^2}{kT}}(2mkT)^{\frac{3}{2}}\frac{4\pi g}{(2\pi\hbar)^3}\int_0^\infty e^{-y^2}y^2 dy \Rightarrow n \sim T^{\frac{3}{2}}e^{-\frac{mc^2}{kT}}$$

When does freeze-out happen?

- **Happens when KT cools 10-20 times below mc^2, run out of photons to create the particles**
  - **Non-relativisitic decoupling**
- **Except for neutrinos**

particles of energy $E_c=h\nu_c$ unbound by high energy tail of photon bath



If run short of hard photon to unbind => "Freeze-out" => $KT_c \sim \frac{h\nu_c}{25}$

Rule 1. Competition of two processes

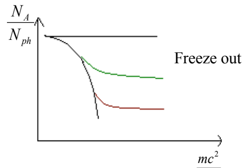
- **Interactions keeps equilibrium:**
  - **E.g., a particle A might undergo the annihilation reaction:**

$$A + \bar{A} \rightarrow \gamma + \gamma$$

- **depends on cross-section** σ and speed v. & most importantly
  - the **number density n of photons** (falls as $t^{(-6/n)}$, Why? Hint $R \sim t^{(-2/n)}$)
- **What insulates: the increasing gap of space between particles due to Hubble expansion $H \sim t^{-1}$.**
- **Question: which process dominates at small time? Which process falls slower?**

- **Rule 2. Survive of the weakest**

- **While in equilibrium, $n_A/n_{ph} \sim \exp(-\theta)$. (Heavier is rarer)**
- **When the reverse reaction rate $\sigma_A\upsilon$ is slower than Hubble expansion rate H(z) , the abundance ratio is frozen $N_A/N_{ph} \sim 1/(\sigma_A\upsilon) / T_{freeze}$**



$\sigma_A\upsilon$ LOW→ (v) smallest interaction, early freeze-out while relativistic

$\sigma_A\upsilon$ HIGH→ later freeze-out at lower T

- **Question: why frozen while n_A, n_{ph} both drop as $T^3 \sim R^{-3}$.**
- **$\rho_A \sim n_{ph}/(\sigma_A\upsilon)$, if $m \sim T_{freeze}$**

Effects of freeze-out

- Number of particles change (reduce) in this phase transition,
  - (photons increase slightly)
- Transparent to photons or neutrinos or some other particles
- This defines a "last scattering surface" where optical depth to future drops below unity.

Number density of non-relativistic particles to relativistic photons

- Reduction factor ~ exp(- θ), θ=mc²/kT, which drop sharply with cooler temperature.
- Non-relativistic particles (relic) become *much rarer* by exp(-θ) as universe cools below mc²/θ, $\theta \sim 10\text{–}25$.
  - So rare that infrequent collisions can no longer maintain coupled-equilibrium.
  - So Decouple = switch off = the chain is broken = Freeze-out

After freeze-out

- Particle numbers become conserved again.
- Simple expansion.
  - number density falls with expanding volume of universe, but Ratio to photons kept constant.

Small Collision cross-section

- Decouple non-relativistically once $kT<mc^2$. Number density ratio to photon drops steeply with cooling exp(- mc^2/kT).
  - wimps (Cold DM) etc. decouple (stop creating/annihilating) while non-relativistic. Abundance of CDM $\Omega \sim 1/\sigma_A \upsilon$
- $T_c \sim 10^9$K NUCLEOSYNTHESIS (100s)
- $T_c \sim 5000$K RECOMBINATION (0.3 Myrs) (z=1000)

For example,

- **Antiprotons freeze-out $t=(1000)^{-6}$ sec,**
- **Why earlier than positrons freeze-out t=1sec ?**
  - **Hint: anti-proton is ~1000 times heavier than positron.**
  - **Hence factor of 1000 hotter in freeze-out temperature**
- **Proton density falls as R^{-3} now, conserving numbers**
- **Why it falls exponentially $\exp(-\theta)$ earlier on**
  - **where $\theta = mc^2/kT \sim R$.**
  - **Hint: their numbers were in chemical equilibrium, but not conserved earlier on.**

SKIP SKIP SKIP
why fewer neutrons in universe than protons

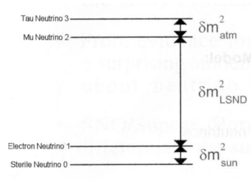
- **Before 1 s, lots of neutrinos and electrons keep the abundance of protons and neutrons about equal through**
  - **$n + \upsilon \leftrightarrow p + e^-$**
- **After 1 s free-moving neutrons (which is slightly more massive than protons) start to decay with half life ~ 10.3 min compared to proton ~ 10^{32} yr.**
  - **$n \rightarrow p + e^- + \upsilon$**
- **Some are locked into D.**
  - **p+n -> D + photon**

smallest Collision cross-section

- **neutrinos (Hot DM) decouple from electrons (due to very weak interaction) while still hot (relativistic 0.5 Mev ~ kT >mc^2 ~ 0.02-2 eV)**
-
- **Presently there are 3 x 113 neutrinos and 452 CMB photons per cm^3 . Details depend on**
  - **Neutrinos have 3 species of spin-1/2 fermions while photons are 1 species of spin-1 bosons**
  - **Neutrinos are a wee bit colder, 1.95K vs. 2.7K for photons [during freeze-out of electron-positions, more photons created]**

Counting neutrinos

- **There are at least 3 species of neutrinos: electron, muon, tau, perhaps more (called sterile neutrinos). Their masses are slightly different, all very light, they mix and oscillate,**



- At early times energy density of photons are high enough to produce particle pairs
  - the number density of photons was so high, and typical photons were so energetic
  - PHOTON+PHOTON←→PARTICLE +ANTI-PARTICLE
- The kinds of particles and anti-particles that are created depends on photon energy spectrum
  - Particularly, depends on the average energy per photon, which depends on the temperature.
  - If the photon energy is less than $m_p c^2$ then m_p can't be created;
  - as universe cools, more massive particles ceased to be created, while less massive particles were still allowed to be created.

NEUTRINO DECOUPLE as Hot DM

- Neutrinos are kept in thermal equilibrium by scattering (weak interaction):

$$\upsilon + e^- \leftrightarrow \upsilon + e^-$$

- This interaction freezes out when the temperature drops to kT_υ ~MeV~ rest mass electrons
  - Because very few electron-positions left afterwards (they become photons)
  - Neutrinos Move without scattering by electrons after 1 sec.
- Argue that Neutrinos have Relativistic speeds while freezing out
  - kT_υ >> rest mass of neutrinos(~eV)
  - They are called Hot Dark Matter (HDM)

SKIP SKIP SKIP
A worked-out exercise

$A + \bar{A} \rightarrow \gamma + \gamma$

Show at last scattering surface Optical depth $\tau = \int_0^z \sigma v \eta n_{ph}(z) \frac{dt}{dz} dz$

$\sim \int_0^z \sigma v \eta (1+z)^3 \frac{d(1+z)^{-n/2}}{dz} dz$

$\sim \sigma v \eta (1+z)^{3-n/2} \sim \sigma v \eta T^{3-n/2} \sim 1$.

where n=4 for radiation era.

Given that Freeze-out fraction $\eta \sim \exp(-\frac{\Delta mc^2}{kT})$

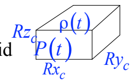
and assume decouple at $kT \sim mc^2/\ln(1/\eta)$,

Argue cosmic abundance

$\Omega \sim \eta m \sim T^{-1} m/(\sigma v) \sim (\sigma v)^{-1}$

Evolution of Sound Speed

Expand a box of fluid



Sound Speed $C_s^2 = \frac{\partial P / \partial(vol)}{\partial \rho / \partial(vol)} = \frac{\partial P / \partial R}{\partial \rho / \partial R}$

$Vol = R^3(t)\, x_c y_c z_c \propto R^3(t)$

Coupled radiation-baryon relativistic fluid

	Radiation	Matter
Where fluid density $\rho(t) =$	ρ_r	ρ_m
Fluid pressure $P(t) =$	$\frac{c^2}{3}\rho_r$	$\frac{\rho_m}{\mu} \cdot KT_m$

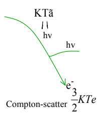
Matter number density — Random motion energy Non-Relativistic IDEAL GAS

Note $\rho_r \propto R^{-4}$

$\rho_m \propto R^{-3}$ Neglect $\frac{1}{\mu} KT_m << c^2$

Show $C_s^2 = c^2/3\,/(1+Q)$, $Q = (3\,\rho_m)\,/(4\,\rho_r)$, ➔ C_s drops
- **from c/sqrt(3) at radiation-dominated era**
- **to c/sqrt(5.25) at matter-radiation equality**

Coupled Photon-Baryon Fluid



Compton-scatter $\frac{3}{2}KTe$
electrons in γ – bath

Keep electrons hot Te ~ Tr until redshift z

$Tr \approx 1500 \times \left(\frac{1+z}{500}\right)$

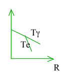
Temperature and Sound Speed of Decoupled Baryonic Gas

After decoupling (z<500),
Cs ~ 6 (1+z) m/s because

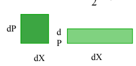
$d^3P\, d_x^3$ invarient phase space volume

So: $P \propto x^{-1} \propto R^{-1}$

$\frac{3}{2} \times T_e = mv^2/2 \propto R^{-2}$ $\quad Te \sim 1500\,K? \left(\frac{1+z}{500}\right)^2$



Until reionization z ~ 10 by stars quasars



$T_e \propto C_s^2 \propto R^{-2}$

What have we learned?

Where are we heading?

- **Sound speed of gas before/after decoupling**

Topics Next:

- **Growth of [chpt 11 bankruptcy of uniform universe]**
  - Density Perturbations (how galaxies form)
  - peculiar velocity (how galaxies move and merge)
- **CMB fluctuations (temperature variation in CMB)**
- **Inflation (origin of perturbations)**

Peculiar Motion

- **The motion of a galaxy has two parts:**

$$\vec{v} = \frac{d}{dt}\left[R(t)\theta(t)\right]$$

$$= \dot{R}(t).\theta + R(t)\dot{\theta}(t)$$

$R(t)\theta(t)$: Proper length vector

$\dot{R}(t).\theta$: Uniform expansion v_o

$R(t)\dot{\theta}(t)$: Peculiar motion δv

Damping of peculiar motion (in the absence of overdensity)

- **Generally peculiar velocity drops with expansion.**

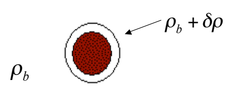
$$R^2\dot{\theta} = R*(R\dot{\theta}) = \text{constant}\sim\text{"Angular Momentum"}$$

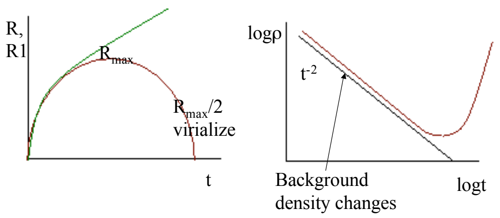
- **Similar to the drop of (non-relativistic) sound speed with expansion**

$$\delta v = R(t)\dot{x}_c = \frac{\text{constant}}{\text{R(t)}}$$

Non-linear Collapse of an Overdense Sphere

- **An overdense sphere is a very useful non linear model as it behaves in exactly the same way as a closed sub-universe.**
- **The density perturbations need not be a uniform sphere: any spherically symmetric perturbation will clearly evolve at a given radius in the same way as a uniform sphere containing the same amount of mass.**





$$\rho_b = \frac{1}{6\pi G t^2}$$

Gradual Growth of perturbation

$$\rightarrow \frac{\delta\rho}{\rho} = \frac{3c^2}{8\pi G}\frac{1}{\rho R^2} \propto \begin{cases} R^2 & \text{(mainly radiation } \rho \propto R^{-4}) \\ R & \text{(mainly matter } \rho \propto R^{-3}) \end{cases}$$

Perturbations Grow!

Verify δ changes by a factor of 10 between z=10 and z=100? And a factor of 100 between $z=10^5$ and $z=10^6$?

Equations governing Fluid Motion

$$\nabla^2\phi = 4\pi G\rho \qquad \text{(Poissons Equation)}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{d\ln\rho}{dt} = -\vec{\nabla}.\vec{v} \qquad \text{(Mass Conservation)}$$

$$\frac{dv}{dt} = -\vec{\nabla}\phi - c_s^2\vec{\nabla}\ln\rho \qquad \text{(Equation of motion)}$$

$$\frac{\nabla P}{\rho} \quad \text{since } \partial P = c_s^2 \partial\rho$$

Decompose into unperturbed + perturbed

- **Let**

$$\rho = \rho_o + \delta\rho$$
$$v = v_o + \delta v = \dot{R}\chi_c + R\dot{\chi}_c$$
$$\phi = \phi_o + \delta\phi$$

- **We define the Fractional Density Perturbation:**

$$\delta = \frac{\delta\rho}{\rho_o} = \delta(t)\exp(-i\vec{k}\bullet\vec{x}),$$
$$|\vec{k}| = 2\pi/\lambda, \quad \text{where } \lambda = R(t)\lambda_c$$
$$\vec{k}\bullet\vec{x} = \vec{k}_c\bullet\vec{x}_c \qquad x(t) = R(t)\chi_c$$

- **Motion driven by gravity: due to an overdensity:** $\vec{g}_o(t) + \vec{g}_1(\theta, t)$

$$\rho(t) = \rho_o(1 + \delta(\theta, t))$$

- **Gravity and overdensity by Poisson's equation:**
- **Continuity equation:**

$$-\vec{\nabla} \bullet g_1 = 4\pi G \rho_o \delta$$

Peculiar motion δv and peculiar gravity g_1 both scale with δ and are in the same direction.

$$-\vec{\nabla} \bullet \delta\vec{v} = \frac{d}{dt}\left(\delta(\theta, t)\right)$$

The over density will rise if there is an inflow of matter

THE equation for structure formation

- **In matter domination**
- **Equation becomes**

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{R}}{R}\frac{\partial \delta}{\partial t} = (4\pi G \rho_o + c_s^2 \nabla^2)\delta$$

$-c_s^2 k^2$

Gravity has the tendency to make the density perturbation grow exponentially.

Pressure makes it oscillate

- **Each eq. is similar to a forced spring**

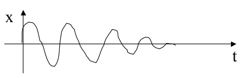
$$\frac{d^2 x}{dt^2} = \frac{F}{m} - \omega^2 x - \mu \frac{dx}{dt}$$

Restoring

Term due to friction

$$\frac{d^2 x}{dt^2} + \mu \frac{dx}{dt} + \omega^2 x = \frac{F(t)}{m}$$

(Displacement for Harmonic Oscillator)



e.g., Nearly Empty Pressure-less Universe

$$\rho \sim 0$$

$$\frac{\partial^2 \delta}{\partial t^2} + \frac{2}{t}\frac{\partial \delta}{\partial t} = 0, \quad H = \frac{\dot{R}}{R} = \frac{1}{t} \quad (R \propto t)$$

$$\delta \propto t^0 = \text{constant}$$

$$\rightarrow \text{no growth}$$

What have we learned? Where are we heading?

- **OverDensity grows as**
  - **R (matter) or R^2 (radiation)**
- **Peculiar velocity points towards overdensities**
- **Topics Next: Jeans instability**

The Jeans Instability

- **Case 1- no expansion** $\dot{R} = 0$
  - **the density contrast δ has a wave-like form**

for the harmonic oscillator equation

$$\delta = \delta_o \exp(i\vec{k}.\vec{r} - i\omega t)$$

where we have the dispersion relation

$$\frac{\partial^2 \delta}{\partial t^2} + 2 * 0 * \frac{\partial \delta}{\partial t} = -\omega^2 \delta$$

$$\omega^2 \equiv c_s^2 k^2 - 4\pi G\rho$$

Pressure support ($c_s^2 k^2$)   gravity ($4\pi G\rho$)

- **At the (proper) JEANS LENGTH scale we switch from**
  - **Oscillations for shorter wavelength modes to**
  - **the exponential growth of perturbations for longer wavelength**

$$\lambda_J = c_s \tau, \quad \text{where timescale } \tau = \sqrt{\frac{\pi}{G\rho}}$$

- **$\lambda < \lambda_J$, $\omega^2 > 0$ → oscillation of the perturbation.**

- **$\lambda \geq \lambda_J$, $\omega^2 \leq 0$ → exponential growth/decay**

$$\delta \propto \exp(\pm \Gamma t) \text{ where } \Gamma = \sqrt{-\omega^2}$$

Jeans Length in background of constant or falling density

- **Background of Constant density :**
  - **Application: Collapse of clouds, star formation.**
  - **Timescale:**

$$\tau = (G\rho/\pi)^{-\frac{1}{2}}$$

$\sim$ dynamical collapse time

for region of uniform density ρ.

- **Background of Falling density**
  - **Expanding universe $G\rho \sim t^{-2}$,**
  - **Instantaneous Jeans length ~ $c_s t$**

Jeans Instability

- Case 2: on very large scale $\lambda >> \lambda_J = c_s\, t$ of an Expanding universe
  - Neglect Pressure (restoring force) term
  - Grow as delta ~ R ~ $t^{2/3}$ for long wavelength mode if Omega_m=1 universe.

$$c_s^2 k^2 << 4\pi G\rho = c_s^2 k_J^2$$

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} = \underline{\underline{4\pi G\rho_m \delta}}$$

$$2/(3t) \qquad\qquad 2/(3t^2)$$

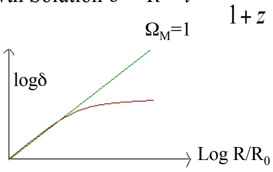
E.g.,

- Einstein de Sitter Universe

$$\Omega_M = 1, H = \frac{\dot R}{R} = \frac{2}{3t}$$

$$\text{Verify Growth Solution } \delta \propto R \propto t^{\frac{2}{3}} \propto \frac{1}{1+z}$$

- Generally



Case III: Relativistic (photon) Fluid

- equation governing the growth of perturbations being:

$$\Rightarrow \frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \delta.\left(\frac{32\pi G\rho}{3} - k^2 c_s^2\right)$$

$$1/t \qquad\qquad 1/t^2$$

- Oscillation solution happens on small scale $2\pi/k = \lambda < \lambda_J$
- On larger scale, growth as

$$\Rightarrow \delta \propto t \propto R^2 \quad \text{for length scale } \lambda >> \lambda_J \sim c_s t$$

SKIP SKIP SKIP Jeans Mass Depends on the Species of the Fluid that dominates

- If Photon dominates:

$$M_J^\gamma = \rho_\gamma(t)\frac{4\pi}{3}\left(\frac{\lambda_J}{2}\right)^3 \propto \frac{1}{6t^2}\left[\underline{\underline{\frac{c}{\sqrt{3}}t}}\right]^3 \propto t^1 \propto (1+z)^{-2}$$

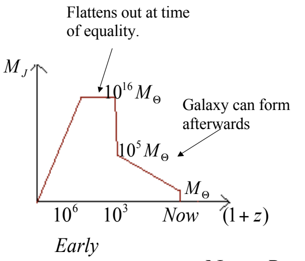
$c_s t$=distance travelled since big bang

- If Dark Matter dominates & decoupled from photon:

$$M_J^D = \rho_D(t)\frac{4\pi}{3}\left(\frac{\lambda_J}{2}\right)^3 \propto t^{-2}\left[c_s t\right]^3 \propto t^{-1}$$

non-relativistic cooling of random motion $c_s \propto 1/R \propto t^{-2/3}$

- SKIP SKIP SKIP Jeans Mass past and now



Note: $R \propto (1+z)^{-1}$

SKIp SKIp SKIP: Dark Matter Overdensity Growth Condition

- **GROW [~Collapse] only if**
  - **During matter-domination ($t > t_{eq}$) [chpt 11.4] or**
  - **during radiation domination, but on proper length scales larger than**
    - **sound horizon ($\lambda > c_s\ t$) [chtp12.1] &**
    - **free-streaming length of relativistic dark matter ($\lambda > c\ t_{fs}$) [chpt 13.3]**

Lec 8

- **What have we learned: [chpt 11.4]**
  - **Conditions of gravitational collapse (=growth)**
  - **Stable oscillation (no collapse) within sound horizon if pressure-dominated**
- **Where are we heading:**
  - **Cosmic Microwave Background [chpt 15.4]**
    - **As an application of Jeans instability**
  - **Inflation in the Early Universe [chpt 20.3]**

Theory of CMB Fluctuations

- **Linear theory of structure growth predicts that the perturbations:**

$$\delta_D \text{ in dark matter } \frac{\delta\rho_D}{\rho_D}$$

$$\delta_B \text{ in baryons } \frac{\delta\rho_B}{\rho_B}$$

$$\delta_r \text{ in radiation } \frac{\delta\rho_r}{\rho_r} \quad \text{Or} \quad \tilde{\delta}_r = \frac{3}{4}\delta_r = \frac{\delta n_\gamma}{n_\gamma}$$

will follow a set of coupled Harmonic Oscillator equations.

- **The solution of the Harmonic Oscillator [within sound horizon] is:**

$$\delta(t) = A_1 \cos kc_s t + A_2 \sin kc_s t + A_3$$

- **Amplitude is sinusoidal function of $k\, c_s\, t$**
  - if k=constant and oscillate with t
  - or t=constant and oscillate with k.

$$\frac{d^2}{dt^2}\begin{pmatrix}\delta_D\\ \delta_B\\ \tilde{\delta}_r\end{pmatrix} + 2H(t)\frac{d}{dt}\begin{pmatrix}\delta_D\\ \delta_B\\ \tilde{\delta}_r\end{pmatrix} + k^2\begin{pmatrix}c_{s,D}^2\delta_D\\ c_{s,B}^2\delta_B\\ c_{s,r}^2\tilde{\delta}_r\end{pmatrix} = \nabla^2\Psi = -k^2\Psi$$

- **Where ψ is the perturbation in the gravitational potential, with SKIp SKIp SKIP**

$$\Psi_{x,t} \propto \Psi(t)\exp(i\vec{k}.\vec{x})$$

Gravitational Coupling

$$\Psi = 4\pi G\delta\rho_D + 4\pi G\delta\rho_B + 8\pi G\delta\rho_r$$
$$= 4\pi G\rho_{crit} \times \left[\Omega_D\delta_D + \Omega_B\delta_B + 2\Omega_r\delta_r\right]$$

- **We don't observe the baryon overdensity δ_B directly**
- **-- what we actually observe is temperature fluctuations.**

$$\frac{\Delta T}{T} = \frac{\Delta n_\gamma}{3n_\gamma} = \frac{\delta_B}{3} = \frac{\tilde{\delta}_R}{3}$$

$$n_\gamma \sim R^{-3} \propto T^3$$
$$\varepsilon_\gamma \sim n_\gamma kT \propto T^4$$

- **The driving force is due to dark matter over densities.**
- **The observed temperature is:**

$$\left(\frac{\Delta T}{T}\right)_{obs} = \frac{\delta_B}{3} + \frac{\psi}{c^2}$$

Effect due to having to climb out of gravitational well

- **The observed temperature also depends on how fast the Baryon Fluid is moving.**

Velocity Field $\nabla v = -\frac{d\delta_B}{dt}$

$$\left(\frac{\Delta T}{T}\right)_{obs} = \frac{\delta_B}{3} + \frac{\psi}{c^2} \pm \frac{v}{c}$$

Doppler Term

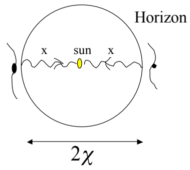
Inflation in Early Universe [chtp 20.3]

Consider universe goes through a phase with

$\rho(t) \sim R(t)^{-n}$

$R(t) \sim t^{q}$ *where* q=2/n

- **Problems with normal expansion theory (n=2,3,4):**
  - **What is the state of the universe at t→0? Pure E&M field (radiation) or exotic scalar field?**
  - **Why is the initial universe so precisely flat?**
  - **What makes the universe homogeneous/similar in opposite directions of horizon?**
- **Solutions: Inflation, i.e., n=0 or n<2**
  - **Maybe the horizon can be pushed to infinity?**
  - **Maybe there is no horizon?**
  - **Maybe everything was in Causal contact at early times?**



Why are these two galaxies so similar without communicating yet?

$$\frac{\varepsilon_K(z)}{\varepsilon(z)} = \frac{\varepsilon_K(0) \times R^{-2}}{\varepsilon(0) \times R^{-n}} \sim R^{n-2} \sim 0 \text{ at } t = 0$$

Why is the curvature term so small (universe so flat) at early universe if radiation dominates n=4 >2?

What have we learned?

- **What determines the patterns of CMB at last scattering**
  - **Analogy as patterns of fine sands on a drum at last hit.**
- **The need for inflation to**
  - **Bring different regions in contact**
  - **Create a flat universe naturally.**

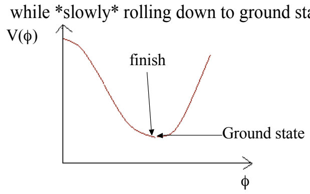
Inflationary Physics

- **Involve quantum theory to z~10^{32} and perhaps a scalar field φ(x,t) with energy density**

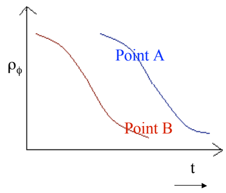
$$\rho_\phi = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi) \sim R(t)^{-n}, \text{where n<<1}$$

fluctuate between neighbouring points [A,B]

while *slowly* rolling down to ground state



- S
- **A slightly different finishing time (Quantum Effect) of inflation at different positions leads to slight perturbations to curvatures, which seed structure formation.**



- **Speculative at best.**

Inflation broadens Horizon

- **Light signal travelling with speed c on an expanding sphere R(t), e.g., a fake universe R(t)=1lightyr (t/1yr)^q**
  - **Emitted from time t_i**
  - **By time t=1yr will spread across (co-moving coordinate) angle x_c**

Horizon in co-moving coordinates

$$x_c = \int_{t_i}^{1} \frac{\mathrm{cdt}}{\mathrm{R(t)}} = \int_{t_i}^{1} \frac{\mathrm{cdt}}{\mathrm{t^q}} = \frac{(1^{1-q} - t_i^{1-q})}{(1-q)}$$

Normally $x_c < \frac{1}{(1-q)}$ is finite if q=2/n<1

(e.g., n=3 matter-dominate or n=4 photon-dominate)

INFLATION phase $x_c = \frac{(t_i^{1-q} - 1)}{(q-1)}$ can be very large for very small t_i if q=2/n>1

(e.g., $t_i = 0.01, q = 2, x_c = 99 >> \pi$, Inflation allows we see everywhere)

Inflation dilutes the effect of initial curvature of universe

$$\frac{\varepsilon_K(R)}{\varepsilon\ (R)} = \frac{\varepsilon_K(R_i)}{\varepsilon(R_i)} \left(\frac{R}{R_i}\right)^{n-2} \sim 0 \text{ (for n<2) sometime after } R>>R_i$$

even if initially the universe is curvature-dominated $\frac{\varepsilon_K(R_i)}{\varepsilon(R_i)} = 1$

E.g.

If a toy universe starts with $\frac{\varepsilon_K(R_i)}{\varepsilon(R_i)} = 0.1$ inflates from $t_i=10^{-40}$sec to t_f=1sec with n=1,

and then expand normally with n=4 to t=1 year,

SHOW at this time the universe is far from curvature-dominated.

Exotic Pressure drives Inflation

$$P = -\frac{d(\rho c^2 R^3)}{d(R^3)}$$

=>

$$\frac{\rho}{3} + \frac{P}{c^2} = -\frac{d(\rho R^2)}{3RdR} = \frac{n-2}{3}\rho \quad \text{if } \rho \sim R^{-n}$$

=>

$P/\rho c^2$=(n-3)/3

Inflation $n < 2$ requires exotic (negative) pressure,

define w=$P/\rho c^2$, then w = (n-3)/3<0,

Verify negligble pressure for cosmic dust (matter),

Verify for radiation P=$\rho c^2 / 3$

Verify for vaccum P=-ρc^2

What Have we learned?

- How to calculate Horizon.
- The basic concepts and merits of inflation
- Pressure of various kinds (radiation, vacuum, matter)

Expectations for my part of the Exam

- Remember basic concepts (or analogies)
  - See list
- Can apply various scaling relations to do *some* of the short questions at the lectures.
  - See list
- *Relax*.
  - thermal history and structure formation are advanced subjects with lots of details. Don't worry about details and equations, just be able to recite the big picture.
- *If you like*, you can read reference texts to have deeper understanding of the lectured material.
  - Only material on this Final Notes is examinable.

Why Analogies in Cosmology

- Help you memorizing
  - Cosmology calls for knowledge of many areas of physics.
  - Analogies help to you memorize how things move and change in a mind-boggling expanding 4D metric.
- *Help you reason*, avoid "more equations, more confusions".
  - During the exam, You might be unsure about equations and physics,
    - the analogies *help you reason* and *recall* the right scaling relations, and get the big picture right.
- *Months after the exam*,
  - Analogies go a long way

List of keys

- Scaling relations among
  - Redshift z, wavelength, temperature, cosmic time, energy density, number density, sound speed
  - Definition formulae for pressure, sound speed, horizon
  - Metrics in simple 2D universe.
- Describe in words the concepts of
  - Fundamental observers
  - thermal decoupling
    - Common temperature before,
    - Fixed number to photon ratio after
    - Hot and Cold DM.
  - gravitational growth.
    - Over-density,
    - direction of peculiar motion driven by over-density, but damped by expansion
    - pressure support vs. grav. collapse

•Enjoy Prof. Horne's Lectures

Tutorial

- Consider a micro-cosmos of N-ants inhabiting an expanding sphere of radius $R=R_0\ (t/t_0)^q$, where presently we are at $t=t_0$ =1year, $R=R_0$ =1m. Let q=1/2, N=100, and the ants has a cross-length σ=1cm for collision. Let each ant keep its random angular momentum per unit mass J=1m*1(m/yr) with respect to the centre of the sphere.
  - What is the present rate of expansion dR/dt/R = in units of 1/yr,
  - How does the ant random speed, ant surface density, change as function of cosmic time?
  - Light emitted by ant-B travels a half circle and reaches ant-A now, what redshift was the light emitted?
  - What is the probability that the ant-A would encounter another ant from time t_1 to time t_2. How long has it travelled? Calculate assume t_1 =1/2 yr, t_2 =2yr.

E.g.

- As in previous universe but with n=3, Argue that the horizon of a non-relativistic moving ant at time t=1yr is also finite.
- Assuming the ant moves with 1cm/sec now, but was faster earlier on, estimate the age of universe when it was moving relativistically? Estimate how much it has moved from time zero to t=1 yr. What fraction of the length was in the relativistic phase?

- Show the age of the universe is t=1sec at $z\sim10^{10}$; assume crudely that at matter-radiation equality $z=10^3$ and age $t=10^6$ yr
  - Argue that a void in universe now originates from an under-dense perturbation at $z=10^{10}$ with δ about 10^{-17}.
  - The edge of the void are lined up by galaxies. What direction is their peculiar gravity and peculiar motion?
- A patch of sky is presently hotter in CMB by 3 micro Kelvin than average. How much was it hotter than average at the last scattering (z=1000)?