AS 4022: Cosmology

H Zhao

Online notes:

star-www.st-and.ac.uk/~hz4/cos/cos.html

take your own notes (including blackboard lectures

AS 4022 Cosmology

Text (intro): Andrew Liddle: Intro to Modern Cosmology

(advanced): John Peacock: Cosmological Physics

Malcolm Longaire: galaxy formation (chapters 1,2,9,10)

Web Lecture Notes: John Peacock, Ned Wright

Why Study Cosmology?

- **Fascinating questions:**
  - Birth, life, destiny of our Universe
  - Hot Big Bang --> (75% H, 25% He) observed in stars!
  - Formation of structure (galaxies …)

- **Technology -> much recent progress:**
  - Precision cosmology: uncertainties of 50% --> 2%

- **Deep mysteries remain:**
  - Dark Matter? Dark Energy? General Relativity wrong?

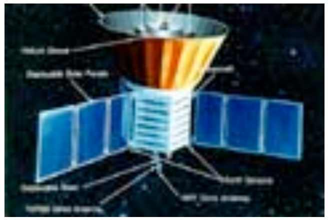
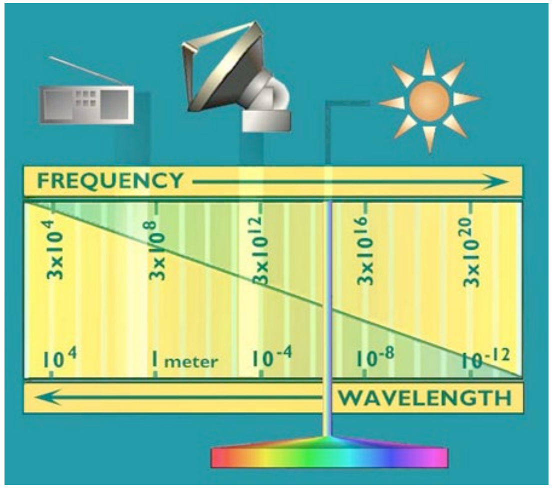
- **Stretches your mind:**
  - Curved expanding spaces, looking back in time, …

Observable Space-Time and Bands

See What is out there? In all Energy bands

Pupil → Galileo's Lens → 8m telescopes → square km arrays

Radio, Infrared ← optical → X-ray, Gamma-Ray (spectrum)

COBE satellites ← Ground → Underground DM detector

Know How were we created? XYZ & T ?

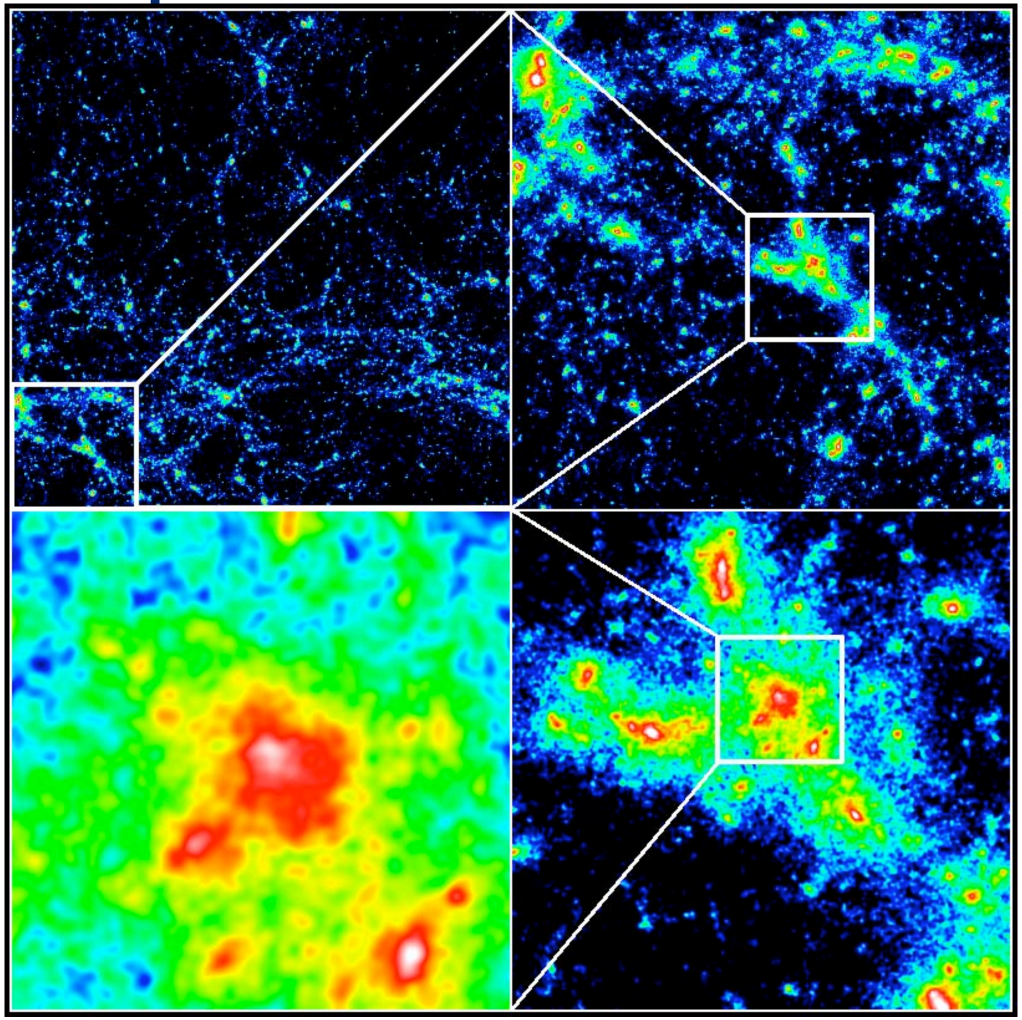
Us, CNO in Life, Sun, Milky Way, … further and further

→ first galaxy → first star → first Helium → first quark

Now → Billion years ago → first second → quantum origin

The Visible Cosmos: a hierarchy of structure and motion

"Cosmos in a computer"



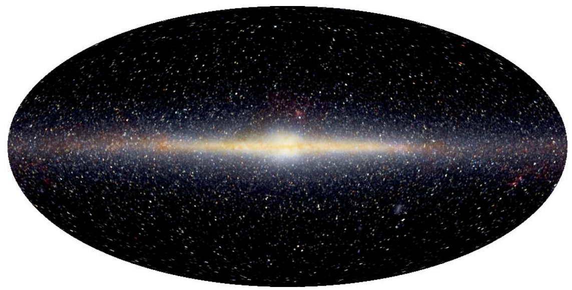
Observe A Hierarchical Universe

Planets

moving around stars;

Stars grouped together,

moving in a slow dance around the center of galaxies.



Cosmic Village

The Milky Way and Andromeda galaxies,

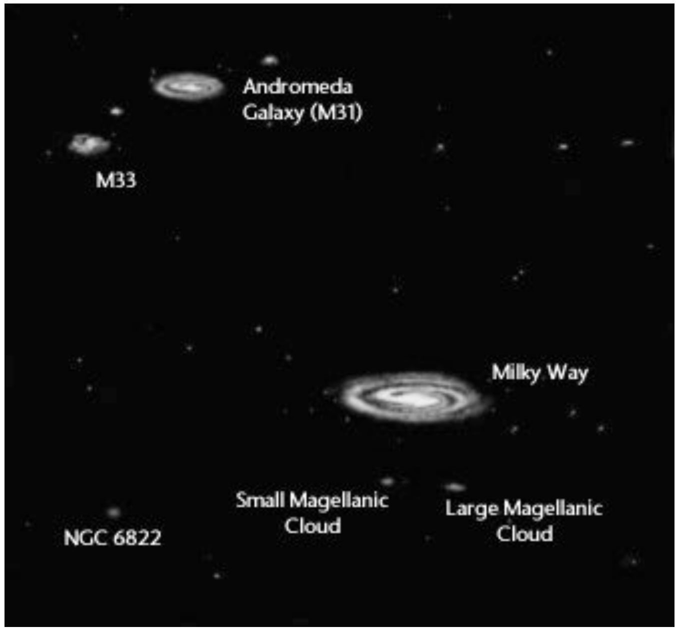
along with about fifteen or sixteen smaller galaxies,
form what's known as the Local Group of galaxies.

The Local Group

sits near the outer edge of a supercluster, the Virgo cluster.
the Milky Way and Andromeda are moving toward each other,
the Local Group is falling into the middle of the Virgo cluster, and

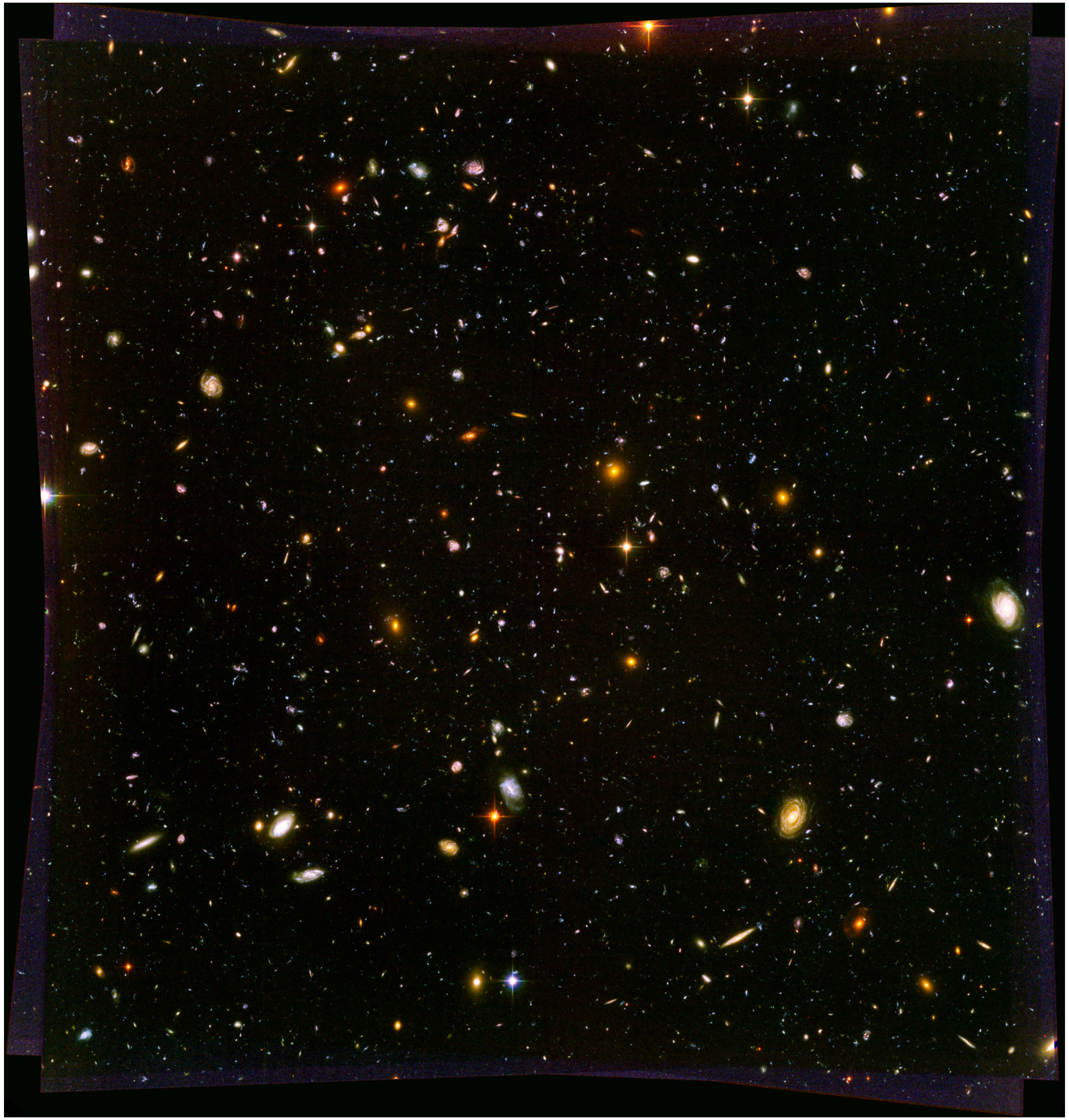
the entire Virgo cluster itself,

is speeding toward a mass
known only as "The Great Attractor."



Hubble Deep Field:

At faint magnitudes, we see **thousands of Galaxies for every star** !



~10^{10} galaxies in the visible Universe

~10^{10} stars per galaxy

~10^{20} stars in the visible Universe

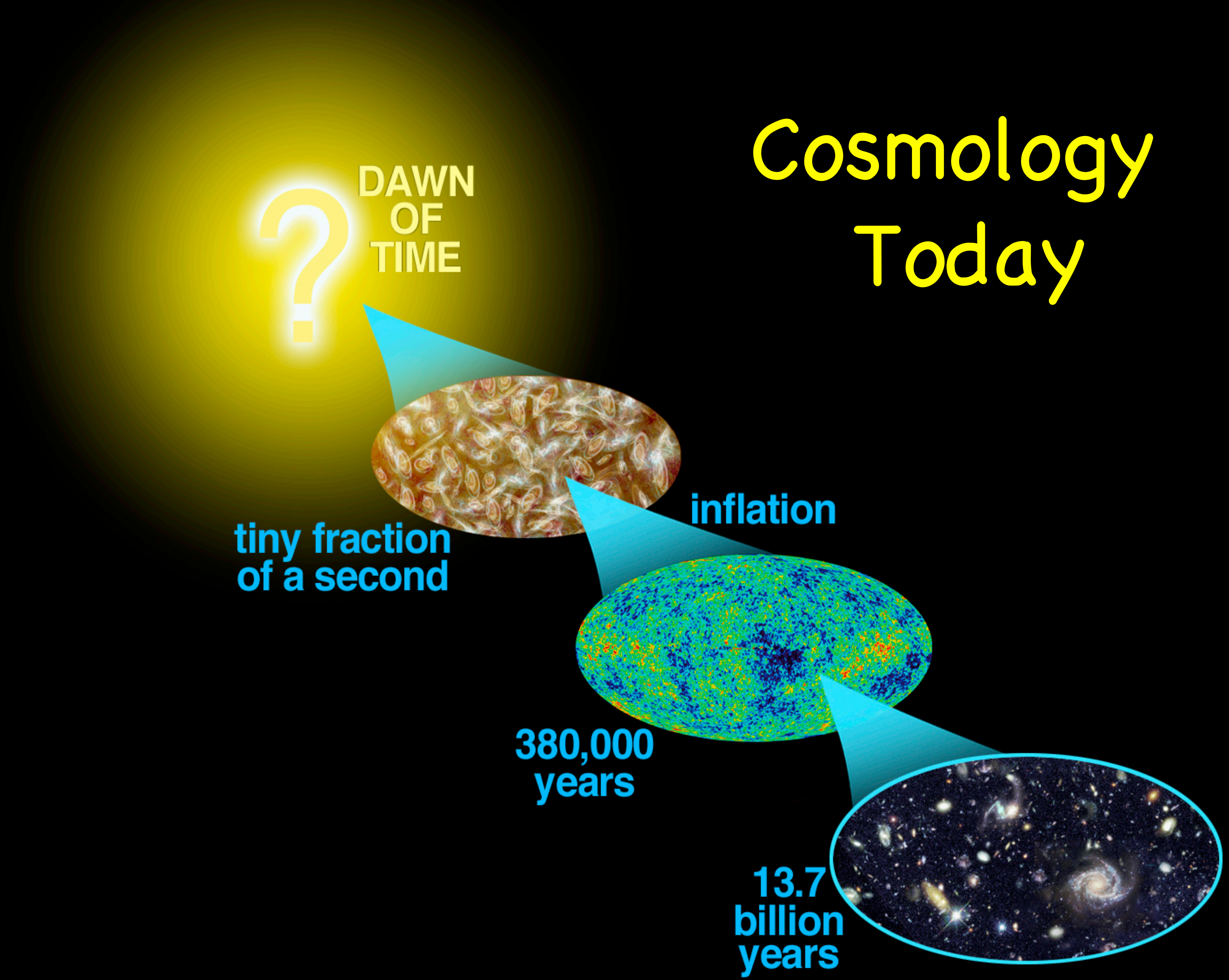
Galaxies themselves

- some 100 billion of them in the observable universe—
- form galaxy clusters bound by gravity as they journey through the void.

But the largest structures of all are superclusters,

- each containing thousands of galaxies
- and stretching many hundreds of millions of light years.
- are arranged in filament or sheet-like structures,
- between which are gigantic voids of seemingly empty space.

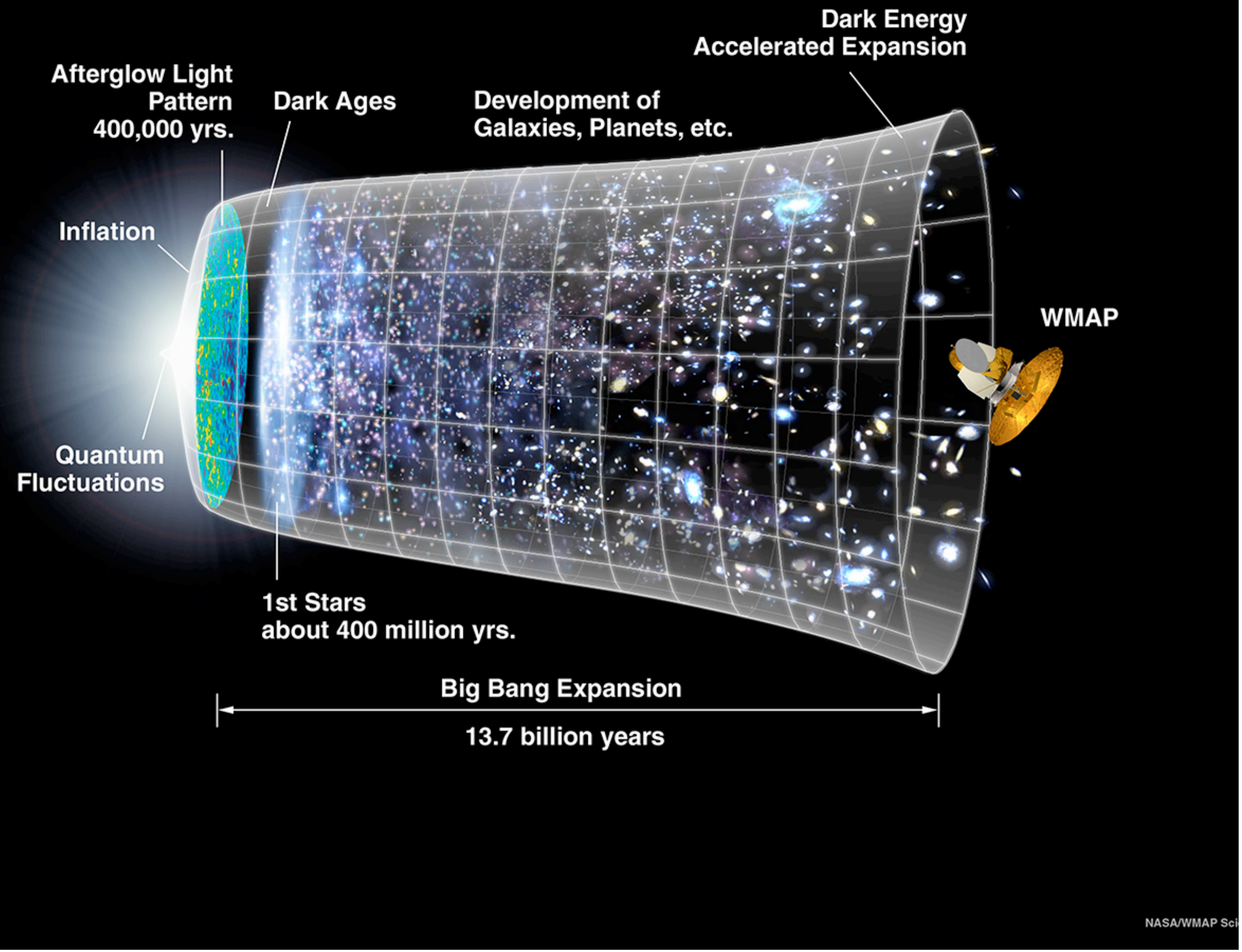
Cosmology Today



1980: Inflation (Alan Guth)

- Universe born from "**nothing**" ?
- A **quantum fluctuation** produces a tiny bubble of "**False Vacuum".**
- High vacuum energy drives **exponential expansion**, also known as **"inflation."**
- Universe expands by huge factor in tiny fraction of second, as false vacuum returns to true vacuum.
- Expansion so fast that **virtual particle-antiparticle pairs** get separated to become **real particles and anti-particles**.
- Stretches out all structures, giving a **flat geometry** and uniform T and ρ, with **tiny ripples**.
- Inflation launches the **Hot Big Bang**!

Accelerating/Decelerating Expansion



Introducing Gravity and DM (Key players)

These structures and their movements

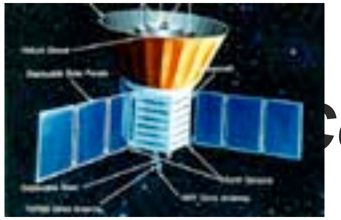
can't be explained purely by the expansion of the universe

must be guided by the gravitational pull of matter.

Visible matter is not enough

one more player into our hierarchical scenario:

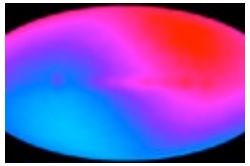
dark matter.

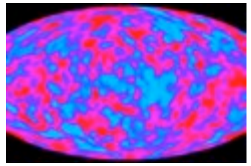


Cosmologists hope to answer these questions:

How old is the universe? H_0

Why was it so smooth? P(k), inflation





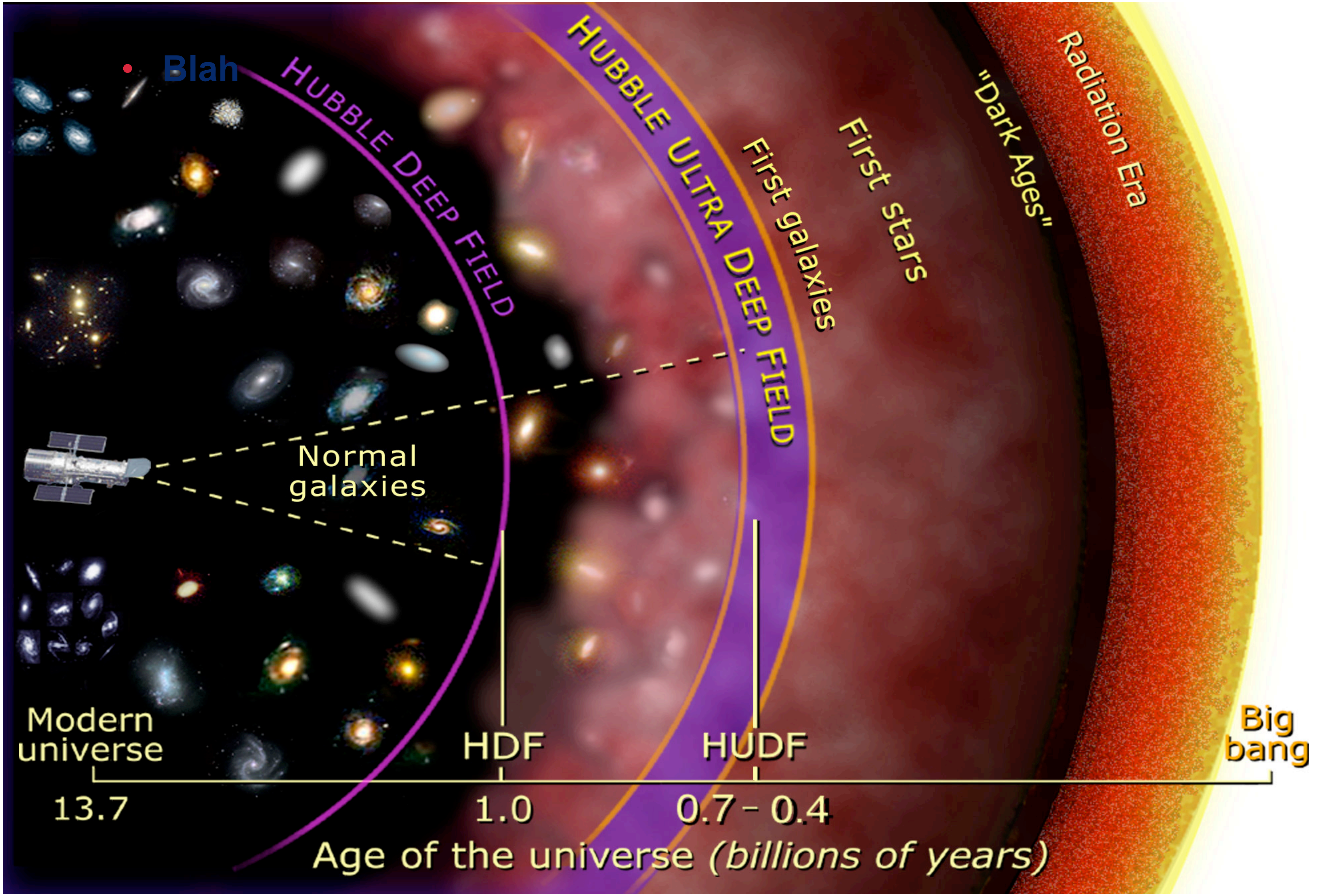
How did structures emerge from smooth? N-body

How did galaxies form? Hydro

Will the universe expand forever? Omega, Lamda

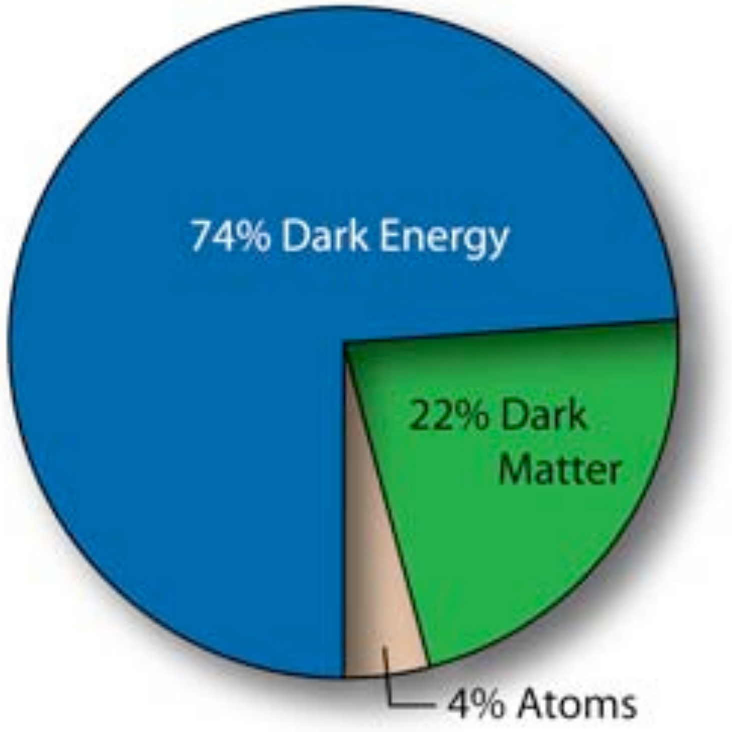
Or will it collapse upon itself like a bubble?

Looking Back in Time



- Blah

Current Mysteries



Dark Matter ?

Holds Galaxies together
Triggers Galaxy formation

Dark Energy ?

Drives Cosmic Acceleration.

Modified Gravity ?

General Relativity wrong ?

main concepts in cosmology

Expansion & Metric

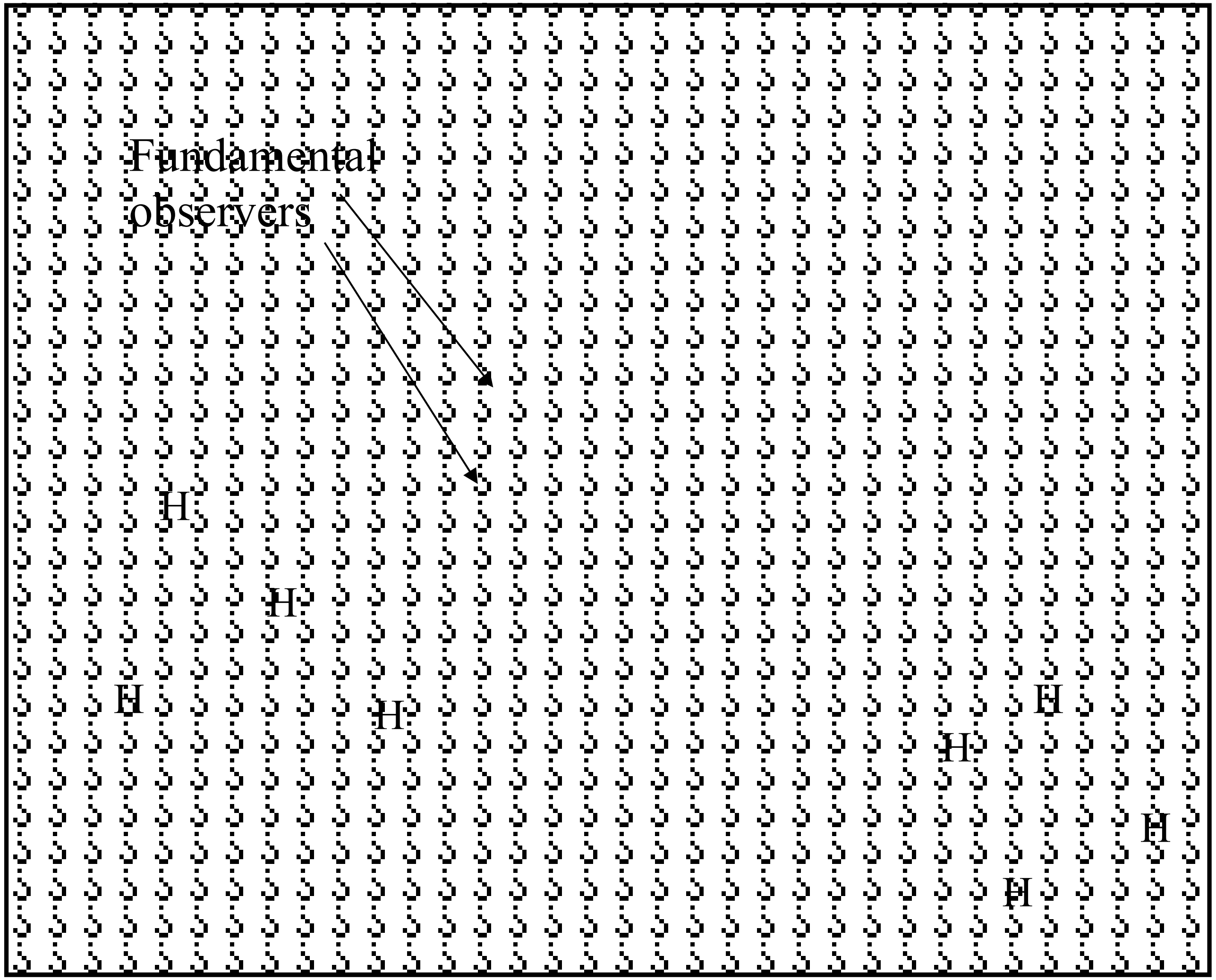
Cosmological Redshift

Energy density

Trafalgar Square

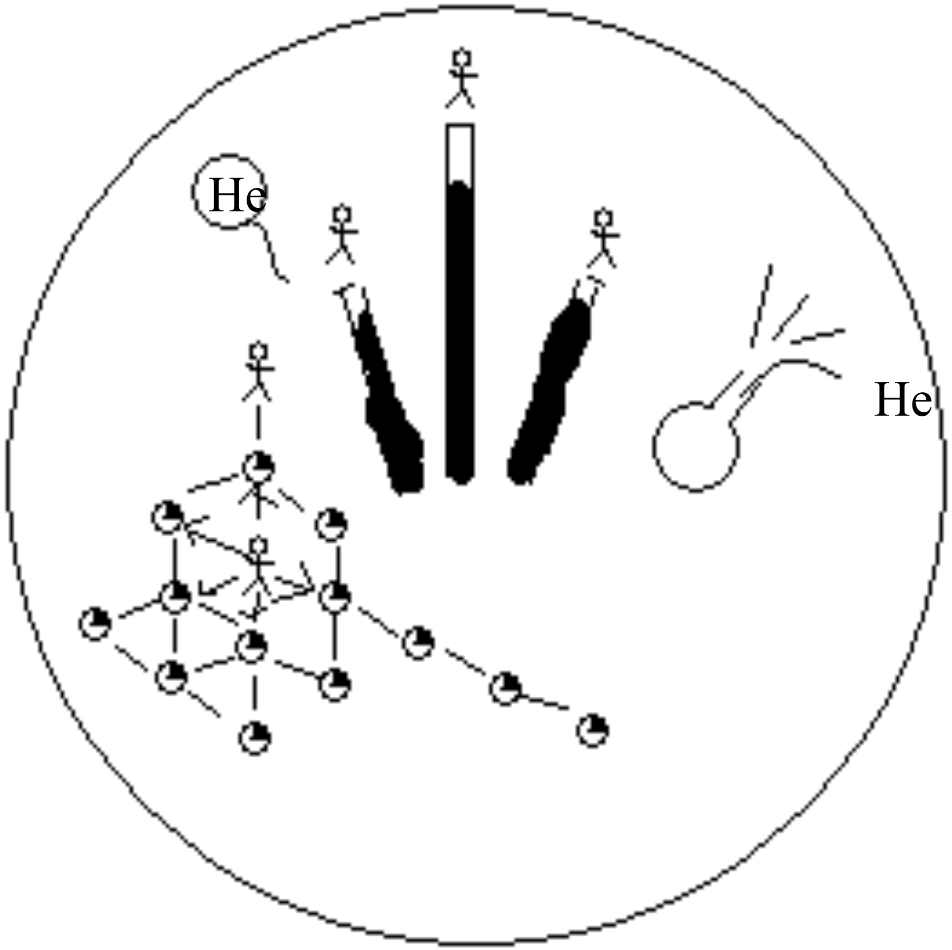
London Jan 1

Set your watches 0h:0m:0s



A comic explanation for cosmic expansion …

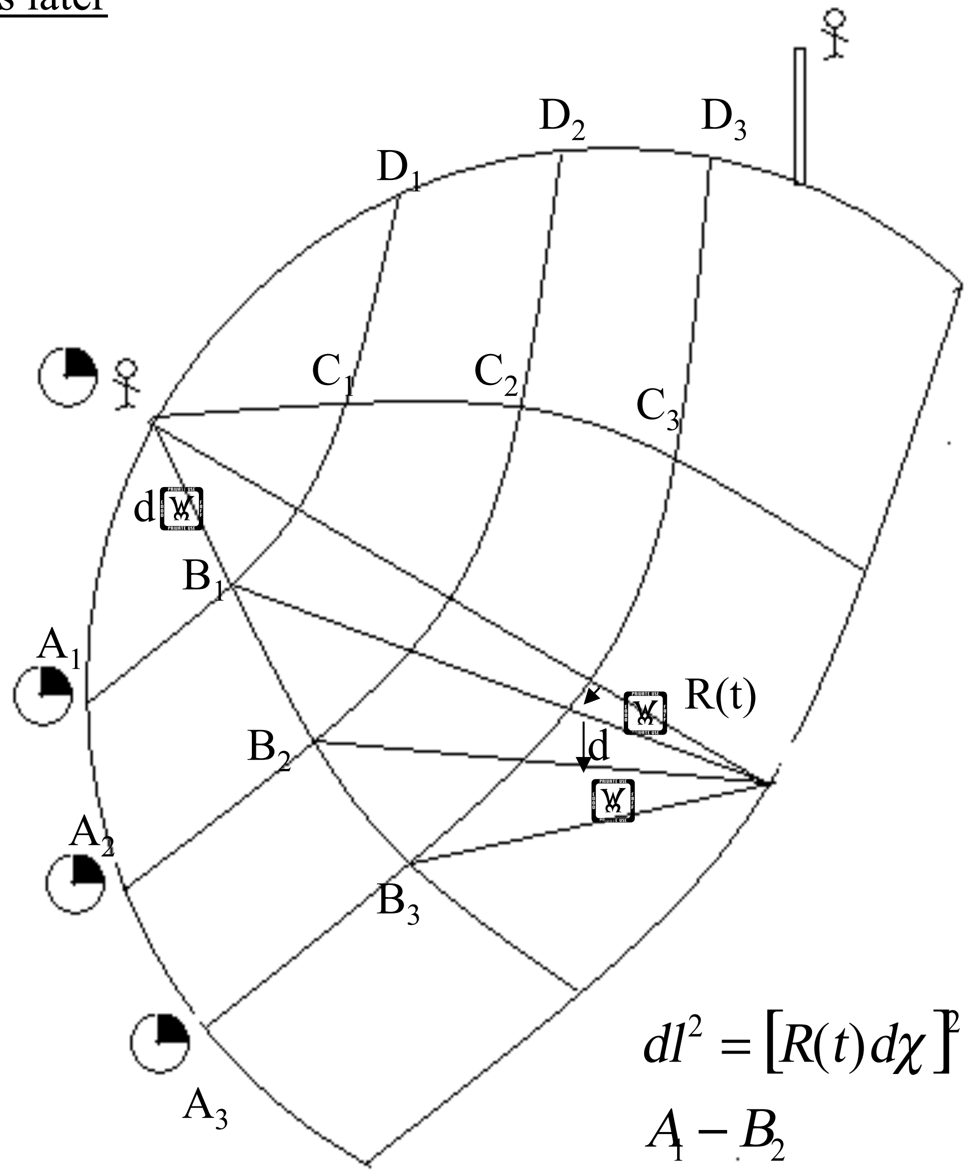
A few mins after New Year
Celebration at Trafalgar Square



Homogeneous
Isotropic Universe

$$Walking \leftrightarrow Elevating \leftrightarrow Earth\ \ Radius\ \ Stretching\ \ R(t)$$

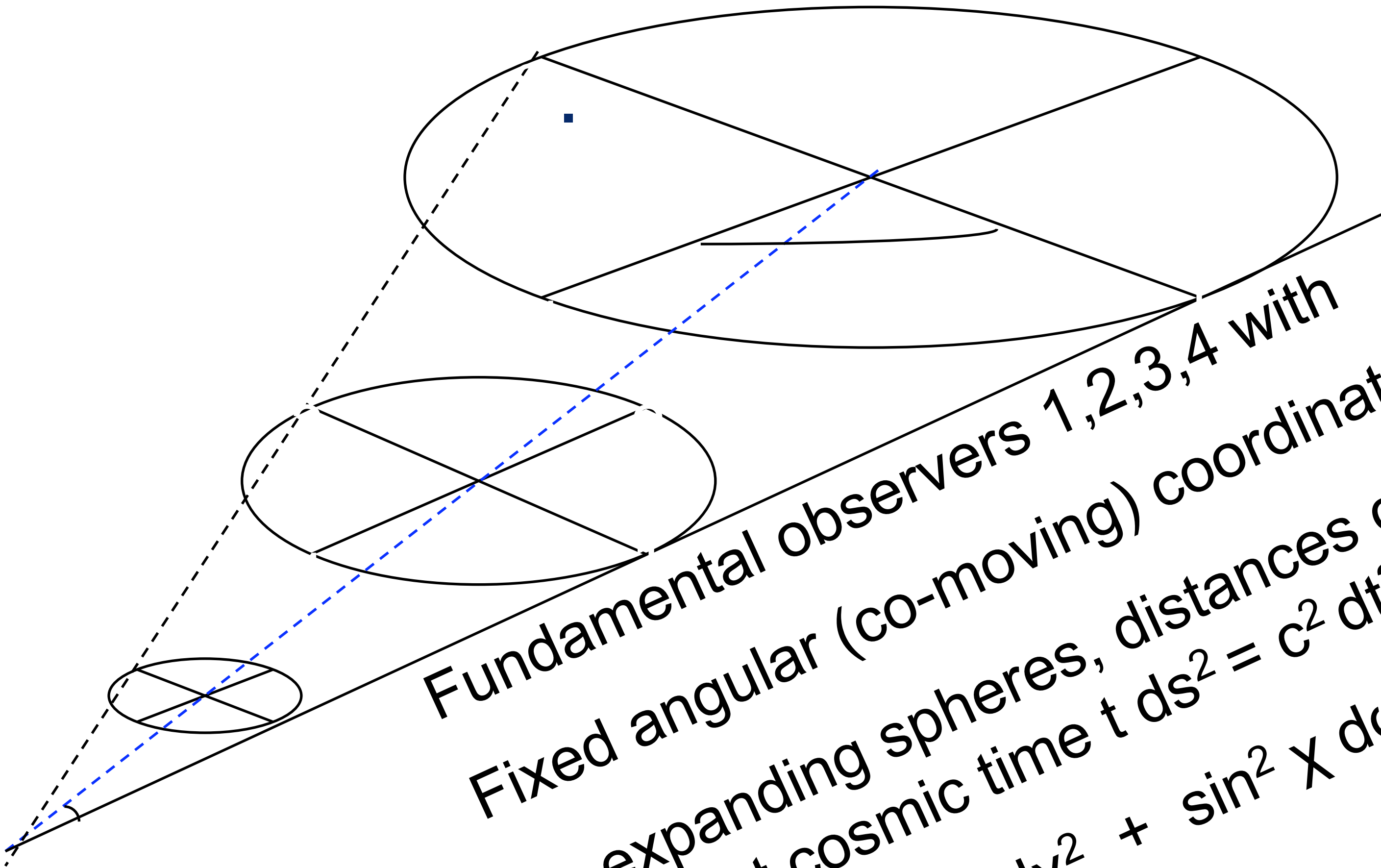
Feb 14   t=45 days later



$$dl^2 = \left[R(t)\,d\chi\right]^2 + \left[R(t)\sin\chi\,d\phi\right]^2$$

$$A_1 - B_2$$

Metric: ant network on expanding sphere

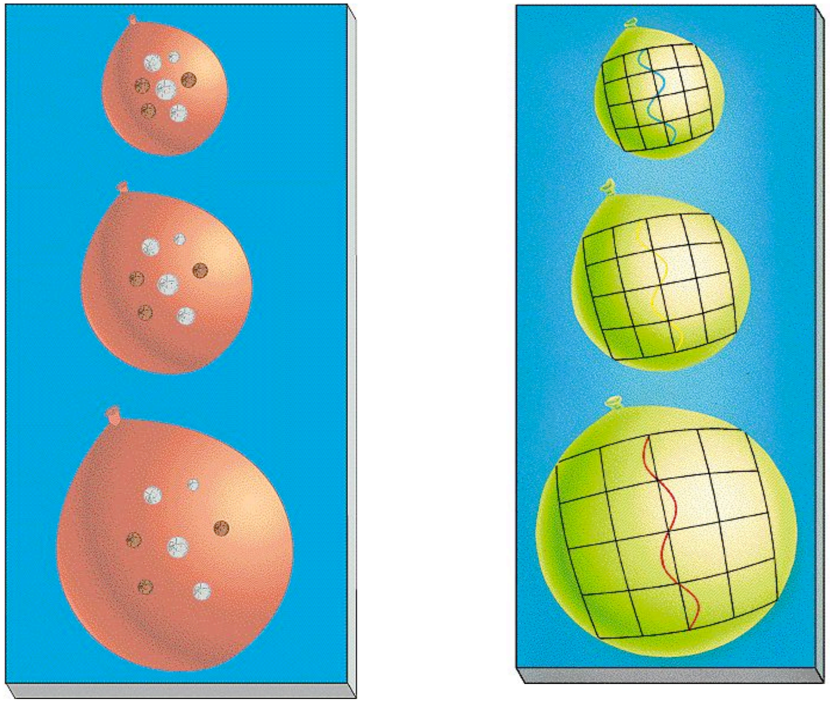


Fundamental observers 1,2,3,4 with Fixed angular (co-moving) coordinates (χ,φ) on expanding spheres, distances given by Metric at cosmic time t $ds^2 = c^2\,dt^2 - dl^2$,

$$dl^2 = R^2(t)\,(d\chi^2 + \sin^2\chi\, d\varphi^2)$$

Stretch of photon wavelength in expanding space

Emitted with intrinsic wavelength λ_0 from Galaxy A at time $t<t_{now}$ in smaller universe $R(t) < R_{now}$



→ Received at Galaxy B now (t_{now}) with λ

$\lambda / \lambda_0 = R_{now} / R(t) = 1+z(t) > 1$

1st main concept: Cosmological Redshift

The space/universe is expanding,

- Galaxies (pegs on grid points) are receding from each other

As a photon travels through space, its wavelength becomes stretched gradually with time.

- Photons wave-packets are like links between grid points

This redshift is defined by:

$$z \equiv \frac{\lambda - \lambda_o}{\lambda_o}$$

$$\frac{\lambda}{\lambda_o} = 1 + z$$

Galaxy Redshift Surveys

Large Scale Structure:

Empty **voids**

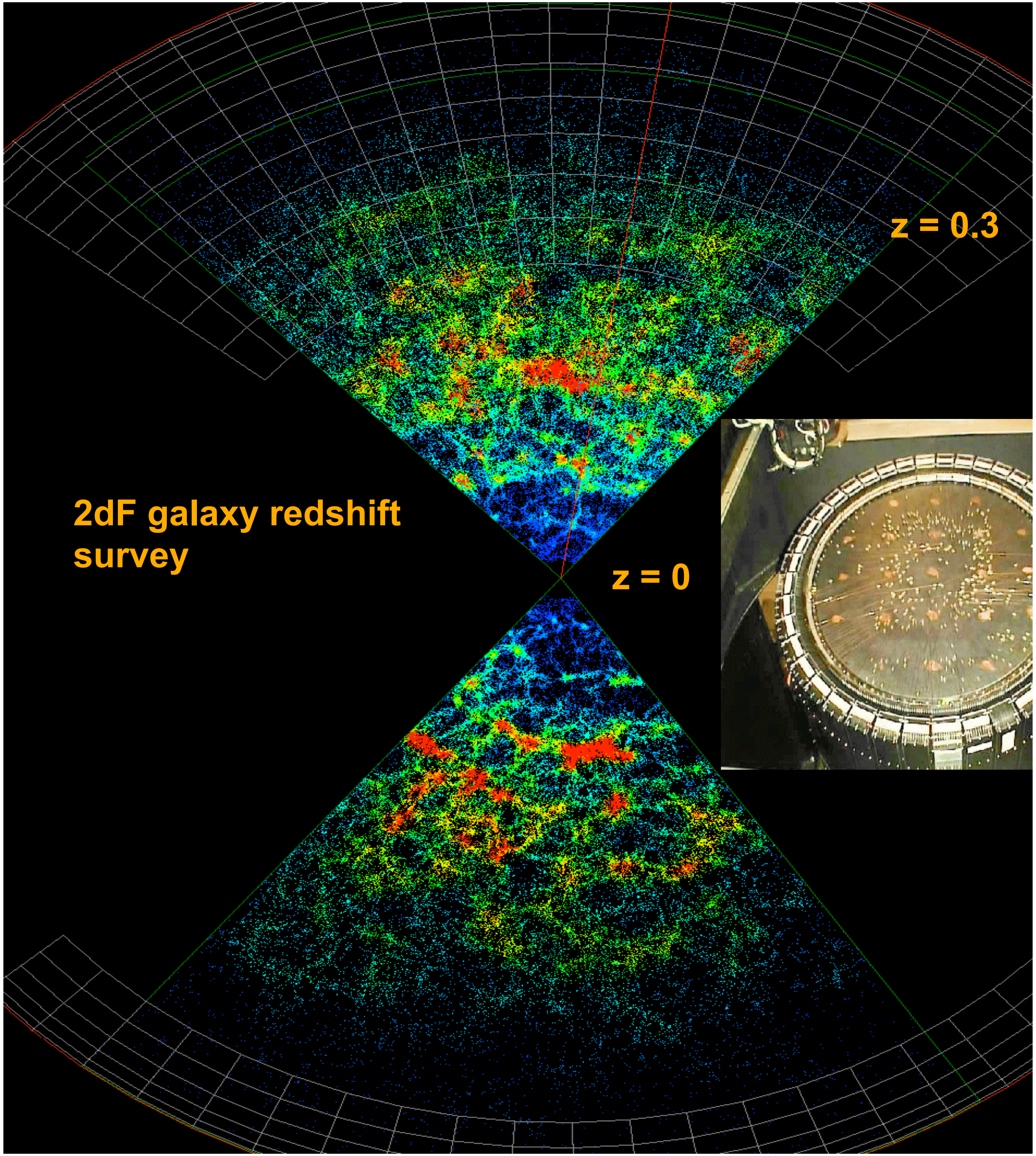
~50Mpc.

Galaxies are in
1. **Walls** between voids.

2. **Filaments** where walls intersect.

3. **Clusters** where filaments intersect.

Like Soap Bubbles !



E.g. Consider a quasar with redshift z=2. Since the time the light left the quasar the universe has expanded by a factor of 1+z=3. At the epoch when the light left the quasar,

What was the distance between us and Virgo (presently 15Mpc)?

What was the CMB temperature then (presently 3K)?

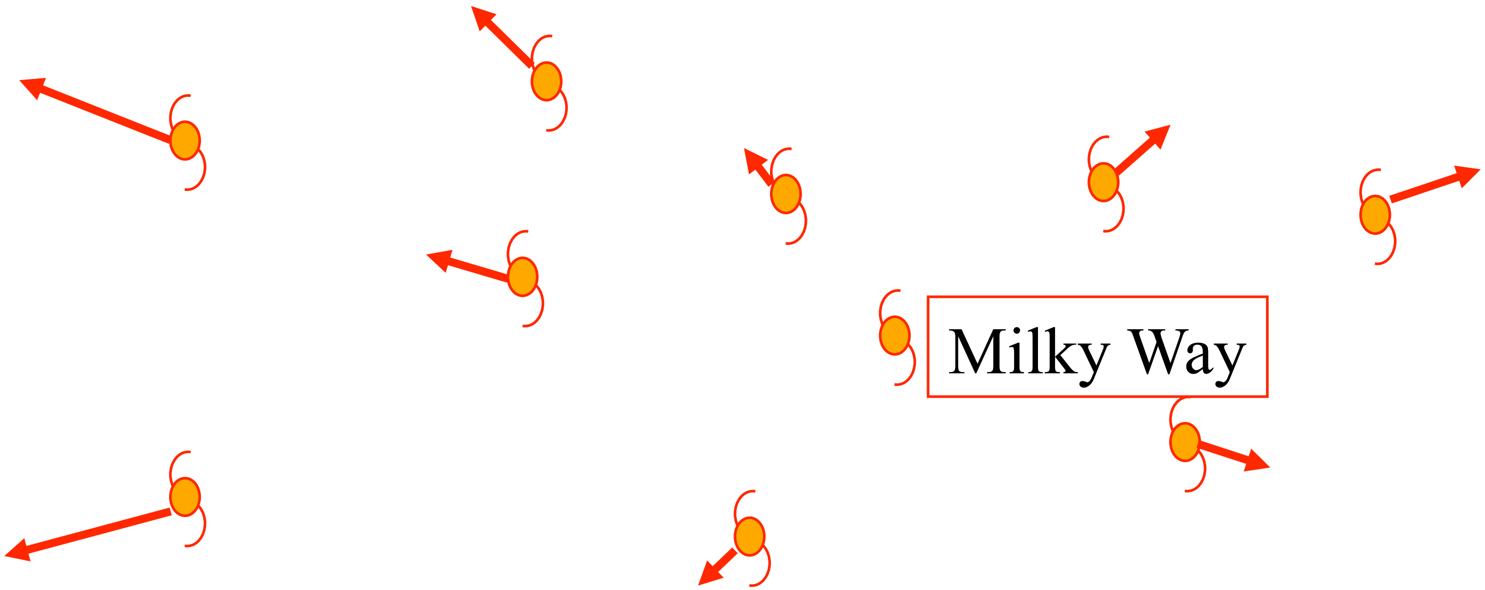
$$1+z=\frac{\lambda_{now}}{\lambda(t)} \quad \text{(wavelength)}$$

$$=\frac{R_{now}}{R(t)} \quad \text{(expansion factor)}$$

$$=\frac{T(t)}{T_{now}} \quad \text{(Photon Blackbody T } \propto 1/\lambda, \, why?)$$

Universal Expansion

Hubble's law appears to violate
The Copernican Principle.
Are we at a special location?

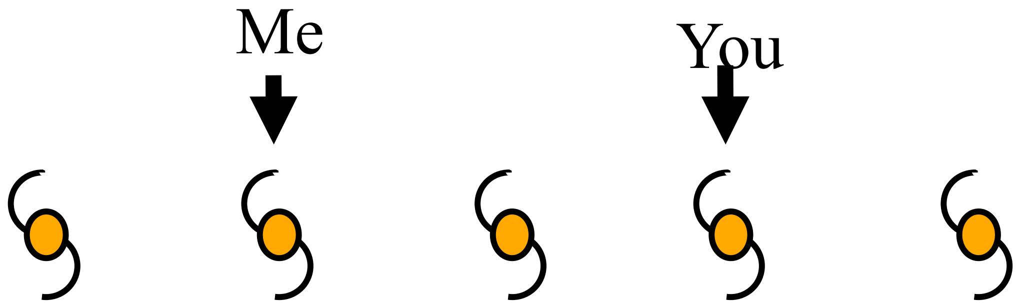


Is everything moving away from us?

Universal Expansion

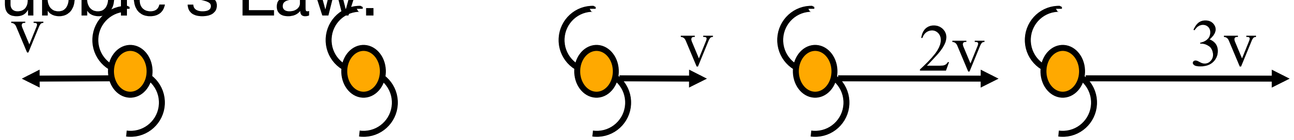
Q : What is so special about our location ?

A : Nothing !

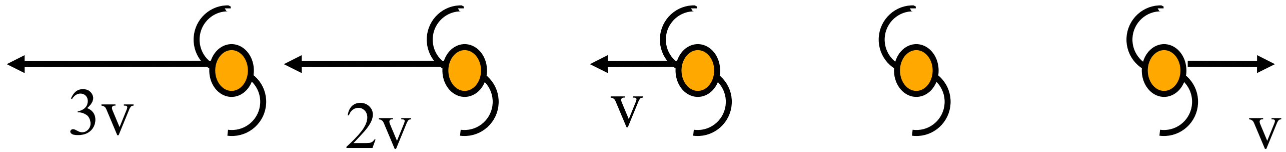


According to Hubble's Law:

I see:



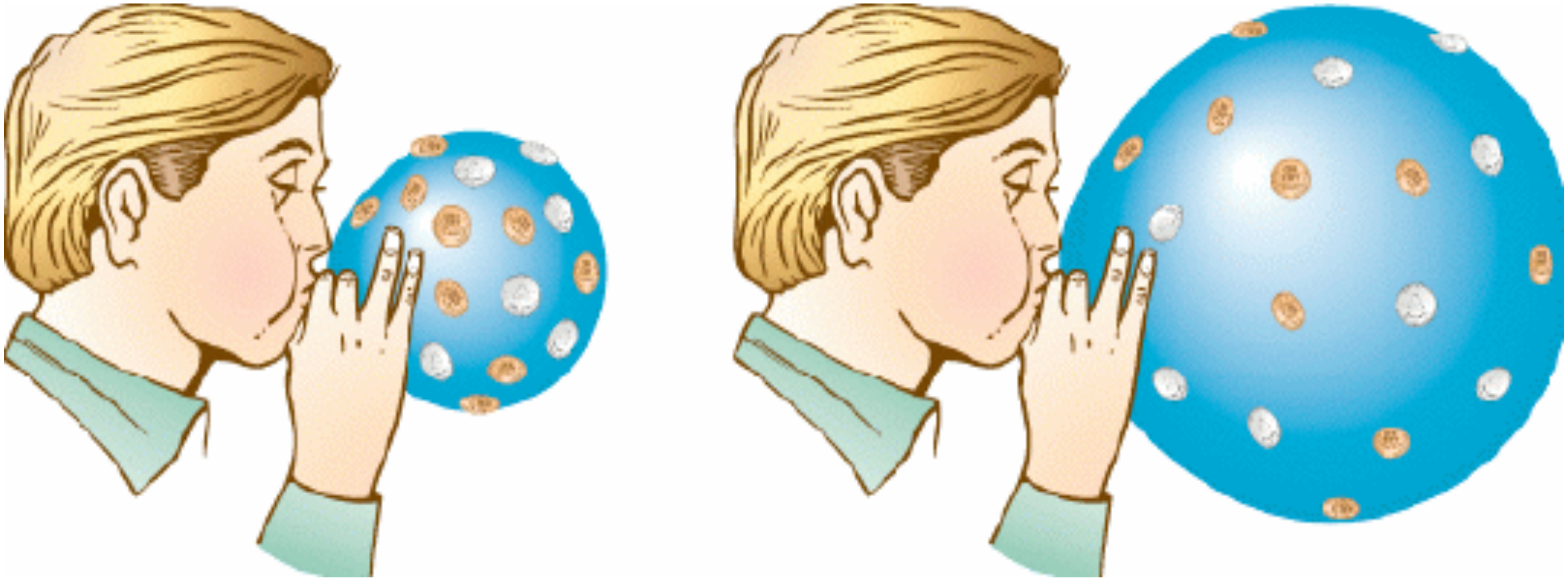
You see:



We all see the same Hubble law expansion.

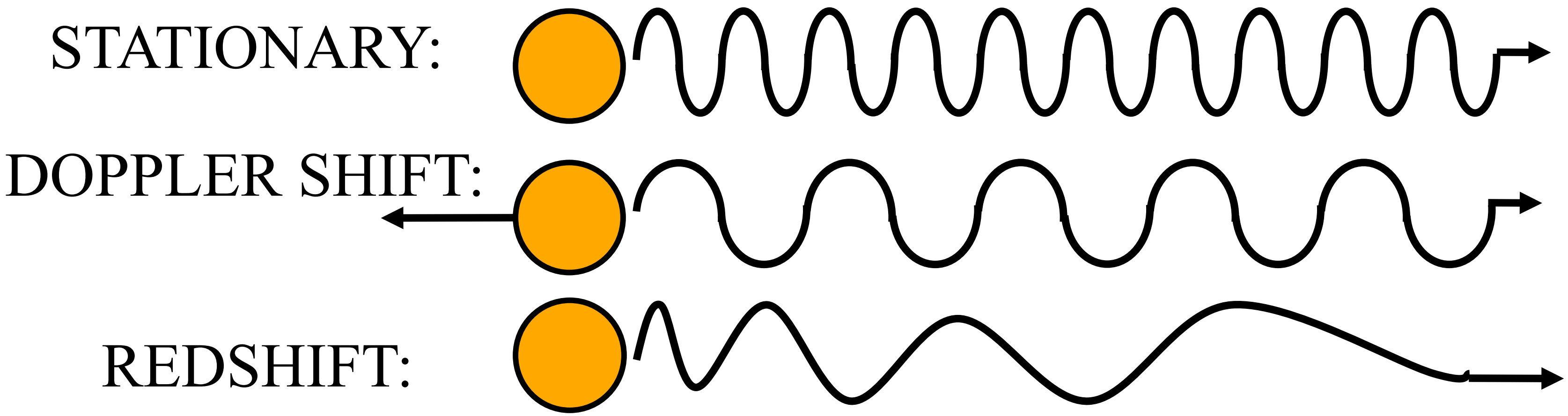
The Universal Expansion

- An observer in any galaxy sees all other galaxies moving away, with the same Hubble law.
- Expansion (or contraction) produces a centre-less but dynamic Universe.



Redshift

- Expansion is a stretching of space.
- The more space there is between you and a galaxy, the faster it appears to be moving away.
- Expansion **stretches the wavelength of light**, causing a galaxy's spectrum to be REDSHIFTED:



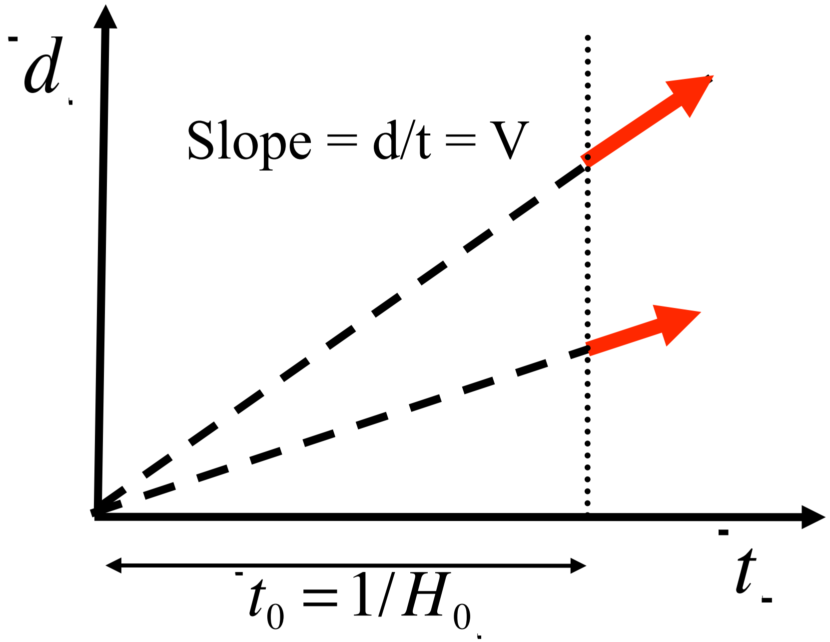
REDSHIFT IS NOT THE SAME AS DOPPLER SHIFT

Hubble Law --> Finite age.

$$V = H_0\, d$$

$$t_0 \approx \frac{d}{V} = \frac{1}{H_0} = \left(\frac{1\ \mathrm{Mpc}}{72\ \mathrm{km/s}}\right)\left(\frac{3\times10^{19}\mathrm{km}}{\mathrm{Mpc}}\right)\left(\frac{1\ \mathrm{yr}}{3\times10^{7}\mathrm{s}}\right)$$

$$\approx 13\times10^{9}\mathrm{yr} = 13\ \mathrm{Gyr.}$$



Concept: The Energy density of Universe

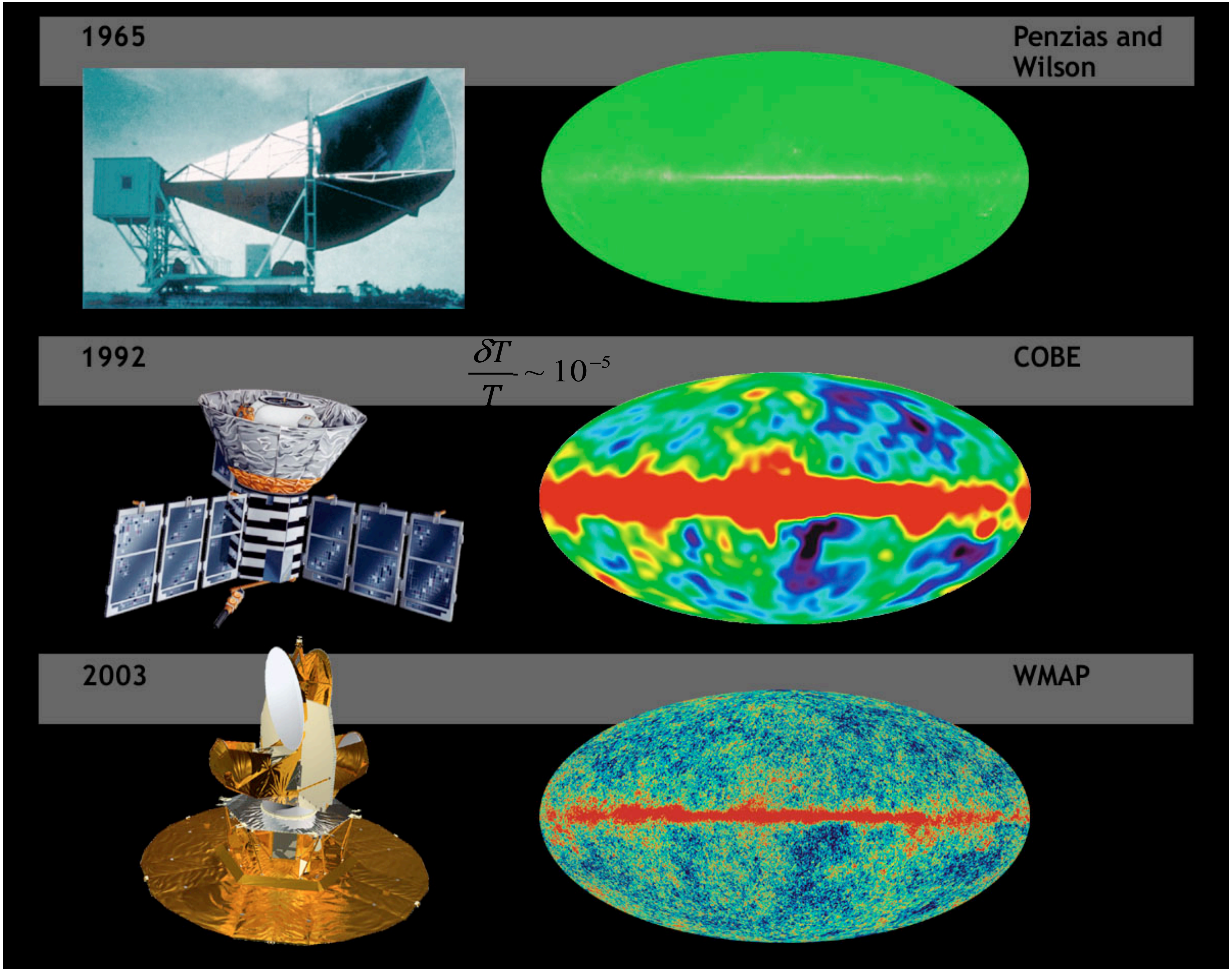
The Universe is made up of three things:

- VACUUM
- MATTER
- PHOTONS (radiation fields)

The total energy density of the universe is made up of the sum of the energy density of these three components.

$$\varepsilon(t) = \varepsilon_{vac} + \varepsilon_{matter} + \varepsilon_{rad}$$

From t=0 to t=10^9 years the universe has expanded by R(t).



1965
Penzias and Wilson
1992
$\frac{\delta T}{T} \sim 10^{-5}$
COBE
2003
WMAP

Recombination Epoch (z~1100)

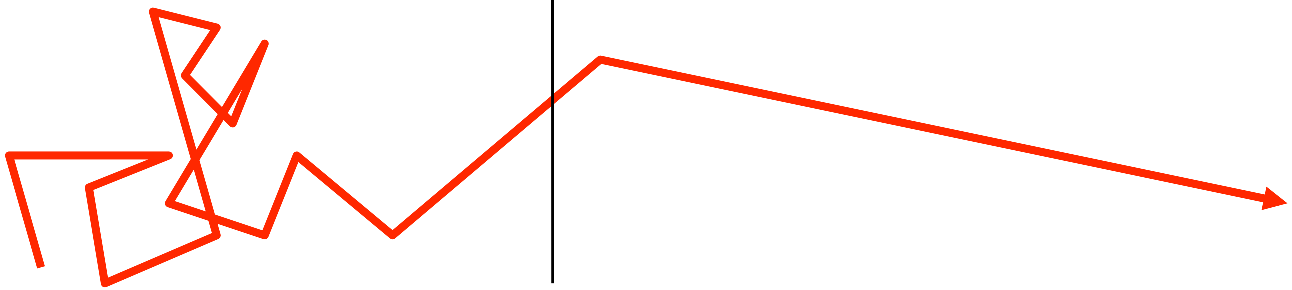
ionised plasma --> neutral gas

- Redshift z > 1100
- Temp T > 3000 K
- H ionised

- electron -- photon Thompson scattering

- z < 1100
- T < 3000 K
- H recombined

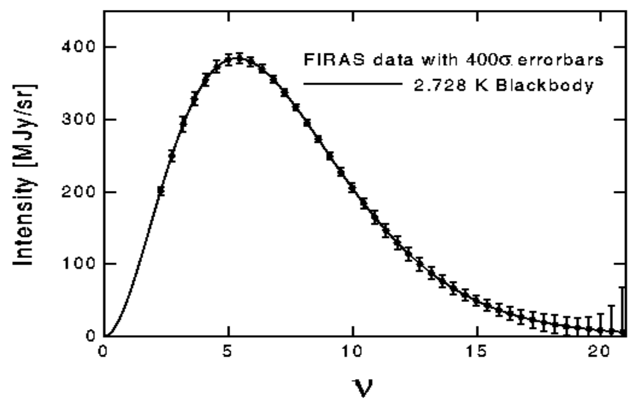
- almost no electrons
- neutral atoms
- photons set free



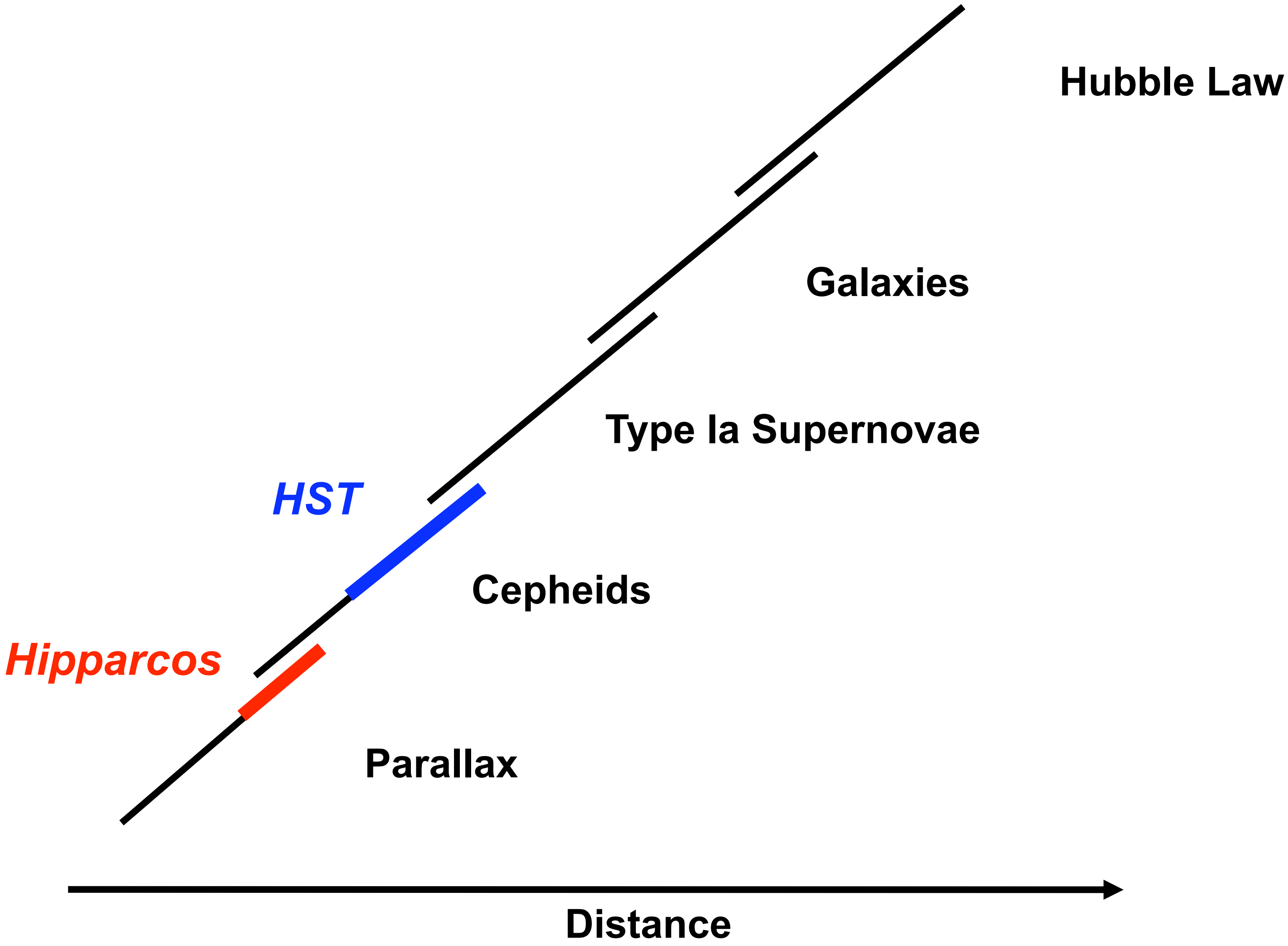
e^- scattering optical depth

$$\tau(z) \approx \left(\frac{z}{1080} \right)^{13}$$

thin surface of last scattering



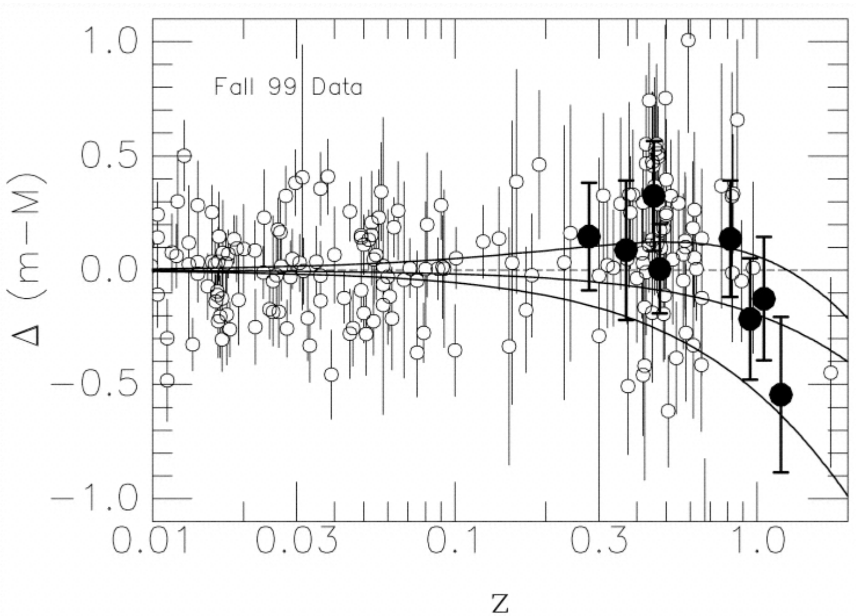
Cosmic Distance Ladder

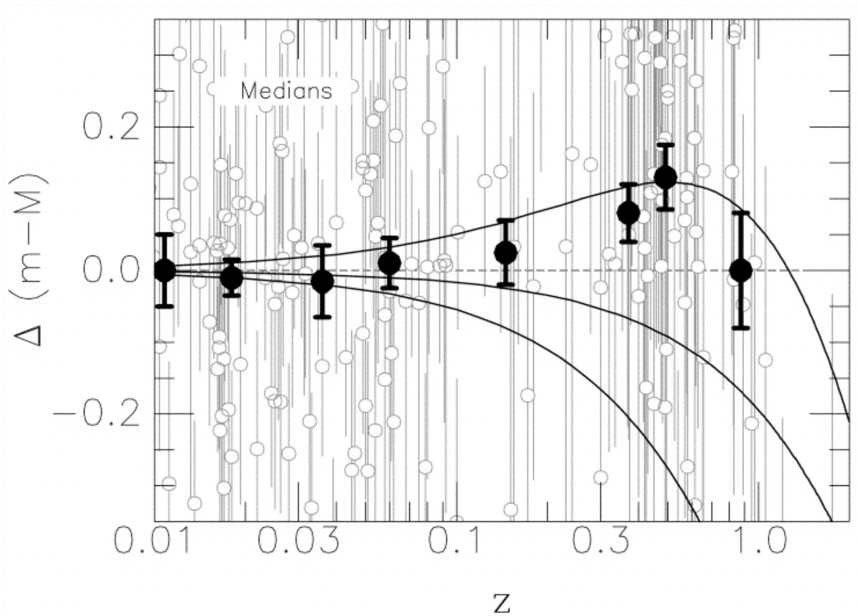


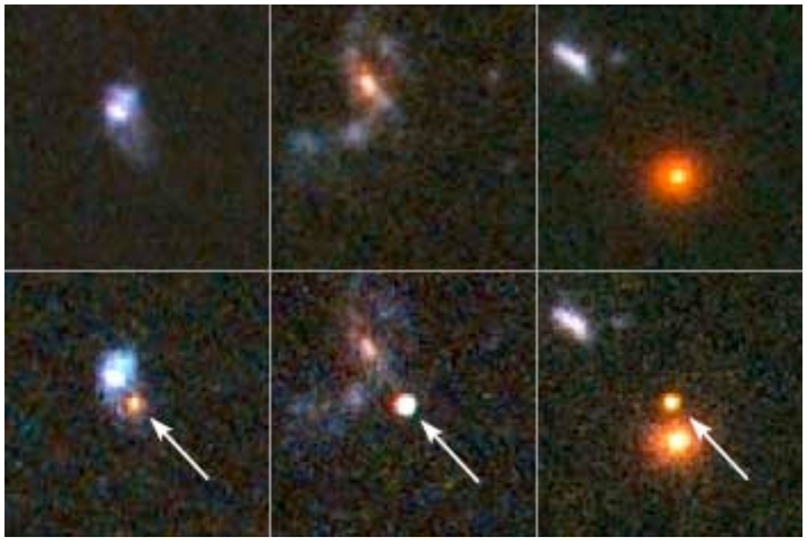
HST Supernova Surveys

Tonry et al. 2004.

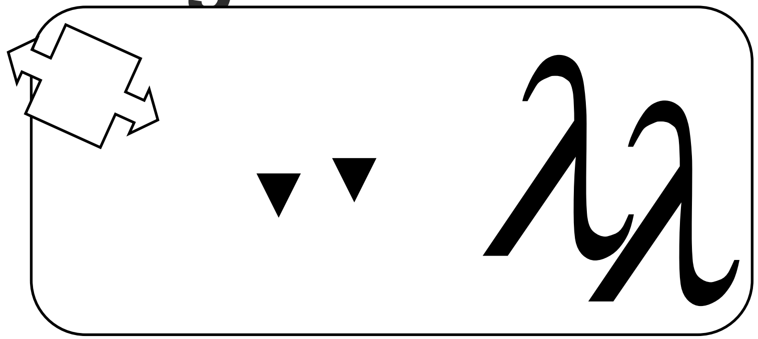
HST surveys to find SN Ia beyond z = 1







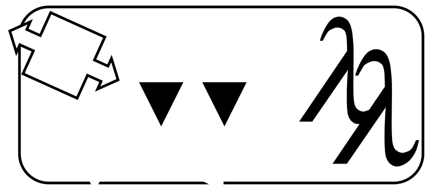
Eq. of State for Expansion & analogy of baking bread



Vacuum~air holes in bread

Matter ~nuts in bread

Photons ~words painted



Verify expansion doesn't change N_{hole}, N_{proton}, N_{photon}

- **No Change with rest energy of a proton, changes energy of a photon**

$$\varepsilon(t) = \rho_{eff}(t)c^2$$

$$\frac{\varepsilon(t)}{c^2} = \rho_{eff}(t)$$

VACUUM ENERGY:

$$\rho = \text{constant} \quad \Rightarrow \quad \mathrm{E}_{vac} \propto R^3$$

MATTER:

$$\rho R^3 = \text{constant}, \quad \Rightarrow m \approx \text{constant}$$

RADIATION: **number of photons N_{ph} = constant**

$$\Rightarrow n_{ph} \approx \frac{N_{ph}}{R^3}$$

$$\text{Wavelength stretches}: \lambda \sim R$$

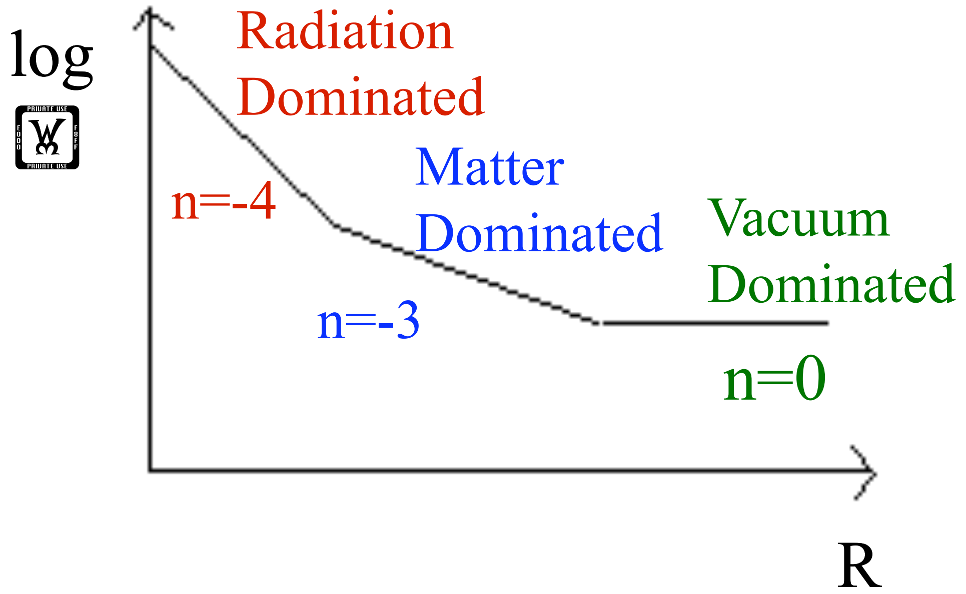
$$\text{Photons:} \mathrm{E} = \mathrm{h}\upsilon = \frac{\mathrm{hc}}{\lambda} \sim \frac{1}{R}$$

$$\Rightarrow \varepsilon_{ph} \sim n_{ph} \times \frac{hc}{\lambda} \sim \frac{1}{R^4}$$

The total energy density is given by:

$$\varepsilon \propto \varepsilon_{vac} + \varepsilon_{matter} + \varepsilon_{ph}$$

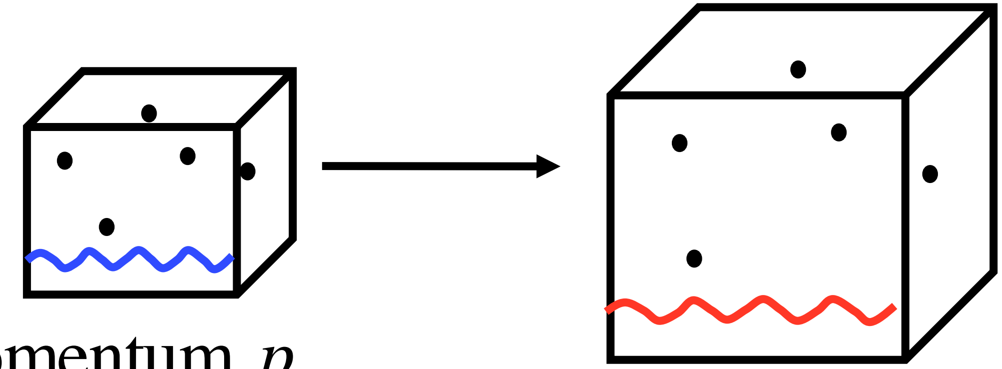
$\varepsilon_{vac} \propto R^{0}$, $\varepsilon_{matter} \propto R^{-3}$, $\varepsilon_{ph} \propto R^{-4}$



Energy Density of expanding box

volume R^3

N particles



particle mass m momentum p

energy $E = h\nu = \sqrt{m^2c^4 + p^2c^2} = m\,c^2 + \dfrac{p^2}{2m} + \ldots$

Cold Matter: ($m > 0,\ p << mc$)

$E \approx m\,c^2 = \text{const}$

$$\varepsilon_M \approx \frac{N\,m\,c^2}{R^3} \propto R^{-3}$$

Radiation: ($m = 0$)

Hot Matter: ($m > 0,\ p >> mc$)

$\lambda \propto R$ (wavelengths stretch) :

$$E = h\,\nu = \frac{h\,c}{\lambda} \propto R^{-1}$$

$$\varepsilon_R = \frac{N\,h\,\nu}{R^3} \propto R^{-4}$$

Precision Cosmology

$$h = 71 \pm 3 \qquad \text{expanding}$$

$$\Omega = 1.02 \pm 0.02 \qquad \text{flat}$$

$$\Omega_b = 0.044 \pm 0.004 \qquad \text{baryons}$$

$$\Omega_M = 0.27 \pm 0.04 \qquad \text{Dark Matter}$$

$$\Omega_\Lambda = 0.73 \pm 0.04 \qquad \text{Dark Energy}$$

$$t_0 = 13.7 \pm 0.2 \times 10^9 \text{ yr} \qquad \text{now}$$

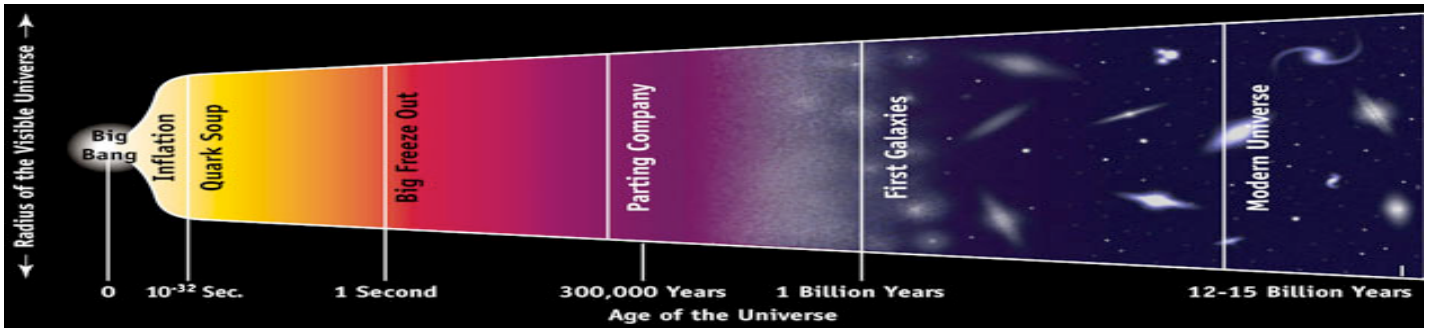
$$t_* = 180^{+220}_{-80} \times 10^6 \text{ yr} \qquad z_* = 20^{+10}_{-5} \qquad \text{reionisation}$$

$$t_R = 379 \pm 1 \times 10^3 \text{ yr} \qquad z_R = 1090 \pm 1 \qquad \text{recombination}$$

(From the WMAP 1-year data analysis)

Brief History of Universe

- **Inflation**
  - Quantum fluctuations of a tiny region
  - Expanded exponentially
- **Radiation cools with expansion $T \sim 1/R \sim t^{-2/n}$**
  - He and D are produced (lower energy than H)
  - Ionized H turns neutral (recombination)
  - Photon decouple (path no longer scattered by electrons)
- **Dark Matter Era**
  - Slight overdensity in Matter can collapse/cool.
  - Neutral transparent gas
- **Lighthouses (Galaxies and Quasars) form**
  - UV photons re-ionize H
  - Larger Scale (Clusters of galaxies) form



Four Pillars of Hot Big Bang

Galaxies moving apart from each other

- Redshift or receding from each other
- Universe was smaller

Helium production outside stars

- Universe was hot, at least 10^9K to fuse 4H → He, to overcome a potential barrier of 1MeV.

Nearly Uniform Radiation 3K Background (CMB)

- Universe has cooled, hence expanded by at least a factor 10^9

Missing mass in galaxies and clusters (Cold Dark Matter: CDM)

- Cluster potential well is deeper than the potential due to baryons
- CMB temperature fluctuations: photons climbed out of random potentials of DM

Acronyms in Cosmology

- **Cosmic Background Radiation (CBR)**
  - **Or CMB (microwave because of present temperature 3K)**
    - Argue **about 10^5 photons fit in a 10cmx10cmx10cm microwave oven. [Hint: 3kT = h c / λ]**

- **CDM/WIMPs: Cold Dark Matter, weakly-interact massive particles**
  - **At time DM decoupled from photons, T ~ 10^{14}K, kT ~ 0.1 mc^2**
    - Argue **that dark particles were**
      - **non-relativistic (v/c << 1), hence "cold".**
      - **Massive (m >> m_{proton} =1 GeV)**

the energy density of universe now consists roughly

- Equal amount of vacuum and matter,
- 1/10 of the matter is ordinary protons, rest in dark matter particles of 10Gev
  - Argue dark-particle-to-proton ratio ~ 1
- Photons (3K ~10^{-4}ev) make up only 10^{-4} part of total energy density of universe (which is ~ proton rest mass energy density)
  - Argue photon-to-proton ratio ~ 10^{-4} GeV/(10^{-4}ev) ~ 10^{9}

Key Points

- **Scaling Relation among**
  - **Redshift: z,**
  - **expansion factor: R**
    - **Distance between galaxies**
  - **Temperature of CMB: T**
    - **Wavelength of CMB photons: lambda**
- **Metric of an expanding 2D+time universe**
  - **Fundamental observers**
    - **Galaxies on grid points with fixed angular coordinates**
- **Energy density in**
  - **vacuum, matter, photon**
  - **How they evolve with R or z**

- **<u>If confused, recall the analogies of</u>**
  - **balloon, bread, a network on red giant star, microwave oven**

Topics
Theoretical and Observational

Universe of uniform density

- Metrics ds, Scale R(t) and Redshift
- EoS for mix of vacuum, photon, matter

Thermal history

- Nucleosynthesis
- He/D/H

Structure formation

- Growth of linear perturbation
- Origin of perturbations
- Relation to CMB

Quest of H0 (obs.)

- Applications of expansion models
- Distances Ladders
  - (GL, SZ)

Quest for Omega (obs.)

- Galaxy/SNe surveys
- Luminosity/Correlation Functions

Cosmic Background

- COBE/MAP/PLANCK etc.
- Parameters of cosmos

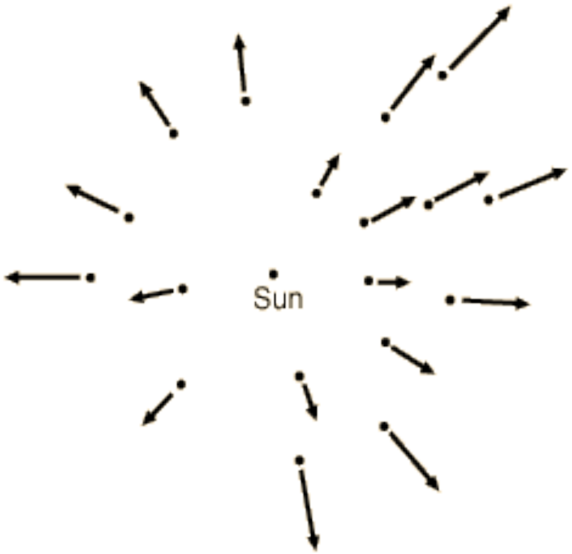
Hongsheng.Zhao (hz4)

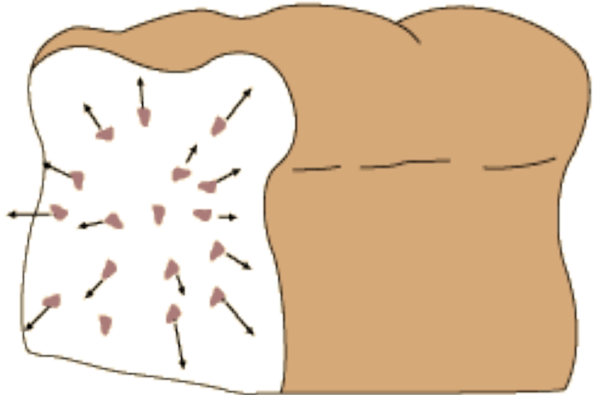
(thanks to slides from Keith Horne)

Cosmology Milestones

- 1925 Galaxy redshifts $\lambda = \lambda_0 \ (1 + z)$ $V = c\, z$
  - Isotropic expansion. (Hubble law $V = H_0 d$)
  - Finite age. ($t_0 = 13 \times 10^9$ yr)
- 1965 Cosmic Microwave Background (CMB)
  - Isotropic blackbody. $T_0 = 2.7$ K
  - Hot Big Bang $T = T_0 (1 + z)$
- 1925 General Relativity Cosmology Models :
  - Radiation era: $R \sim t^{1/2}$ $T \sim t^{-1/2}$
  - Matter era: $R \sim t^{2/3}$ $T \sim t^{-2/3}$
- 1975 Big Bang Nucleosynthesis (BBN)
  - light elements (^{1}H … ^{7}Li) $t \sim 3$ min $T \sim 10^9$ K
  - primordial abundances (75% H, 25% He) as observed!

Isotropic Expansion





Hubble law :

$$V = H_0 \, d$$

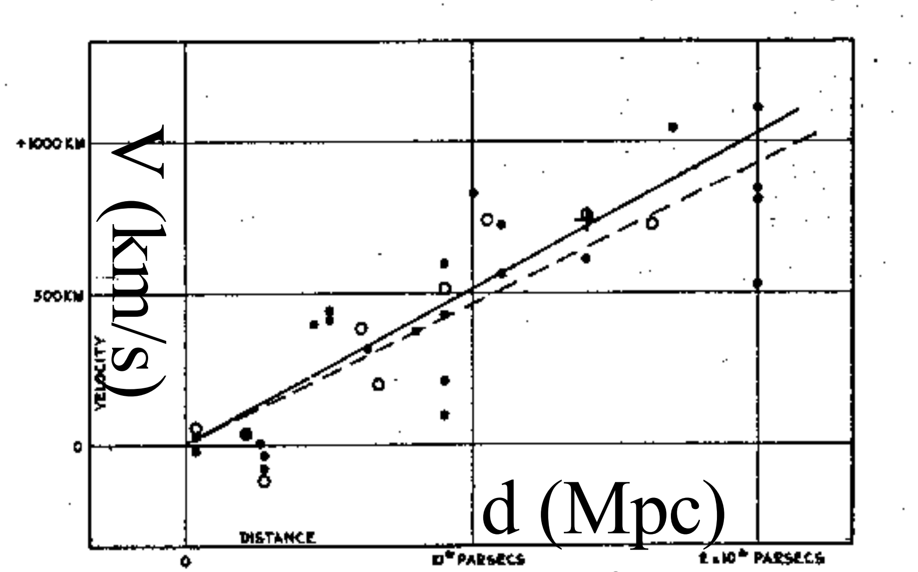


FIGURE 1

Hubble "constant" :

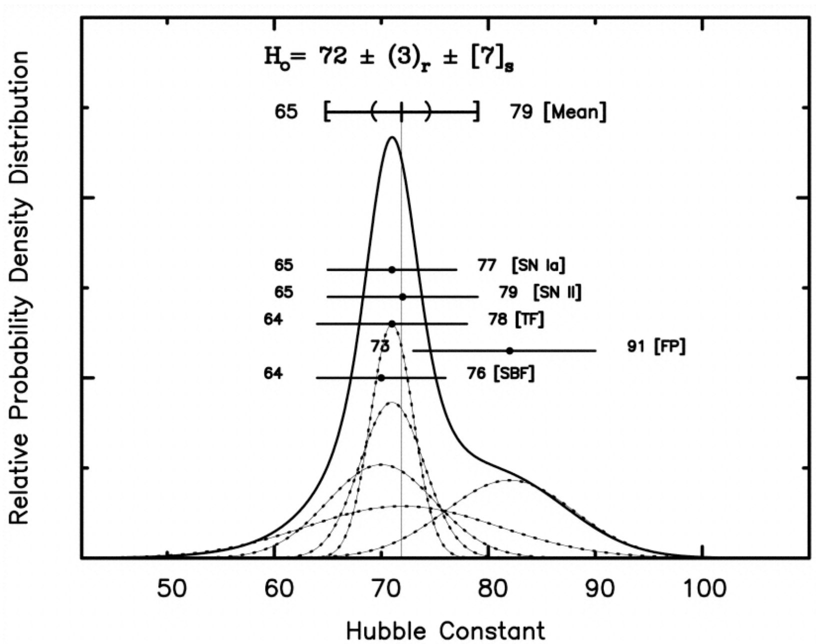
$$H_0 \approx 500 \text{ km s}^{-1}\text{Mpc}^{-1}$$

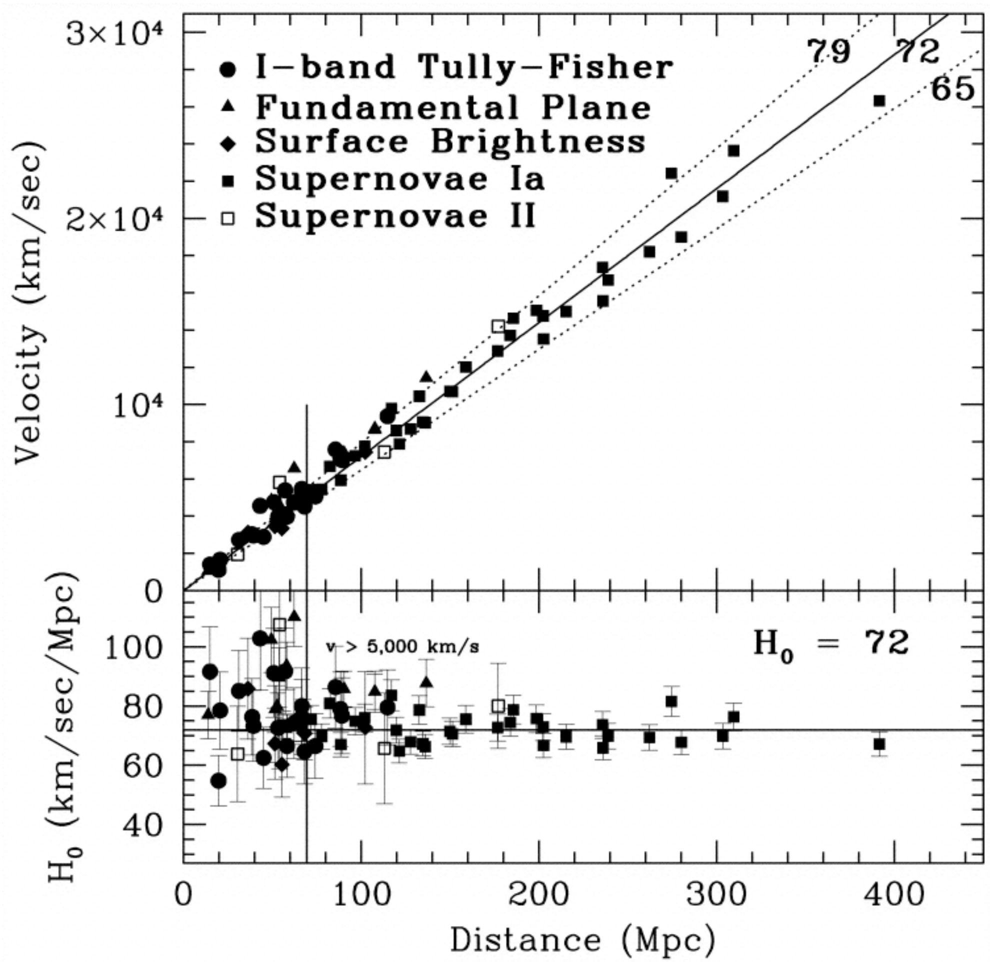
WRONG ! Extinction by interstellar dust was not then known, giving incorrect distances.

H_0 from the HST Key Project

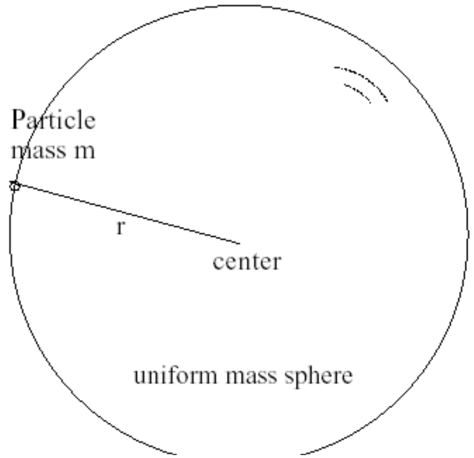
$$H_0 \approx 72 \pm 3 \pm 7 \quad \text{km s}^{-1}\ \text{Mpc}^{-1}$$

Freedman, et al.
2001 ApJ 553, 47.





The rate of expansion of Universe



Consider a sphere of radius $r=R(t)\ \chi$,

If energy density inside is $\rho\ c^2$

➔ Total effective mass inside is

$M = 4\ \pi\rho\ r^3 /3$

Consider a test mass m on this expanding sphere,

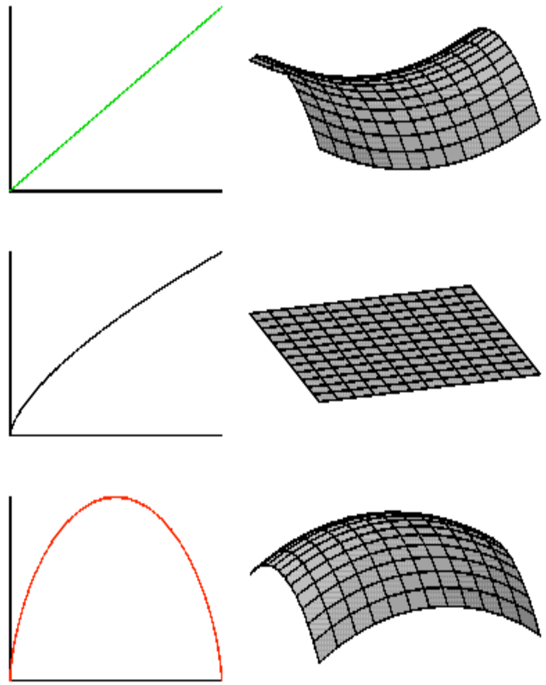
For Test mass its

Kin.Energy + Pot.E. = const E

➔ $m\ (dr/dt)^2/2 - G\ m\ M/r = cst$

➔ $(dR/dt)^2/2\ - 4\ \pi G\ \rho\ R^2/3 = cst$

cst>0, cst=0, cst<0



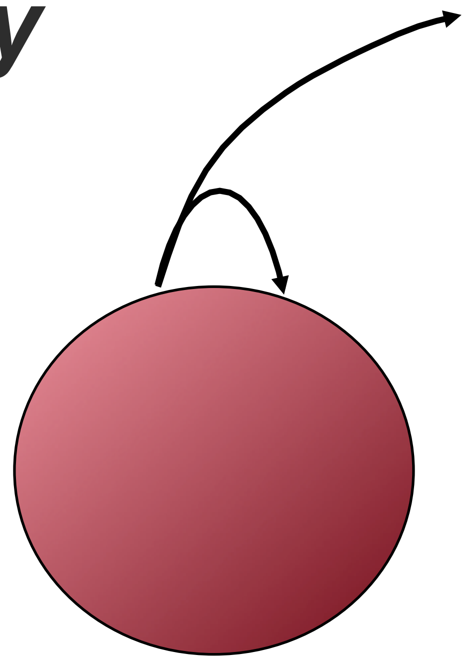
$(dR/dt)^2/2\ = 4\ \pi G\ (\rho + \rho_{cur})\ R^2/3$

where cst is absorbed by $\rho_{cur} \sim R^{(-2)}$

Newtonian Analogy

$$E = \frac{m}{2}\dot{R}^2 - \frac{G\,M\,m}{R}$$

$$V_{esc} = \sqrt{\frac{2\,G\,M}{R}}$$



$$E > 0 \quad V > V_{esc} \quad R \rightarrow \infty \quad V_\infty > 0$$

$$E = 0 \quad V = V_{esc} \quad R \rightarrow \infty \quad V_\infty = 0$$

$$E < 0 \quad V < V_{esc} \quad R \rightarrow 0$$

Critical Density

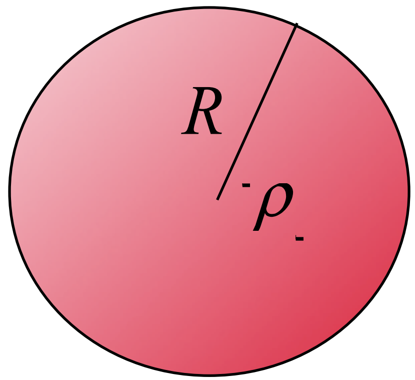
- **Derive using Newtonian analogy:**

escape velocity :

$$V_{esc}^2 = \frac{2GM}{R} = \frac{2G}{R}\left(\frac{4\pi R^3 \rho}{3}\right) = \frac{8\pi G R^2 \rho}{3}$$

Hubble expansion :

$$V = \dot{R} = H_0 R$$



critical density :

$$\left(\frac{V_{esc}}{V}\right)^2 = \frac{8\pi G \rho}{3 H_0^2} \equiv \frac{\rho}{\rho_c}$$

$$\boxed{\rho_c = \frac{3 H_0^2}{8\pi G}}$$

Typical scaling of expansion

$H^2=(dR/dt)^2/R^2=8\pi G\ (\rho_{cur}+\rho_m+\rho_r+\rho_v)/3$

Assume domination by a component $\rho \sim R^{-n}$

Show Typical Solutions Are

$$\rho \propto R^{-n} \propto t^{-2}$$

$$n = 2(\text{curvature constant dominate})$$

$$n = 3(\textit{matter}\ \text{dominate})$$

$$n = 4(\textit{radiation}\ \text{dominate})$$

$$n \sim 0(\text{vaccum dominate}): \ln(R) \sim t$$

Where are we heading?

Have done:

Chpt 1: Introduction

Chpt 2: Metrics, Energy density, Expansion

Malcolm S. Longair's "Galaxy Formation" [Library Short Loan]

Heading to:

Chpt 9-10: Thermal History, particle reaction

Chpt 11: Structure growth

A busy schedule for the universe

Universe crystalizes with a sophisticated schedule, much more confusing than simple expansion!

- many bosonic/fermionic players changing (numbers conserved except in phase transition!)
- p + p^- <-> g+g (baryongenesis)
- e + e^+ <-> g+g, v + e <-> v + e (neutrino decouple)
- n <→ p + e^- + v, p + n <→ D + g (BBN)
- H^+ + e^- <→ H + g, g + e <-> g + e (recombination)

$$\dot{n} + 3Hn = -\langle \sigma \mathrm{v} \rangle (n^2 - n_T^{\,2})$$

Significant Events

Event	T	kT	g_{eff}	z	t
Now	2.7 K	0.0002 eV	3.3	0	13 Gyr
First Galaxies	16 K	0.001 eV	3.3	5	1 Gyr
Recombination	3000 K	0.3 eV	3.3	1100	300,000 yr
$\rho_M = \rho_R$	9500 K	0.8 eV	3.3	3500	50,000 yr
$e^+ e^-$ pairs	$10^{9.7}$ K	0.5 MeV	11	$10^{9.5}$	3 s
Nucleosynthesis	10^{10} K	1 MeV	11	10^{10}	1 s
Nucleon pairs	10^{13} K	1 GeV	70	10^{13}	$10^{-6.6}$ s
E-W unification	$10^{15.5}$ K	250 GeV	100	10^{15}	10^{-12} s
Quantum gravity	10^{32} K	10^{19} GeV	100(?)	10^{32}	10^{-43} s

Here we will try to single out some rules of thumb.

We will caution where the formulae are not valid, exceptions.

You are not required to reproduce many details, but might be asked for general ideas.

What is meant Particle-Freeze-Out?

Freeze-out of equilibrium means NO LONGER in thermal equilibrium.

Freeze-out temperature means a species of particles have the SAME TEMPERATURE as radiation up to this point, then they bifurcate.

Decouple = switch off the reaction chain

= insulation = Freeze-out

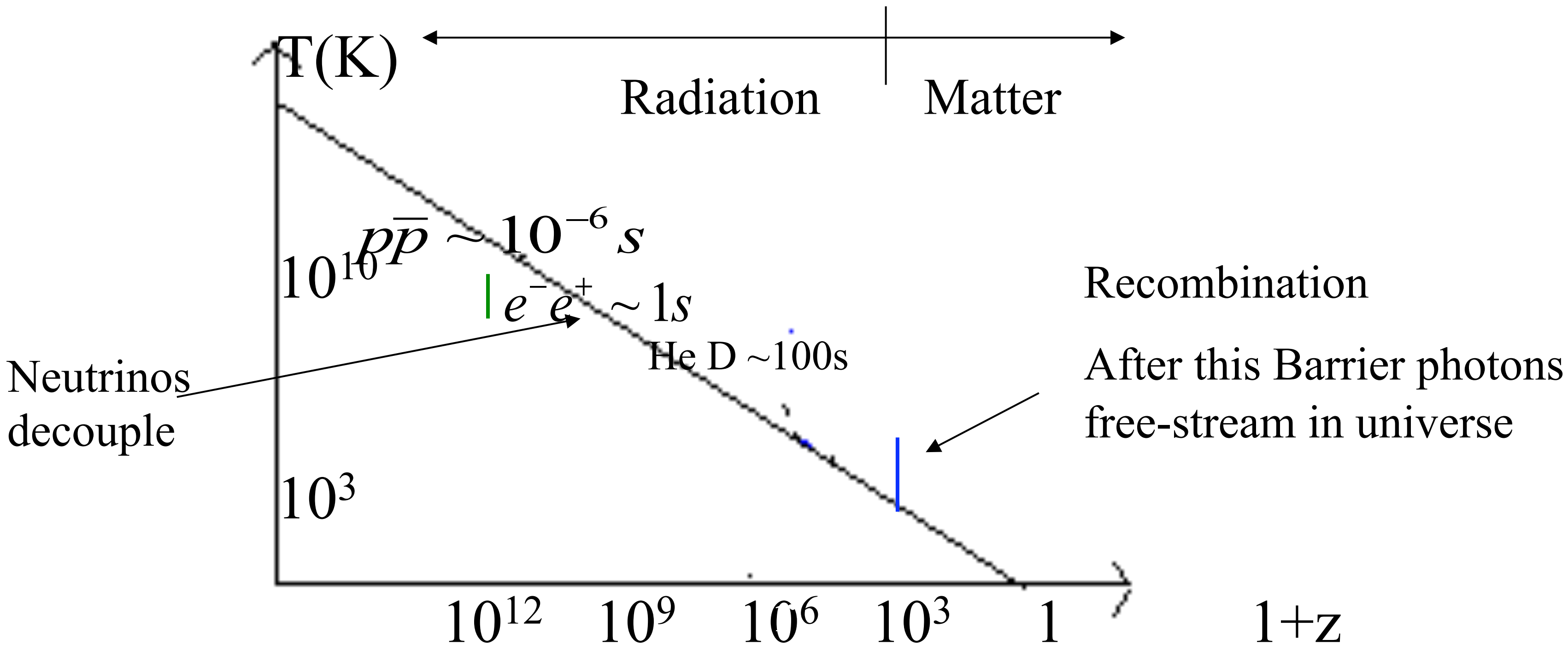
Thermal Schedule of Universe [chpt 9-10]

At very early times, photons are typically energetic enough that they interact strongly with matter so the whole universe sits at a temperature dictated by the radiation.

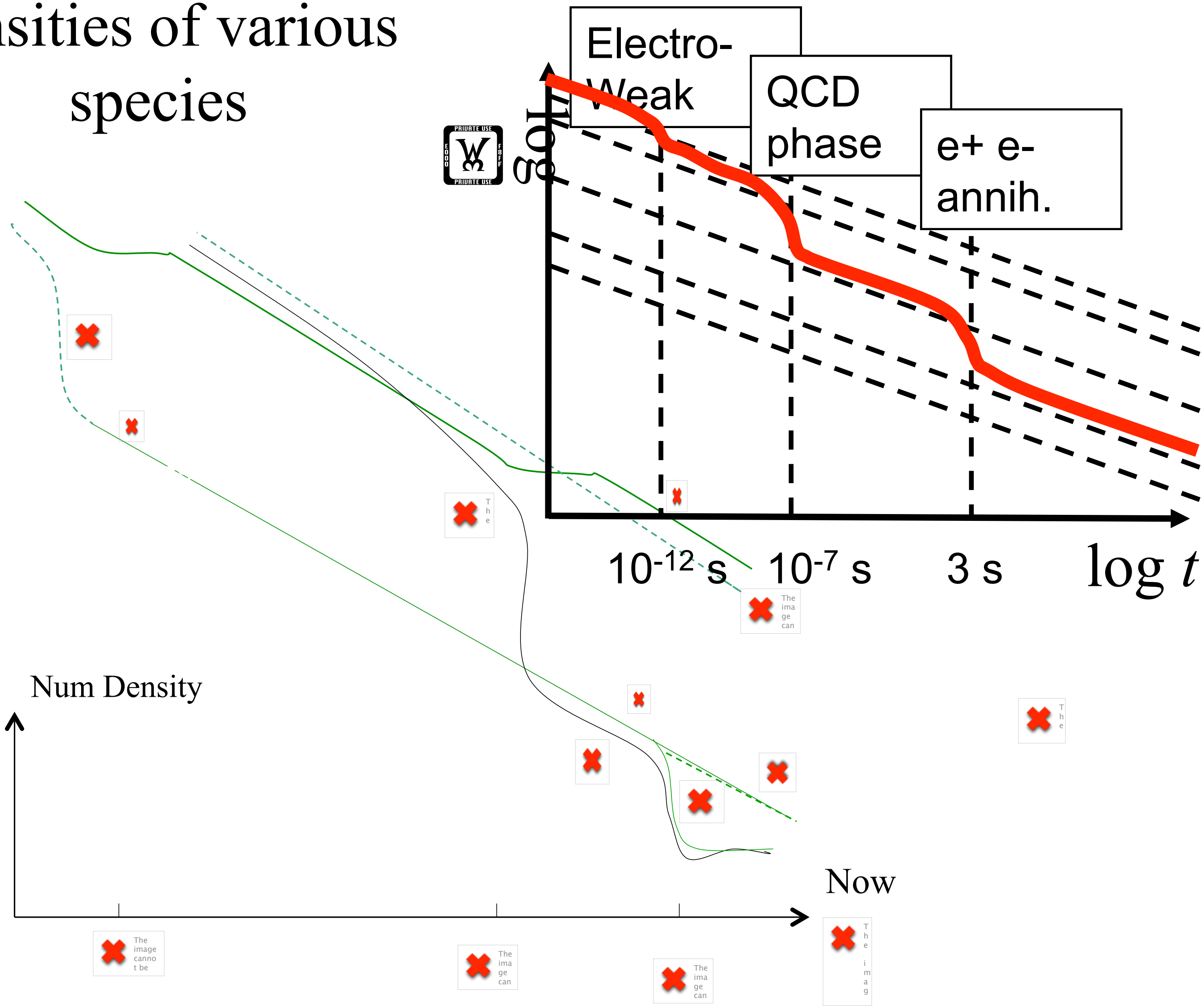
The energy state of matter changes as a function of its temperature and so a number of key events in the history of the universe happen according to a schedule dictated by the temperature-time relation.

Crudely

$(1+z) \sim 1/R \sim (T/3) \sim 10^9 (t/100s)^{(-2/n)} \sim 1000 (t/0.3Myr)^{-2/n}$, $H \sim 1/t$, where n~4 during radiation domination



Evolution of Number Densities of various species



1975: Big Bang Nuclear Fusion

Big Bang + 3 minutes

T ~ 10^9 K

First atomic nuclei forged.

Calculations predict:

75% H and 25% He

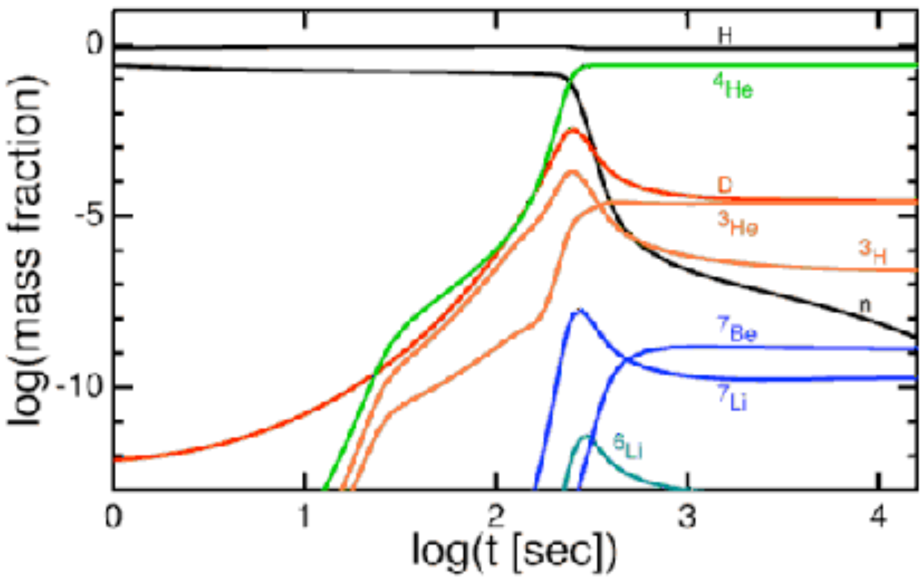
AS OBSERVED !

+ traces of light elements
D, ^{3}H, ^{3}He, ^{7}Be, ^{7}Li

=> normal matter only 4% of critical density.

Helium abundance

Oxygen abundance =>



A general history of a massive particle

Initially mass doesn't matter in hot universe

relativistic, dense (comparable to photon number density $\sim R^{-3} \sim T^3$),

frequent collisions with other species to be in thermal equilibrium and cools with photon bath.

Photon numbers (approximately) conserved, so is the number of relativistic massive particles

Rule 1. Competition of two processes

Interactions keeps equilibrium:

E.g., a particle A might undergo the annihilation reaction:

$$A + \overline{A} \rightarrow \gamma + \gamma$$

depends on cross-section [illegible] and speed v. & most importantly

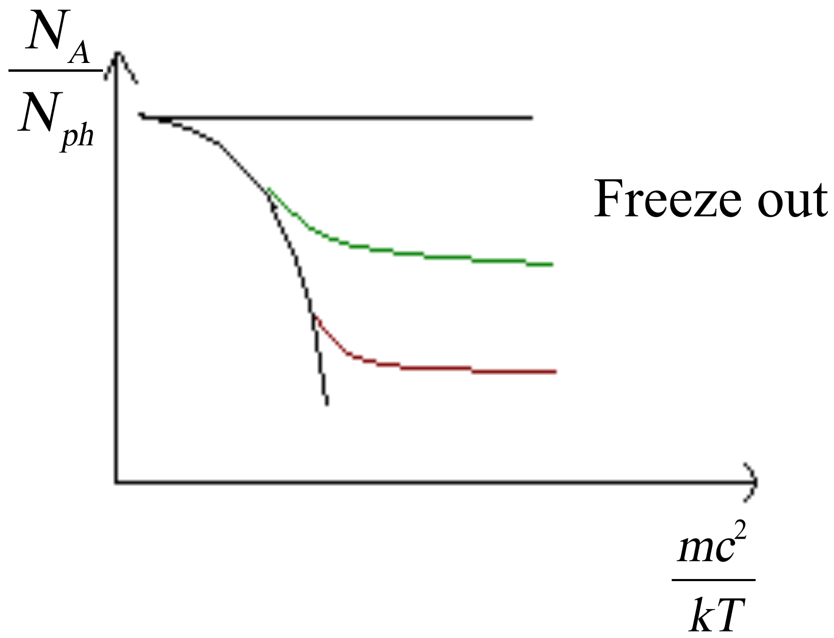
the number density n of photons (falls as $t^{(-6/n)}$, Why? Hint $R \sim t^{(-2/n)}$)

What insulates: the increasing gap of space between particles due to Hubble expansion $H \sim t^{-1}$.

Question: which process dominates at small time? Which process falls slower?

Rule of thumb: Survival of the weakest particle

While in equilibrium, $n_A/n_{ph} \sim \exp(-q)$. $q = mc^2/kT$ →(Heavier is rarer)
At decoupling, $n_A = H_{decouple} /(\square_A \square)$, $n_{ph} \sim T^3{}_{decoupl}$,
Later on the abundance ratio is frozen at this value n_A/n_{ph} ,



$\square_A \square$ LOW→ smallest interaction, early freeze-out while relativistic, Hot Matter

$\square_A \square$ HIGH→ later freeze-out at lower T, reduced abundance, Cold Matter

Question: why frozen while n_A, n_{ph} both drop as $T^3 \sim R^{-3}$.
Energy density of species A $\sim n_{ph}/(\square_A \square)$, if $m \sim kT_{freeze}$

number ratio of non-relativistic particles to photons are const
except for a sudden reduction

Reduction factor ~ exp(- q), q=mc^2/kT, which drops sharply for heavier particles.

Non-relativistic particles (relic) become *much rarer* by exp(-q) as universe cools below mc^2/q,

q ~10−25.

So rare that infrequent collisions can no longer maintain coupled-equilibrium.

For example,

Antiprotons freeze-out $t=10^{-6}$ sec,

Why earlier than positrons freeze-out t=1sec ?

- Hint: anti-proton is ~1000 times heavier than positron.
- Hence factor of 1000 hotter in freeze-out temperature
- t goes as T^2 in radiation-dominated regime

smallest Collision cross-section

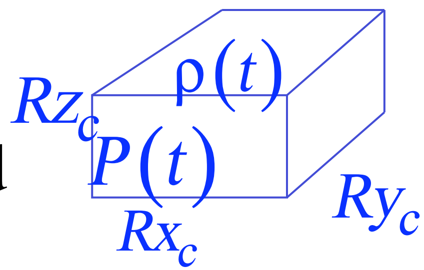
neutrinos (Hot DM) decouple from electrons (due to very weak interaction) while still hot (relativistic 0.5 Mev ~ $kT > mc^2$ ~ 0.02-2 eV)

Presently there are 3 x 113 neutrinos and 452 CMB photons per cm^3 . Details depend on

- Neutrinos have 3 species of spin-1/2 fermions while photons are 1 species of spin-1 bosons
- Neutrinos are a wee bit colder, 1.95K vs. 2.7K for photons [during freeze-out of electron-positions, more photons created]

Evolution of Sound Speed

Expand a box of fluid



Sound Speed

$$C_s^2 \equiv \frac{\partial P / \partial (vol)}{\partial \rho / \partial (vol)},$$
$$= \frac{\partial P / \partial R}{\partial \rho / \partial R}$$

$$Vol = R^3(t) \cdot x_c y_c z_c$$
$$\propto R^3(t)$$

Coupled radiation-baryon relativistic fluid

	Radiation	Matter
Where fluid density $\rho(t) =$	ρ_r	ρ_m
Fluid pressure $P(t) =$	$\frac{c^2}{3}\rho_r$	$\frac{\rho_m}{\mu} \cdot KT_m$

Matter number density

Random motion energy
Non-Relativistic
IDEAL GAS

Note $\rho_r \propto R^{-4}$

$$\rho_m \propto R^{-3}$$

Neglect $\frac{1}{\mu} KT_m << c^2$

Show $C_s^2 = c^2/3\ /(1+Q)$, $Q = (3\ \rho_m)\ /(4\ \rho_r)$, ➔ C_s drops

from c/sqrt(3) at radiation-dominated era

to c/sqrt(5.25) at matter-radiation equality

Sound Speed & Temperature of Gas: $Cs^2 \sim T \sim T_{ph} \sim 1500\text{Kelvin}*(z/500)$ before decouple.

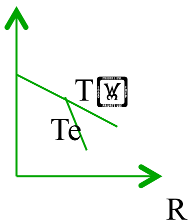
But After decoupling (z<500), Cs ~ 6 (1+z) m/s because

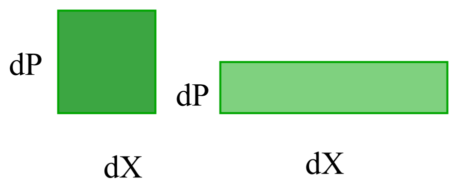
$\underline{d}^3\underline{P}\ \underline{d}_x^3$ invarient phase space volume

So: $P \propto x^{-1} \propto R^{-1}$

$$\frac{3}{2}\times T_e = m\upsilon^2/2 \propto R^{-2}$$

$$Te \sim 1500\,K?\left(\frac{1+z}{500}\right)^2$$





$$T_e \propto C_s^2 \propto R^{-2}$$

Except reionization z ~ 10 by stars quasars

What have we learned?

Where are we heading?

Sound speed of gas before/after decoupling

Topics Next:

Growth of [chpt 11 bankruptcy of uniform universe]

- Density Perturbations (how galaxies form)
- peculiar velocity (how galaxies move and merge)

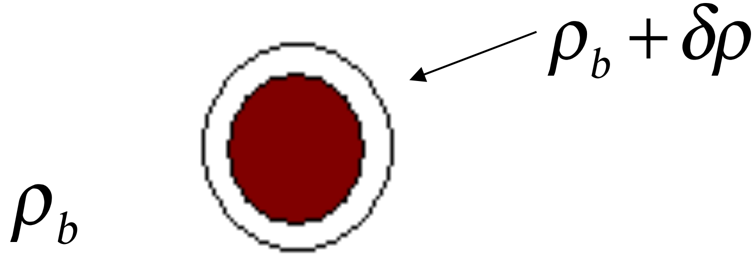
CMB fluctuations (temperature variation in CMB)

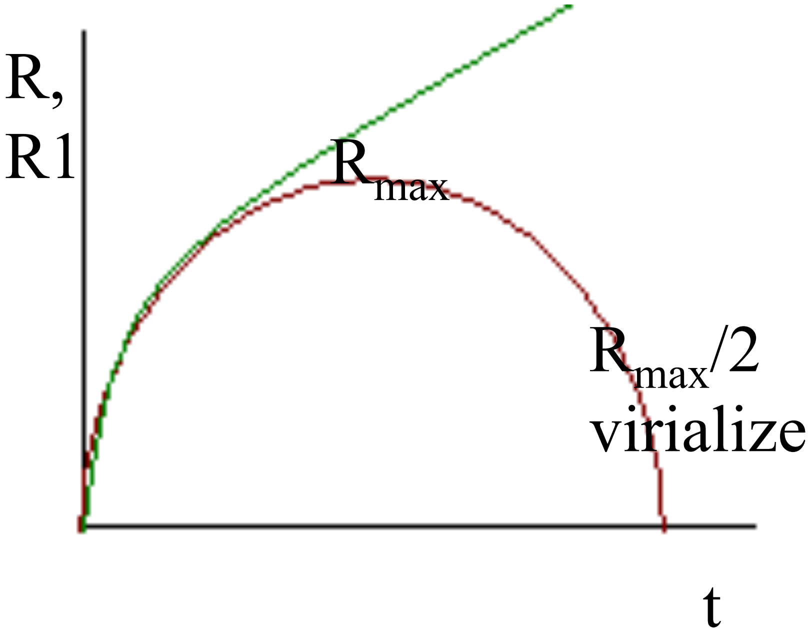
Inflation (origin of perturbations)

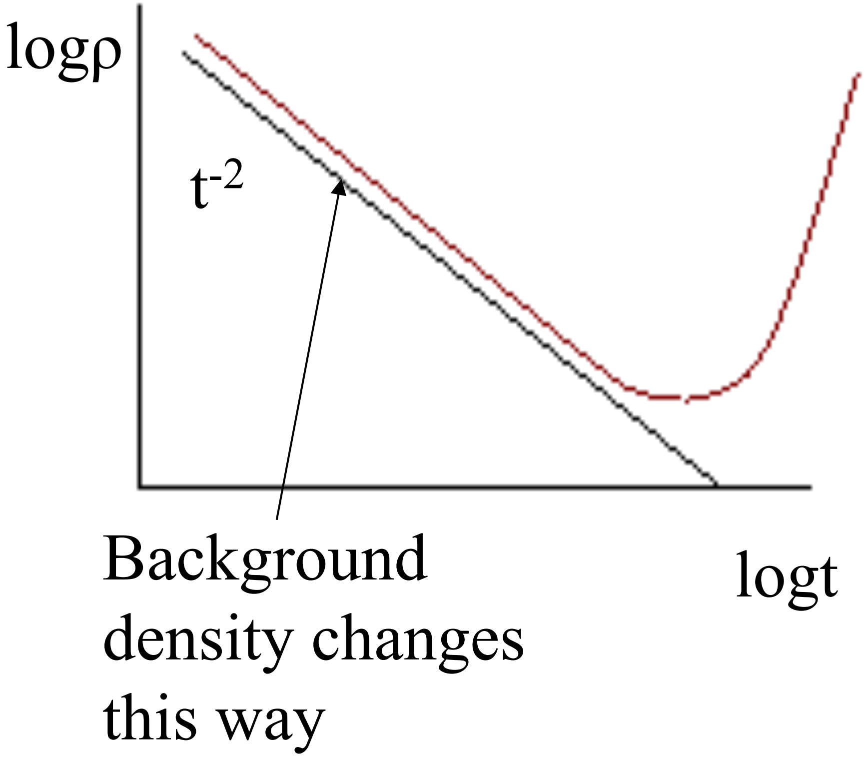
Non-linear Collapse of an Overdense Sphere

An overdense sphere is a very useful non linear model as it behaves in exactly the same way as a closed sub-universe.

The density perturbations need not be a uniform sphere: any spherically symmetric perturbation will clearly evolve at a given radius in the same way as a uniform sphere containing the same amount of mass.







$$\rho_b = \frac{1}{6\pi G t^2}$$

Gradual Growth of perturbation

$$\rightarrow \frac{\delta\rho}{\rho} = \frac{3c^2}{8\pi G}\frac{1}{\rho R^2} \propto \begin{cases} R^2 \text{ (mainly radiation } \rho \propto R^{-4}\text{)} \\ R \text{ (mainly matter } \rho \propto R^{-3}\text{)} \end{cases}$$

Perturbations Grow!

Verify δ changes by a factor of 10 between z=10 and z=100? And a factor of 100 between $z=10^5$ and $z=10^6$?

Peculiar Motion

The motion of a galaxy has two parts:

$$\vec{v} = \frac{d}{dt}\left[R(t)\theta(t)\right]$$

Proper length vector

$$= \underline{\underline{\dot{R}(t).\theta}} + \underline{\underline{R(t)\dot{\theta}(t)}}$$

Uniform expansion v_o

Peculiar motion [illegible] v

Damping of peculiar motion (in the absence of overdensity)

Generally peculiar velocity drops with expansion.

$$R^2\dot{\theta} = R*(R\dot{\theta}) = \text{constant}\sim\text{"Angular Momentum"}$$

Similar to the drop of (non-relativistic) sound speed with expansion

$$\delta v = R(t)\,\dot{x}_c = \frac{\text{constant}}{\text{R(t)}}$$

Equations governing Fluid Motion

$$\nabla^2 \phi = 4\pi G\rho \qquad \text{(Poissons Equation)}$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{dt}} = \frac{d\ln\rho}{dt} = -\vec{\nabla}.\vec{v} \qquad \text{(Mass Conservation)}$$

$$\frac{\mathrm{dv}}{\mathrm{dt}} = -\vec{\nabla}\phi - \underline{\underline{c_s^2 \vec{\nabla} \ln\rho}} \qquad \text{(Equation of motion)}$$

$$\rightarrow \frac{\nabla P}{\rho} \quad \text{since } \partial P = c_s^2 \partial\rho$$

Decompose into unperturbed + perturbed

Let

$$\rho = \rho_o + \delta\rho$$

$$v = v_o + \delta v = \dot{R}\chi_c + R\dot{\chi}_c$$

$$\phi = \phi_o + \delta\phi$$

We define the Fractional Density Perturbation:

$$\delta = \frac{\delta\rho}{\rho_o} = \delta(t)\exp(-i\vec{k}\bullet\vec{x}),$$

$$|\vec{k}| = 2\pi/\lambda, \quad \text{where } \lambda = R(t)\lambda_c$$

$$\vec{k}\bullet\vec{x} = \vec{k}_c\bullet\vec{x}_c \qquad x(t) = R(t)\chi_c$$

Motion driven by gravity: $\vec{g}_o(t) + \vec{g}_1(\theta, t)$

due to an overdensity: $\rho(t) = \rho_o(1 + \delta(\theta, t))$

Gravity and overdensity by Poisson's equation:

$$-\vec{\nabla} \bullet g_1 = 4\pi G \rho_o \delta$$

Continuity equation:

Peculiar motion δv and peculiar gravity g_1 both scale with δ and are in the same direction.

$$-\vec{\nabla} \bullet \delta \vec{v} = \frac{d}{dt}\left(\delta(\theta, t)\right)$$

The over density will rise if there is an inflow of matter

THE equation for structure formation

In matter domination

Equation becomes

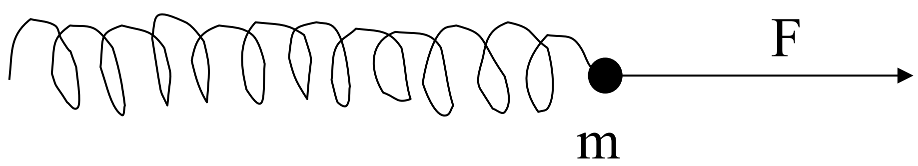
$$-c_s^2 k^2$$

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{R}}{R}\frac{\partial \delta}{\partial t} = (4\pi G\rho_o + c_s^2 \nabla^2)\delta$$

Gravity has the tendency to make the density perturbation grow exponentially.

Pressure makes it oscillate

Each eq. is similar to a forced spring



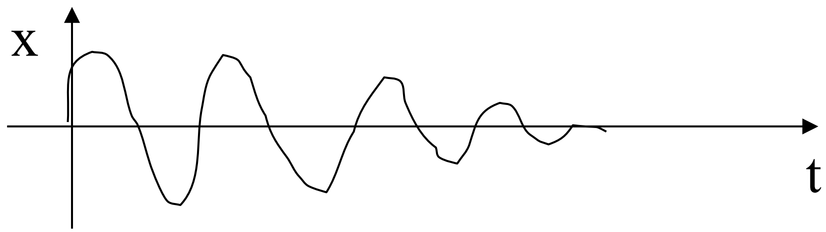
Restoring

$$\frac{d^2x}{dt^2} = \frac{F}{m} - \omega^2 x - \underline{\underline{\mu \frac{dx}{dt}}}$$

Term due to friction

$$\frac{d^2x}{dt^2} + \mu \frac{dx}{dt} + \omega^2 x = \frac{F(t)}{m}$$

(Displacement for Harmonic Oscillator)



e.g., Nearly Empty Pressure-less Universe

$$\rho \sim 0$$

$$\frac{\partial^2 \delta}{\partial t^2} + \frac{2}{t}\frac{\partial \delta}{\partial t} = 0, \quad H = \frac{\dot{R}}{R} = \frac{1}{t} \quad (R \propto t)$$

$$\delta \propto t^0 = \text{constant}$$

$\rightarrow$ no growth

What have we learned?
Where are we heading?

OverDensity grows as

R (matter) or R^2 (radiation)

Peculiar velocity points towards overdensities

Topics Next: Jeans instability

Tutorial: Jeans Instability (no expansion)

Case 1- no expansion

$$\dot{R} = 0$$

- the density contrast δ has a wave-like form

for the harmonic oscillator equation

$$\delta = \delta_o \exp(i\vec{k}.\vec{r} - i\omega t)$$

where we have the dispersion relation

$$\frac{\partial^2 \delta}{\partial t^2} + 2 * 0 * \frac{\partial \delta}{\partial t} = -\omega^2 \delta$$

$$\omega^2 \equiv \underline{\underline{c_s^2 k^2}} - \underline{\underline{4\pi G\rho}}$$

Pressure support

gravity

At the (proper) JEANS LENGTH scale we switch from

Oscillations for shorter wavelength modes to

the exponential growth of perturbations for longer wavelength

$$\lambda_J = c_s \tau, \quad \text{where timescale } \tau = \sqrt{\frac{\pi}{G\rho}}$$

$\lambda < \lambda_J$, $\omega^2 > 0$ → oscillation of the perturbation.

$\lambda \geq \lambda_J$, $\omega^2 \leq 0$ → exponential growth/decay

$$\delta \propto \exp(\pm \Gamma t) \ \text{where} \ \Gamma = \sqrt{-\omega^2}$$

Jeans Instability

Case 2: on very large scale $\lambda >> \lambda_J = c_s t$ of an Expanding universe

Neglect Pressure (restoring force) term

Grow as delta ~ R ~ $t^{2/3}$ for long wavelength mode if Omega_m=1 universe.

$$c_s^2 k^2 << 4\pi G\rho = c_s^2 k_J^2$$

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} = \underline{\underline{4\pi G\rho_m \delta}}$$

$$2/(3t) \qquad\qquad 2/(3t^2)$$

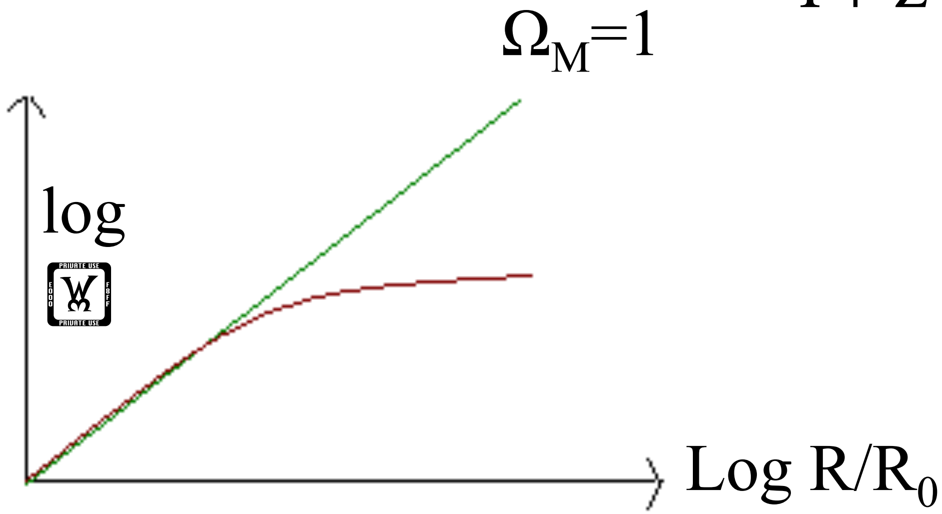
E.g.,

Einstein de Sitter Universe

$$\Omega_M = 1, H = \frac{\dot{R}}{R} = \frac{2}{3t}$$

Verify Growth Solution $\delta \propto R \propto t^{\frac{2}{3}} \propto \frac{1}{1+z}$

Generally



Case III: Relativistic (photon) Fluid

equation governing the growth of perturbations being:

1/t 1/t²

$$\Rightarrow \frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \delta.\left(\frac{32\pi G\rho}{3} - k^2 c_s^2\right)$$

Oscillation solution happens on small scale $2\pi/k = \lambda < \lambda_J$

On larger scale, growth as

$$\Rightarrow \delta \propto t \propto R^2 \quad \text{for length scale } \lambda >> \lambda_J \sim c_s t$$

Lec 8

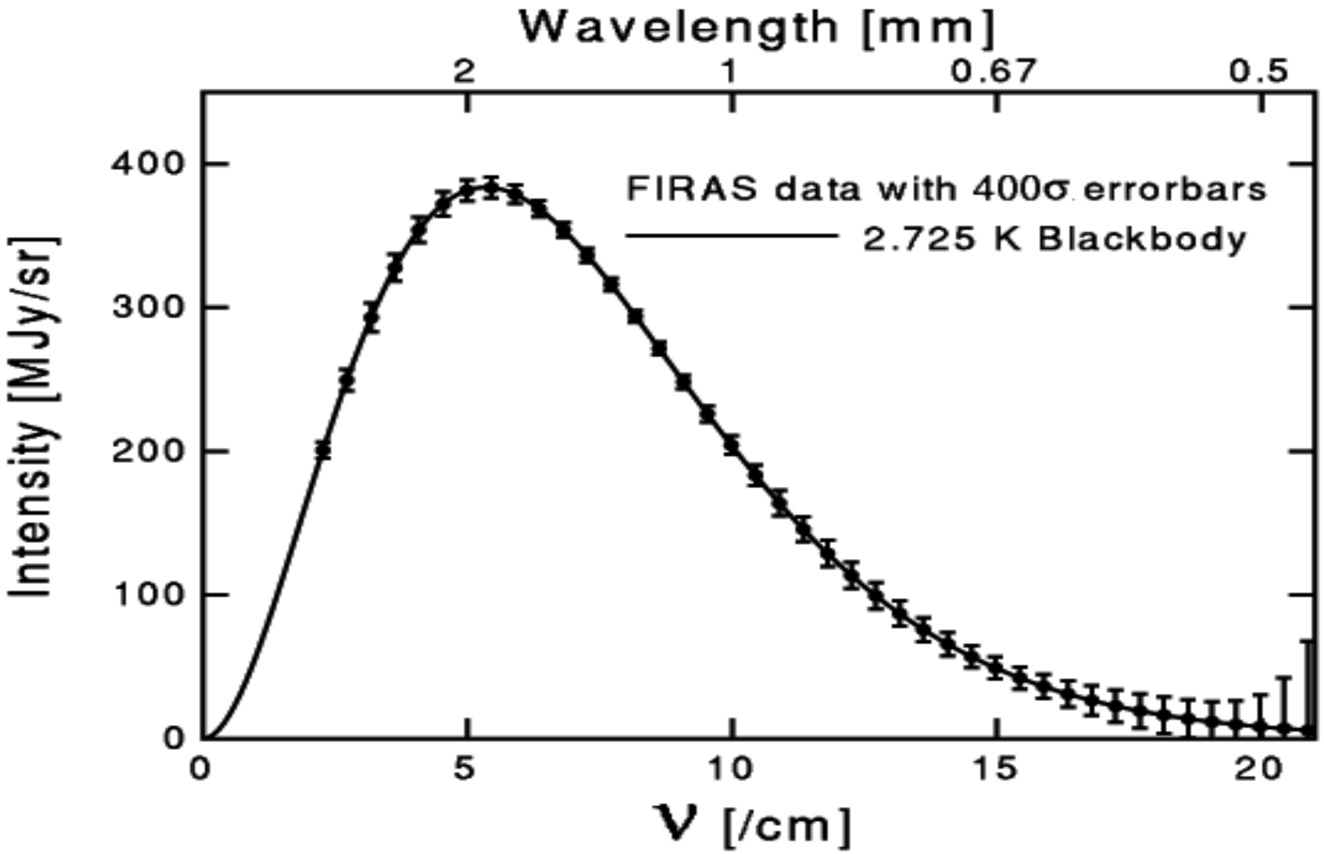
What have we learned: [chpt 11.4]

- Conditions of gravitational collapse (=growth)
- Stable oscillation (no collapse) within sound horizon if pressure-dominated

Where are we heading:

- Cosmic Microwave Background [chpt 15.4]
  - As an application of Jeans instability
- Inflation in the Early Universe [chpt 20.3]

COBE spectrum of CMB

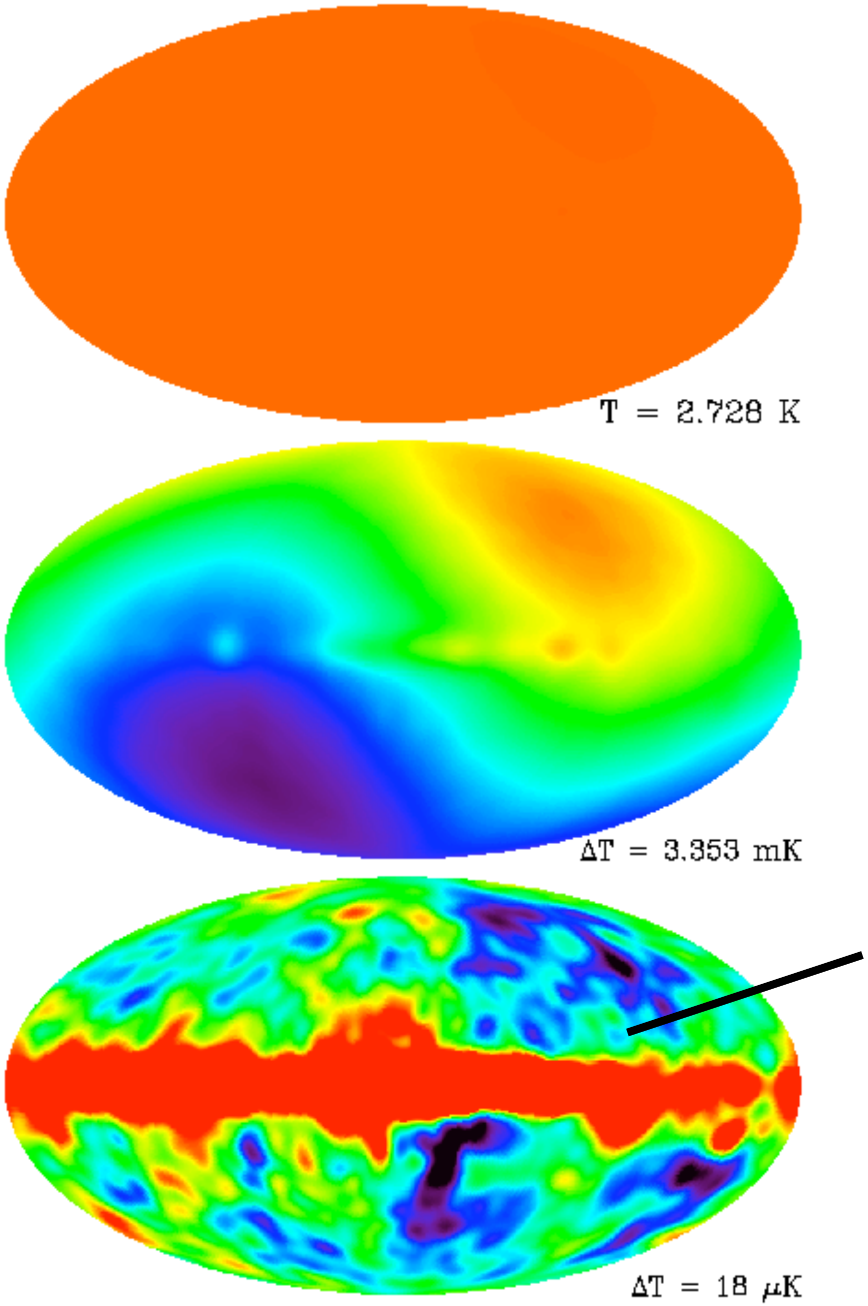


A perfect Blackbody !

No spectral lines -- strong test of Big Bang.
Expansion preserves the blackbody spectrum.

$$T(z) = T_0\,(1+z) \qquad T_0 \sim 3000\ K \quad z \sim 1100$$

Cosmic Microwave Background



Almost isotropic

$T = 2.728$ K

Dipole anisotropy

$$\frac{V}{c} = \frac{\Delta\lambda}{\lambda} = \frac{\Delta T}{T} \approx 10^{-3}$$

Our velocity:

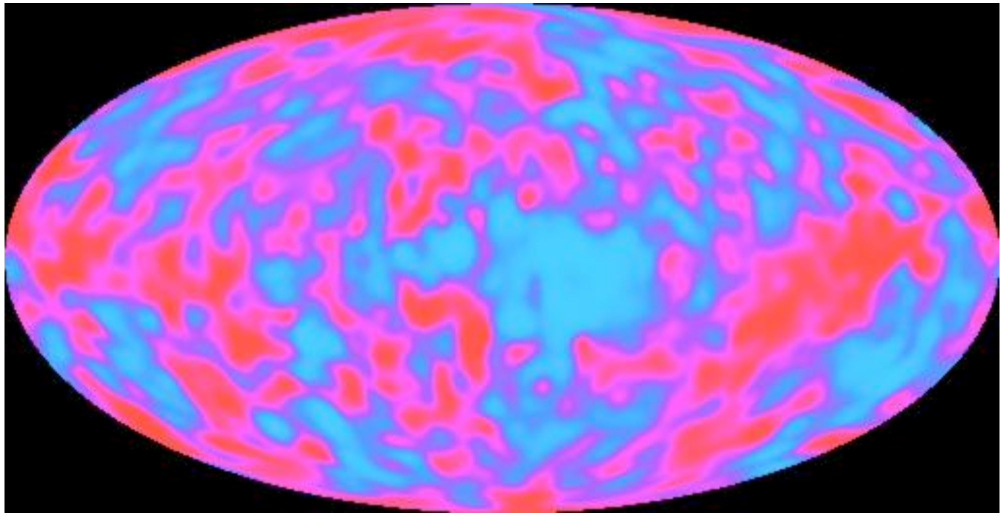
$V \approx 600$ km/s

Milky Way sources

$+$ anisotropies $\frac{\Delta T}{T} \sim 10^{-5}$

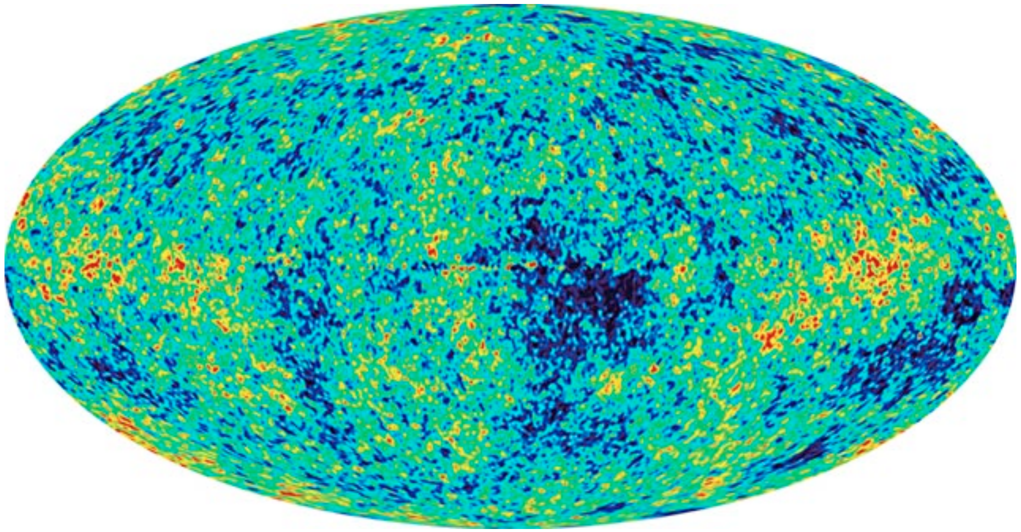
CMB Anisotropies

COBE
1994



$$\frac{\Delta T}{T} \sim 10^{-5}$$

WMAP
2004



$$\Delta\theta \sim 1^{o}$$

Snapshot of Universe at z = 1100

Seeds that later form galaxies.

Theory of CMB Fluctuations

Linear theory of structure growth predicts that the perturbations:

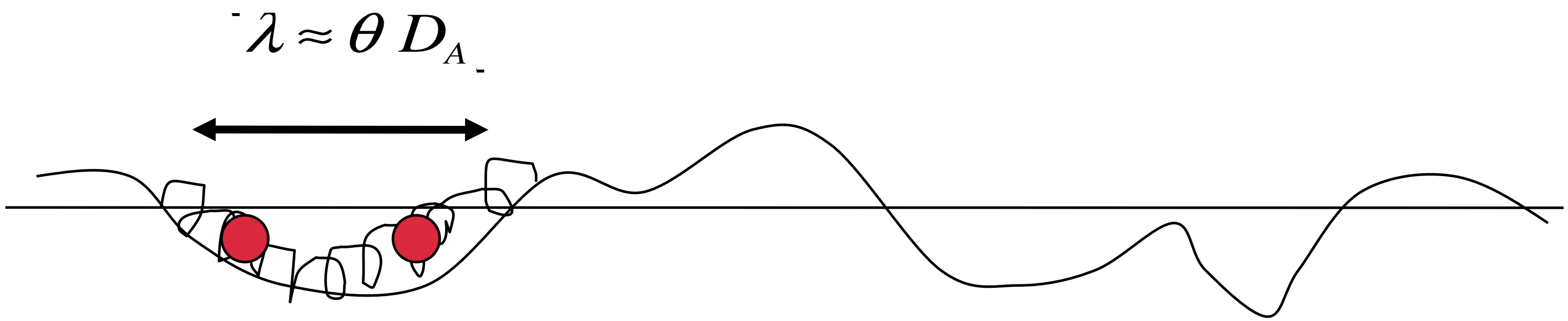
$$\delta_D \text{ in dark matter } \frac{\delta\rho_{\mathrm{D}}}{\rho_{\mathrm{D}}}$$

$$\delta_B \text{ in baryons } \frac{\delta\rho_{\mathrm{B}}}{\rho_{\mathrm{B}}}$$

$$\delta_r \text{ in radiation } \frac{\delta\rho_{\mathrm{r}}}{\rho_r} \quad \text{Or} \quad \tilde{\delta}_r = \frac{3}{4}\delta_r = \frac{\delta n_\gamma}{n_\gamma}$$

will follow a set of coupled Harmonic Oscillator equations.

Acoustic Oscillations



Dark Matter potential wells - many sizes.

photon-electron-baryon fluid

fluid falls into DM wells

photon pressure pushes it out again

oscillations starting at t = 0 (post-inflation)

stopping at z = 1100 (recombination)

The solution of the Harmonic Oscillator [within sound horizon] is:

$$\delta(t) = A_1 \cos kc_s t + A_2 \sin kc_s t + A_3$$

Amplitude is sinusoidal function of $k\, c_s\, t$

if k=constant and oscillate with t

or t=constant and oscillate with k.

Resonant Oscillations

size of potential well λ

oscillation period $P \approx \frac{\lambda}{c_s}$

sound speed $c_s = \frac{c}{\sqrt{3}}$

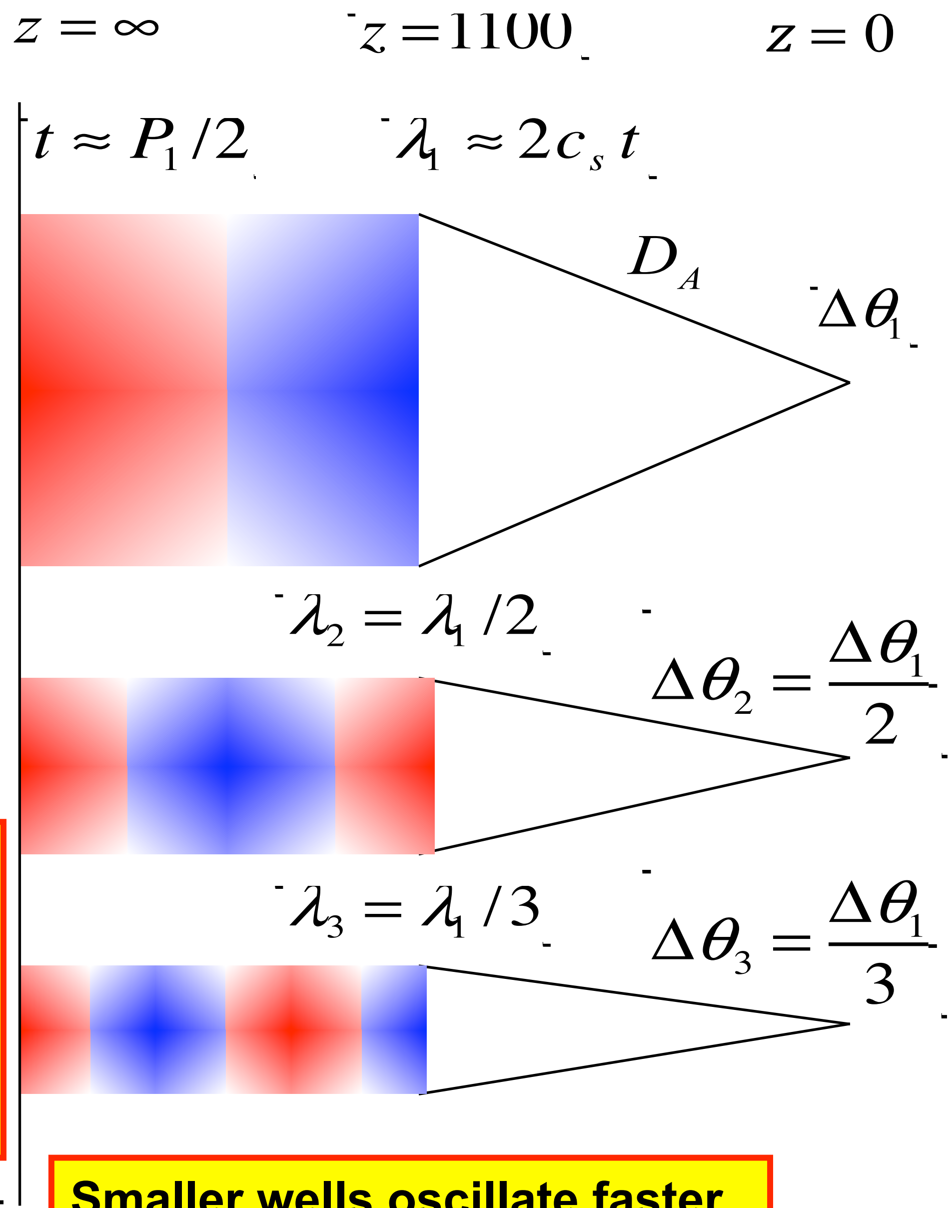
temperature oscillations

$$\Delta T(t) = \Delta T(0)\cos(2\pi\, t/P)$$

$$\max|\Delta T| \text{ at } t = \frac{n\,P}{2} \sim \frac{n\,\lambda}{2c_s}$$

angular size

$$\Delta\theta_n = \frac{\lambda_n}{D_A} = \frac{\Delta\theta_1}{n} \qquad \Delta\theta_1 \approx \frac{2c_s t}{D_A} \sim 0.8^\circ$$



Smaller wells oscillate faster.

We don't observe the baryon overdensity δ_B directly

-- what we actually observe is temperature fluctuations.

$$\frac{\Delta T}{T} = \frac{\Delta n_\gamma}{3n_\gamma}$$

$$= \frac{\delta_B}{3} = \frac{\tilde{\delta}_R}{3}$$

$$n_\gamma \sim R^{-3} \propto T^3$$

$$\varepsilon_\gamma \sim n_\gamma kT \propto T^4$$

The driving force is due to dark matter over densities.

The observed temperature is:

$$\left(\frac{\Delta T}{T}\right)_{obs} = \frac{\delta_B}{3} + \frac{\psi}{c^2}$$

ψ/c^2: Effect due to having to climb out of gravitational well

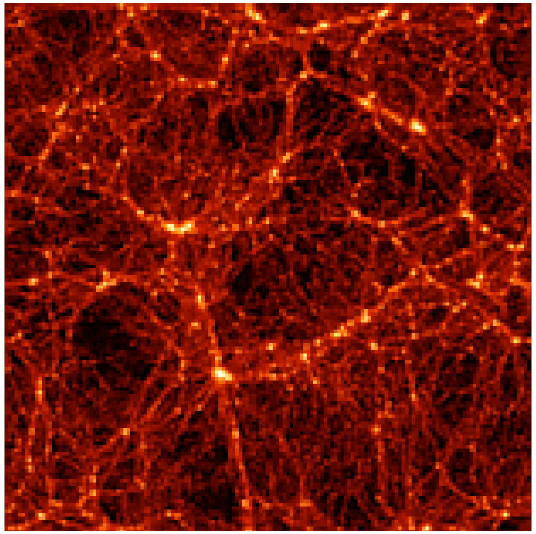
The observed temperature also depends on how fast the Baryon Fluid is moving.

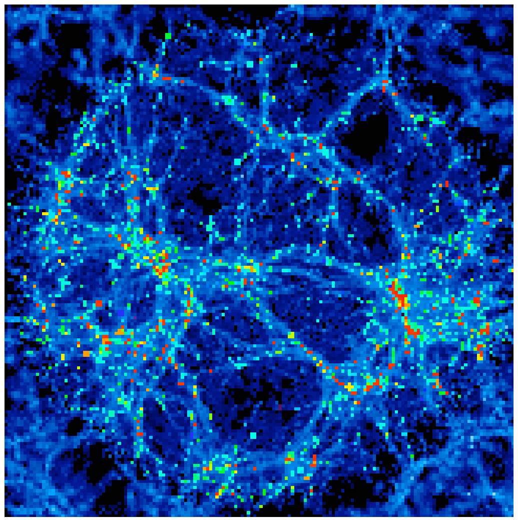
Velocity Field $\nabla v = -\frac{d\delta_B}{dt}$

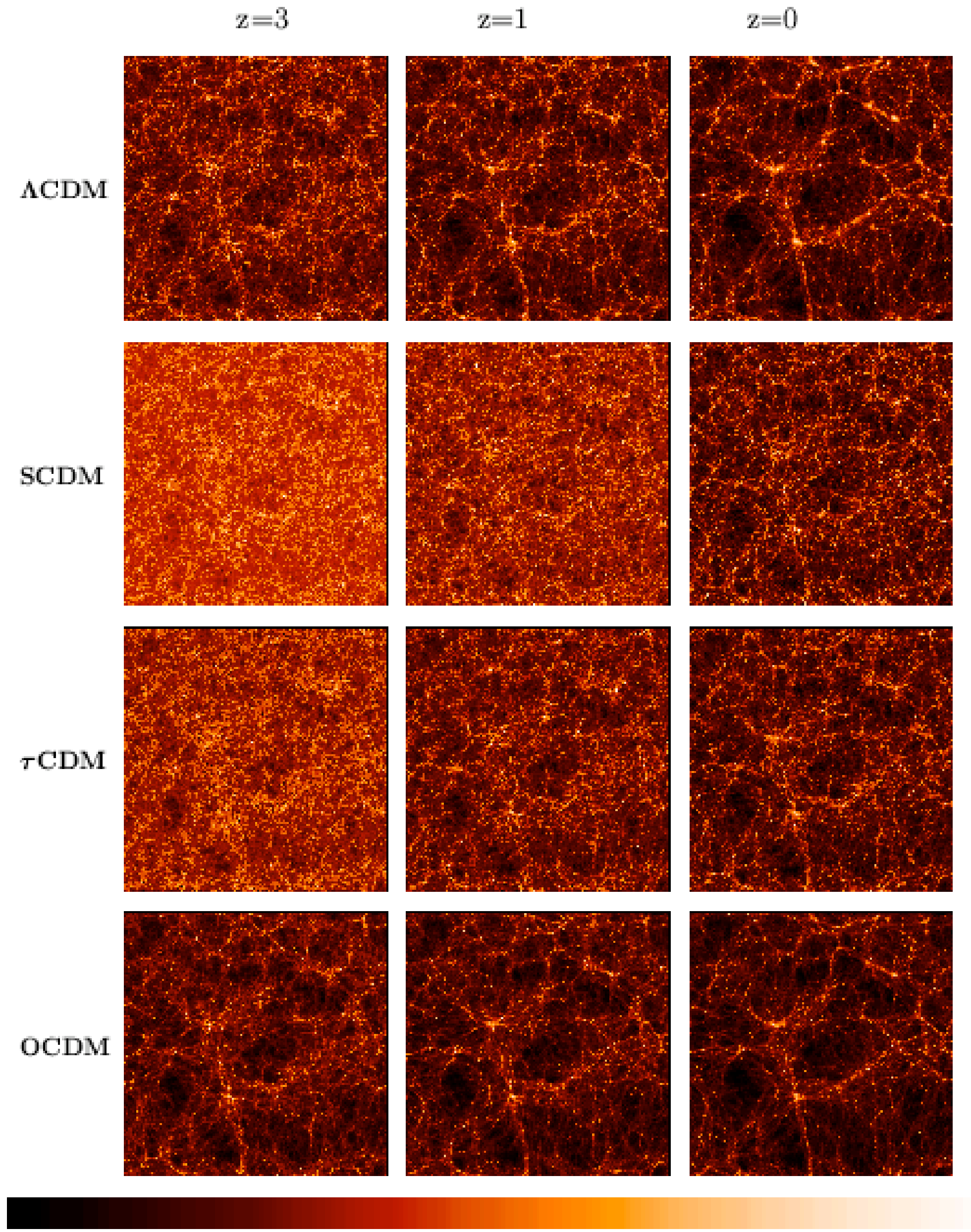
$$\left(\frac{\Delta T}{T}\right)_{obs} = \frac{\delta_B}{3} + \frac{\psi}{c^2} \pm \frac{v}{c}$$

Doppler Term

Supercomputer Simulations







Inflation in Early Universe [chtp 20.3]

Consider universe goes through a phase with

$\rho(t) \sim R(t)^{-n}$

$R(t) \sim t^{q}$ *where* q=2/n

Problems with normal expansion theory (n=2,3,4):

What is the state of the universe at t→0? Pure E&M field (radiation) or exotic scalar field?

Why is the initial universe so precisely flat?

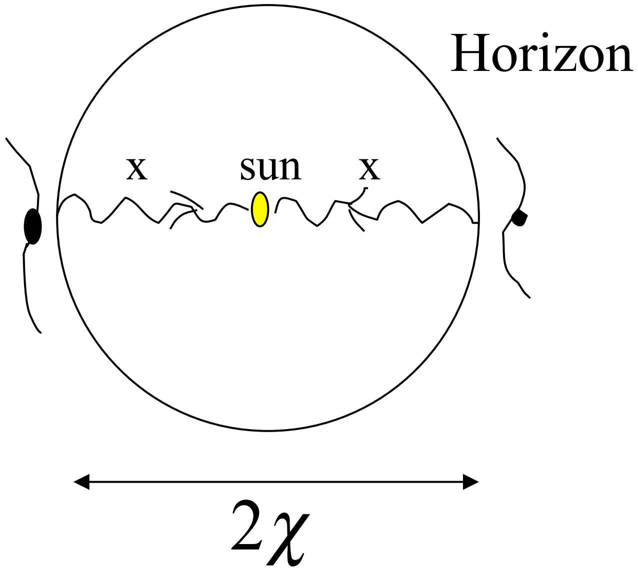
What makes the universe homogeneous/similar in opposite directions of horizon?

Solutions: Inflation, i.e., n=0 or n<2

Maybe the horizon can be pushed to infinity?

Maybe there is no horizon?

Maybe everything was in Causal contact at early times?



Why are these two galaxies so similar without communicating yet?

$$\frac{\varepsilon_K(z)}{\varepsilon\ (z)} = \frac{\varepsilon_K(0)\times R^{-2}}{\varepsilon(0)\times R^{-n}} \sim R^{n-2} \sim 0 \text{ at } t=0$$

Why is the curvature term so small (universe so flat) at early universe if radiation dominates n=4 >2?

What have we learned?

What determines the patterns of CMB at last scattering

- Analogy as patterns of fine sands on a drum at last hit.

The need for inflation to

- Bring different regions in contact
- Create a flat universe naturally.

Inflation broadens Horizon

Light signal travelling with speed c on an expanding sphere R(t), e.g., a fake universe R(t)=1lightyr (t/1yr)^q

Emitted from time t_i

By time t=1yr will spread across (co-moving coordinate) angle x_c

Horizon in co-moving coordinates

$$x_c = \int_{t_i}^{1} \frac{\mathrm{cdt}}{\mathrm{R(t)}} = \int_{t_i}^{1} \frac{\mathrm{cdt}}{\mathrm{t^q}} = \frac{(1^{1-q} - t_i^{1-q})}{(1-q)}$$

Normally $x_c < \frac{1}{(1-q)}$ is finite if q=2/n<1

(e.g., n=3 matter-dominate or n=4 photon-dominate)

INFLATION phase $x_c = \frac{(t_i^{1-q} - 1)}{(q-1)}$ can be very large for very small t_i if q=2/n>1

(e.g., $\mathrm{t_i} = 0.01, q = 2, x_c = 99 >> \pi,$ Inflation allows we see everywhere)

Inflation dilutes the effect of initial curvature of universe

$$\frac{\varepsilon_K(R)}{\varepsilon\ (R)} = \frac{\varepsilon_K(R_i)}{\varepsilon(R_i)}\left(\frac{R}{R_i}\right)^{n-2}$$ ~ 0 (for n<2) sometime after R>>R_i

even if initially the universe is curvature-dominated $\frac{\varepsilon_K(R_i)}{\varepsilon(R_i)} = 1$

E.g.

If a toy universe starts with $\frac{\varepsilon_K(R_i)}{\varepsilon(R_i)} = 0.1$ inflates from $t_i=10^{-40}$sec to $t_f=1$sec with n=1,

and then expand normally with n=4 to t=1 year,

SHOW at this time the universe is far from curvature-dominated.

Exotic Pressure drives Inflation

$$P = -\frac{d(\rho c^2 R^3)}{d(R^3)}$$

=>

$$\frac{\rho}{3} + \frac{P}{c^2} = -\frac{d(\rho R^2)}{3RdR} = \frac{n-2}{3}\rho \quad \text{if } \rho \sim R^{-n}$$

=>

$P/\rho c^2 = (n-3)/3$

Inflation $n < 2$ requires exotic (negative) pressure,

define $w = P/\rho c^2$, then $w = (n-3)/3 < 0$,

Verify negligble pressure for cosmic dust (matter),

Verify for radiation $P = \rho c^2 / 3$

Verify for vaccum $P = -\rho c^2$

What Have we learned?

How to calculate Horizon.

The basic concepts and merits of inflation

Pressure of various kinds (radiation, vacuum, matter)

Expectations for this part

Remember basic concepts (or analogies)

- See list

Can apply various scaling relations to do some of the short questions at the lectures.

- See list

Relax.

- thermal history and structure formation are advanced subjects, just be able to recite the big picture.

Why Analogies in Cosmology

Help you memorizing

Cosmology calls for knowledge of many areas of physics. Analogies help to you memorize how things move and change in a mind-boggling expanding 4D metric.

Help you reason, avoid "more equations, more confusions".

If unsure about equations, e.g. at exams, the analogies *help you recall* the right scaling relations, and get the big picture right.

Years after the lectures,

Analogies go a long way

List of keys

Scaling relations among

- Redshift z, wavelength, temperature, cosmic time, energy density, number density, sound speed
- Definition formulae for pressure, sound speed, horizon
- Metrics in simple 2D universe.

Describe in words the concepts of

- Fundamental observers
- thermal decoupling
  - Common temperature before,
  - Fixed number to photon ratio after
  - Hot and Cold DM.
- gravitational growth.
  - Over-density,
  - direction of peculiar motion driven by over-density, but damped by expansion
  - pressure support vs. grav. collapse

Tutorial

Consider a micro-cosmos of N-ants inhabiting an expanding sphere of radius $R=R_0\ (t/t_0)^q$, where presently we are at $t=t_0$ =1year, $R=R_0$ =1m. Let q=1/2, N=100, and the ants has a cross-length σ=1cm for collision. Let each ant keep its random angular momentum per unit mass J=1m*1(m/yr) with respect to the centre of the sphere.

What is the present rate of expansion dR/dt/R = in units of 1/yr,

How does the ant random speed, ant surface density, change as function of cosmic time?

Light emitted by ant-B travels a half circle and reaches ant-A now, what redshift was the light emitted?

What is the probability that the ant-A would encounter another ant from time t_1 to time t_2. How long has it travelled? Calculate assume t_1 =1/2 yr, t_2 =2yr.

Show the age of the universe is t=1sec at $z\sim10^{10}$; assume crudely that at matter-radiation equality $z=10^3$ and age $t=10^6$ yr

Argue that a void in universe now originates from an under-dense perturbation at $z=10^{10}$ with δ about 10^{-17}.

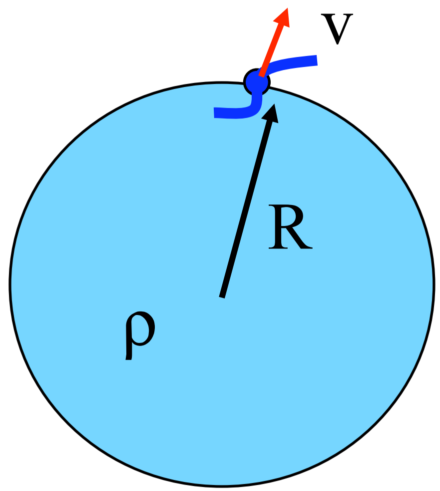
The edge of the void are lined up by galaxies. What direction is their peculiar gravity and peculiar motion?

A patch of sky is presently hotter in CMB by 3 micro Kelvin than average. How much was it hotter than average at the last scattering (z=1000)?

Concepts and scaling laws

- Redshift, comoving coordinates, rho(z), Hubble equation, conservation of particle number.
- Freeze-out, last scattering, structure growth eq.
- Dark Matter candidates, cold/hot
- Flatness/causality problem, horizon

Critical Density



- Newtonian analogy:

escape velocity :

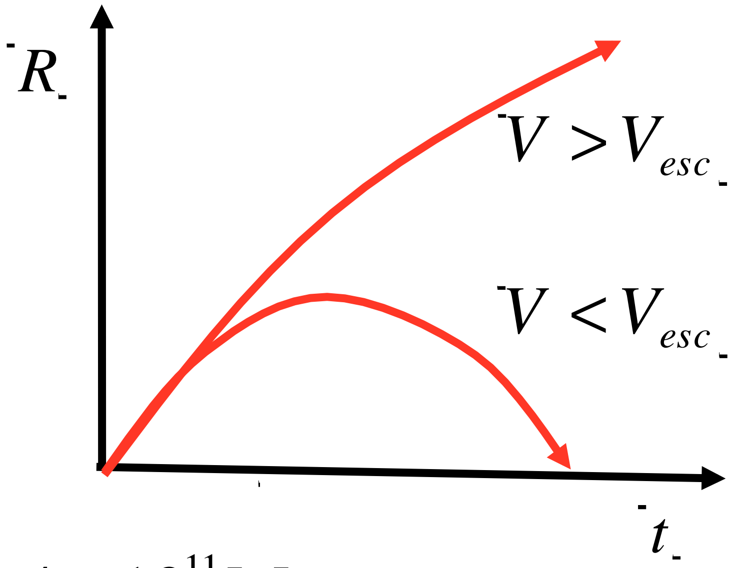
$$V_{esc}^{\ 2} = \frac{2\,G\,M}{R} = \frac{2\,G}{R}\left(\frac{4\pi\,R^3\rho}{3}\right) = \frac{8\pi\,G\,R^2\rho}{3}$$

Hubble expansion :

$$V = H_0\,R$$

critical density :

$$\left(\frac{V_{esc}}{V}\right)^2 = \frac{8\pi\,G\rho}{3\,H_0^{\ 2}} = \frac{\rho}{\rho_c}$$



$$\rho_c \equiv \frac{3\,H_0^{\ 2}}{8\pi\,G} \approx 10^{-26}\mathrm{kg\ m^{-3}} \approx \frac{1.4\times10^{11}\,\mathrm{Msun}}{(\mathrm{Mpc})^3}$$

Cosmological Models

Einstein's gravity theory (General Relativity)
Assume Universe filled with uniform density fluid.
[OK on large scales > 100 Mpc]

Density: $\rho = \Omega \, \rho_c$ Energy density: $\varepsilon = \rho \, c^2$

Critical density: $$\rho_c \equiv \frac{3 H_0^2}{8\pi G} \approx 10^{-26} \text{kg m}^{-3} \approx \frac{1.4\times10^{11}\,\text{Msun}}{(\text{Mpc})^3}$$

3 components:

1. ***Radiation*** $\Omega_R \approx 5\times10^{-5}$
2. ***Matter*** $\Omega_M \sim 0.3$ { "Dark Matter" $\Omega_D \sim 0.26$; baryons $\Omega_B \sim 0.04$
3. "***Dark Energy***" $\Omega_\Lambda \sim 0.7$

Total $\Omega = \Omega_R + \Omega_M + \Omega_\Lambda = 1$

Only ~4% is matter as we know it!

Tutorial: 3 Eras: radiation-matter-vacuum

radiation : $\rho_R \propto R^{-4}$

matter : $\rho_M \propto R^{-3}$

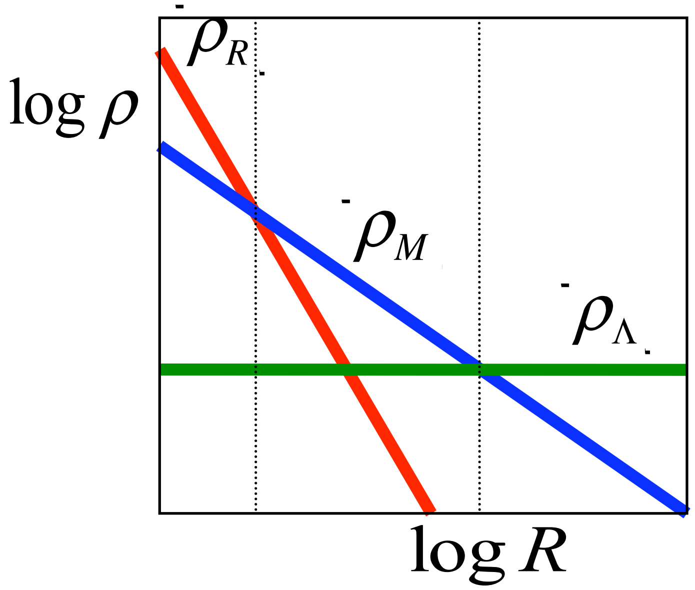
vacuum: $\rho_\Lambda = \text{const}$

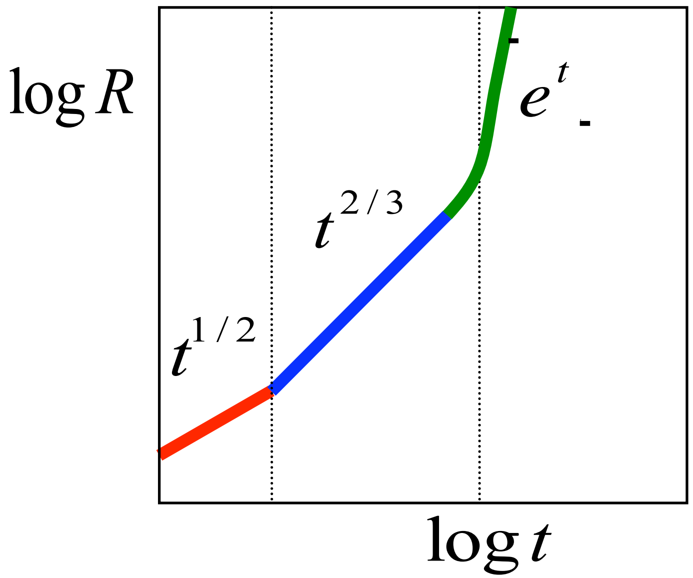
$$a \equiv \frac{R}{R_0} = \frac{1}{1+z}$$

$$\rho = \frac{\rho_{R,0}}{a^4} + \frac{\rho_{M,0}}{a^3} + \rho_\Lambda$$

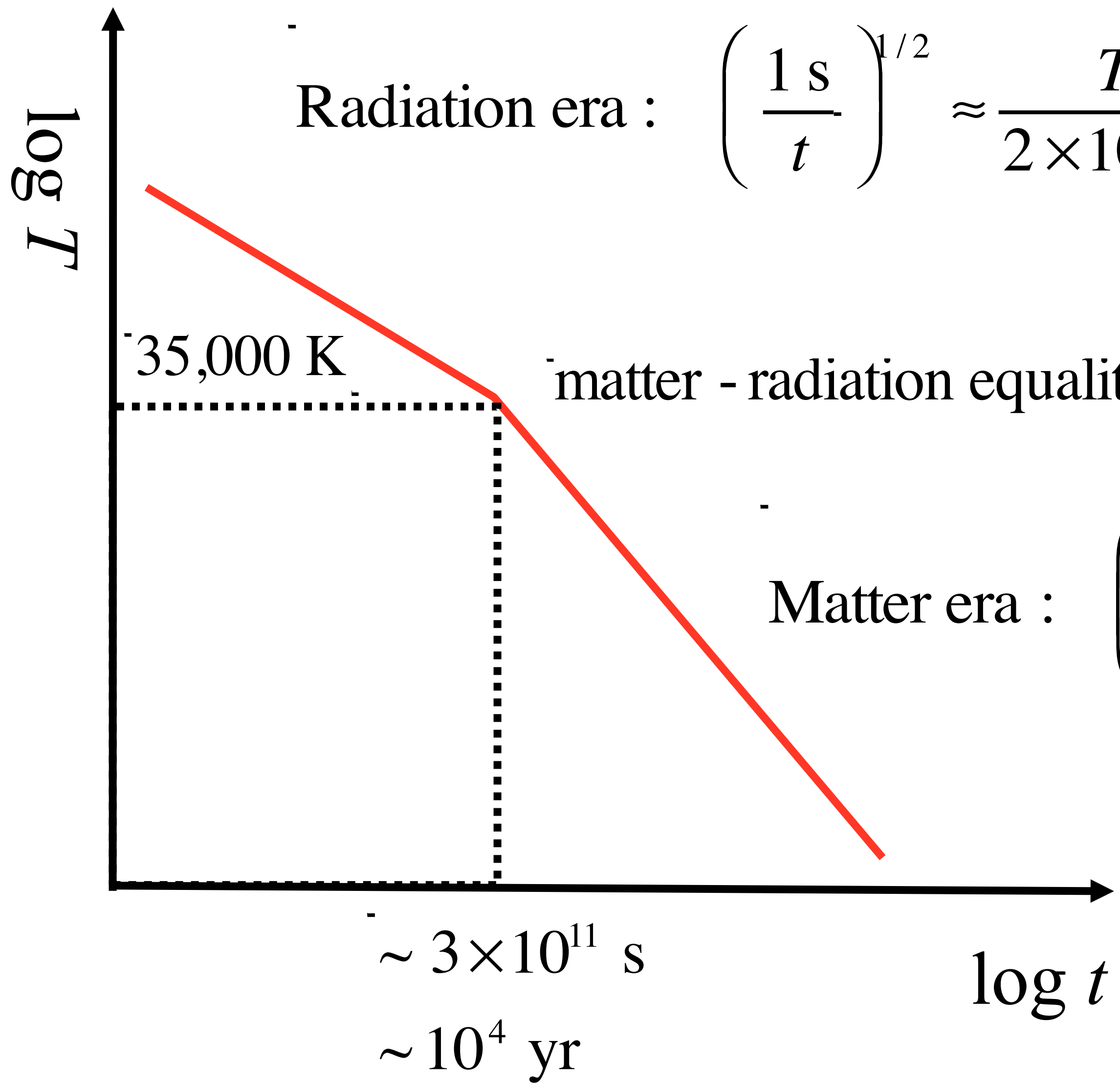
$\rho_R = \rho_M$ at $a \sim 10^{-4}$ $t \sim 10^4$ yr

$\rho_M = \rho_\Lambda$ at $a \sim 0.7$ $t \sim 10^{10}$ yr





Tutorial: Cooling History T(t)



Radiation era : $\left(\frac{1\ \text{s}}{t}\right)^{1/2} \approx \frac{T}{2\times 10^{10}\,\text{K}} = \frac{k\,T}{2\ \text{MeV}}$

matter - radiation equality : $\rho_M = \rho_R$

Matter era : $\left(\frac{10^{10}\ \text{yr}}{t}\right)^{2/3} = \frac{T}{2.7\ \text{K}}$

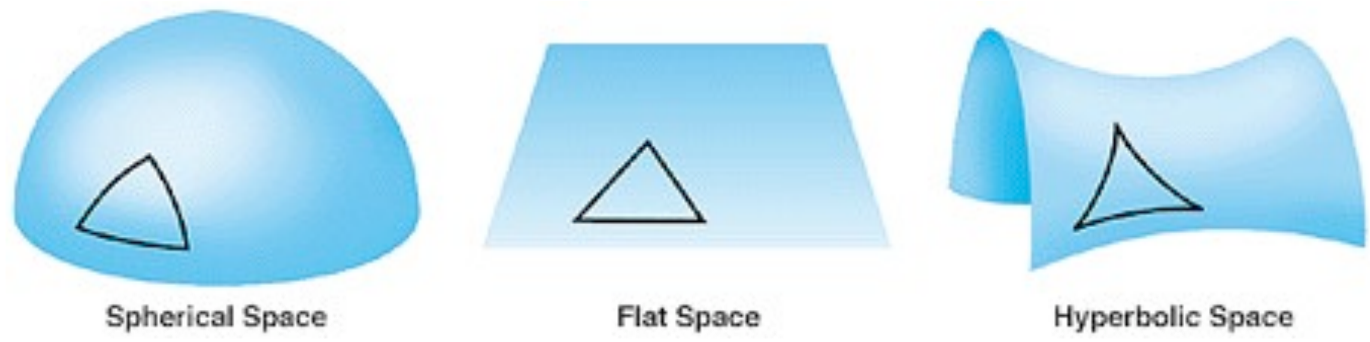
Non-Euclidean Geometry

Curved 3-D Spaces

How Does Curvature affect Distance Measurements ?

Is our Universe Curved?

	Closed	Flat	Open

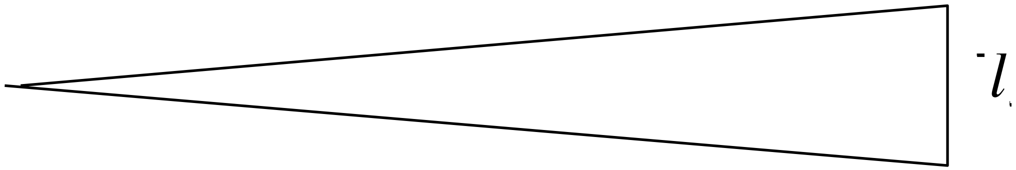


	Closed	Flat	Open
Curvature:	**+**	**0**	**--**
Sum of angles of triangle:	**> 180°**	**= 180°**	**< 180°**
Circumference of circle:	**< 2 π *r***	**= 2 π *r***	**> 2 π *r***
Parallel lines:	**converge**	**remain parallel**	**diverge**
Size:	**finite**	**infinite**	**infinite**
Edge:	**no**	**no**	**no**

Distance Methods

- **Standard Rulers ==> Angular Size Distances**

$$\theta = \frac{l}{D}$$

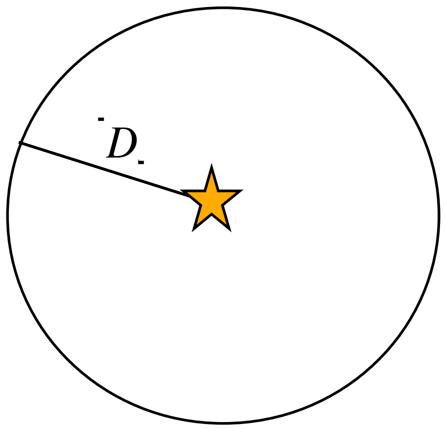


$$D_A = \frac{l}{\theta}$$

(for small angles << 1 radian)

- **Standard Candles ==> Luminosity Distances**

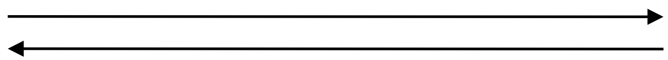
$$F = \frac{\text{energy/time}}{\text{area}} = \frac{L}{4\pi D^2}$$



$$D_L = \left(\frac{L}{4\pi F} \right)^{1/2}$$

- **Light Travel Time**

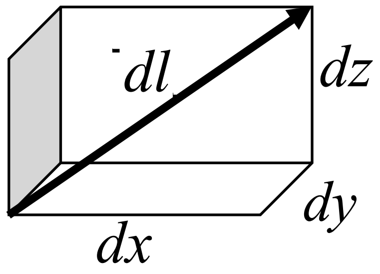
$$t = \frac{\text{distance}}{\text{velocity}} = \frac{2D}{c}$$



$$D_t = \frac{c}{2t}$$

(e.g. within solar system)

Flat Space: Euclidean Geometry



Cartesian coordinates :

$$1\,\mathrm{D}: \quad dl^2 = dx^2$$

$$2\,\mathrm{D}: \quad dl^2 = dx^2 + dy^2$$

$$3\,\mathrm{D}: \quad dl^2 = dx^2 + dy^2 + dz^2$$

$$4\,\mathrm{D}: \quad dl^2 = dw^2 + dx^2 + dy^2 + dz^2$$

Metric tensor : coordinates -> distance

$$dl^2 = (\, dx \quad dy \quad dz \,) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Orthogonal coordinates <--> diagonal metric

$$g_{xx} = g_{yy} = g_{zz} = 1$$

$$g_{xy} = g_{xz} = g_{yz} = 0$$

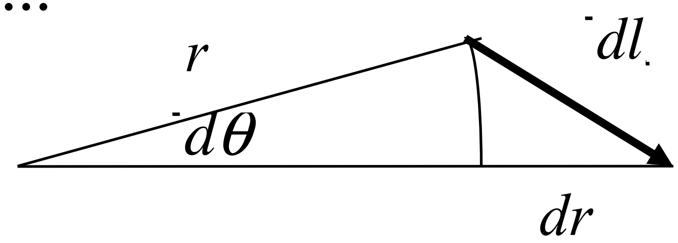
symmetric : $g_{ij} = g_{ji}$

Summation convention :

$$dl^2 = g_{ij}\, dx^i\, dx^j \equiv \sum_i \sum_j g_{ij}\, dx^i\, dx^j$$

Polar Coordinates

Radial coordinate r, angles $\phi, \theta, \alpha, \ldots$



1 D: $\quad dl^2 = dr^2$

2 D: $\quad dl^2 = dr^2 + r^2 \; d\theta^2$

3 D: $\quad dl^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2 \right)$

4 D: $\quad dl^2 = dr^2 + r^2 \left[d\theta^2 + \sin^2\theta \left(d\phi^2 + \sin^2\phi \; d\alpha^2 \right) \right]$

$$dl^2 = dr^2 + r^2 \, d\psi^2 \qquad \text{generic angle: } \; d\psi^2 = d\theta^2 + \sin^2\theta \, d\phi^2 + \ldots$$

$$dl^2 = \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

$g_{rr} = ? \quad g_{r\theta} = ?$

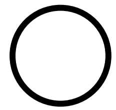
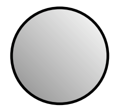
$g_{\theta\theta} = ?$

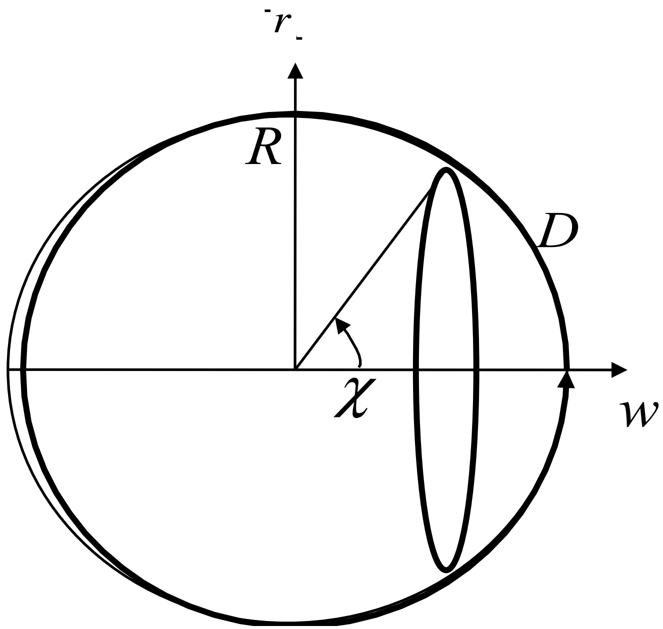
$g_{\phi\phi} = ?$

$g_{\alpha\alpha} = ?$

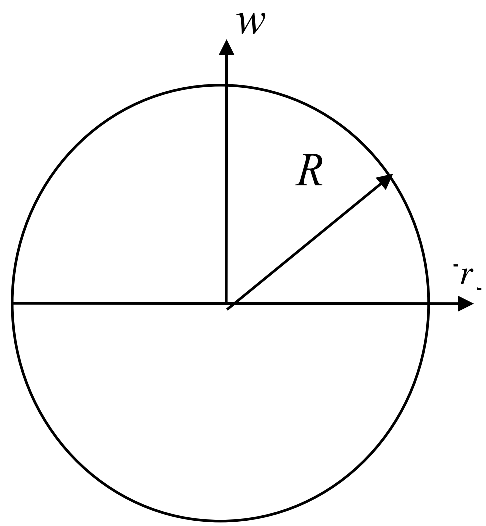
Embedded Spheres

R = radius of curvature

1 – D:	$R^2 = x^2$	0 - D 2 points	
2 – D:	$R^2 = x^2 + y^2$	1 - D circle	
3 – D:	$R^2 = x^2 + y^2 + z^2$	2 - D surface of 3 - sphere	
4 – D:	$R^2 = x^2 + y^2 + z^2 + w^2$	3 - D surface of 4 - sphere	?



Metric for 3-D surface of 4-D sphere



$$4-\text{sphere}: \quad R^2 = x^2 + y^2 + z^2 + w^2$$

i.e. $R^2 = r^2 + w^2$ with $r^2 \equiv x^2 + y^2 + z^2$.

$$0 = 2\,r\,dr + 2\,w\,dw \quad \rightarrow \quad dw^2 = \left(\frac{r\,dr}{w}\right)^2 = \frac{r^2\,dr^2}{R^2 - r^2}$$

$$dl^2 = dw^2 + dr^2 + r^2 d\psi^2 \qquad \text{4 - space metric}$$

$$= \frac{r^2\,dr^2}{R^2 - r^2} + dr^2 + r^2 d\psi^2 \qquad \text{confined to } R^2 = r^2 + w^2$$

$$dl^2 = \frac{dr^2}{1-(r/R)^2} + r^2 d\psi^2 \qquad d\psi^2 = d\theta^2 + \sin^2\theta\, d\phi$$

Metric for a 3 - D space with constant curvature radius R

Non-Euclidean Metrics

$k = -1, 0, +1 \quad (\text{open, flat, closed})$

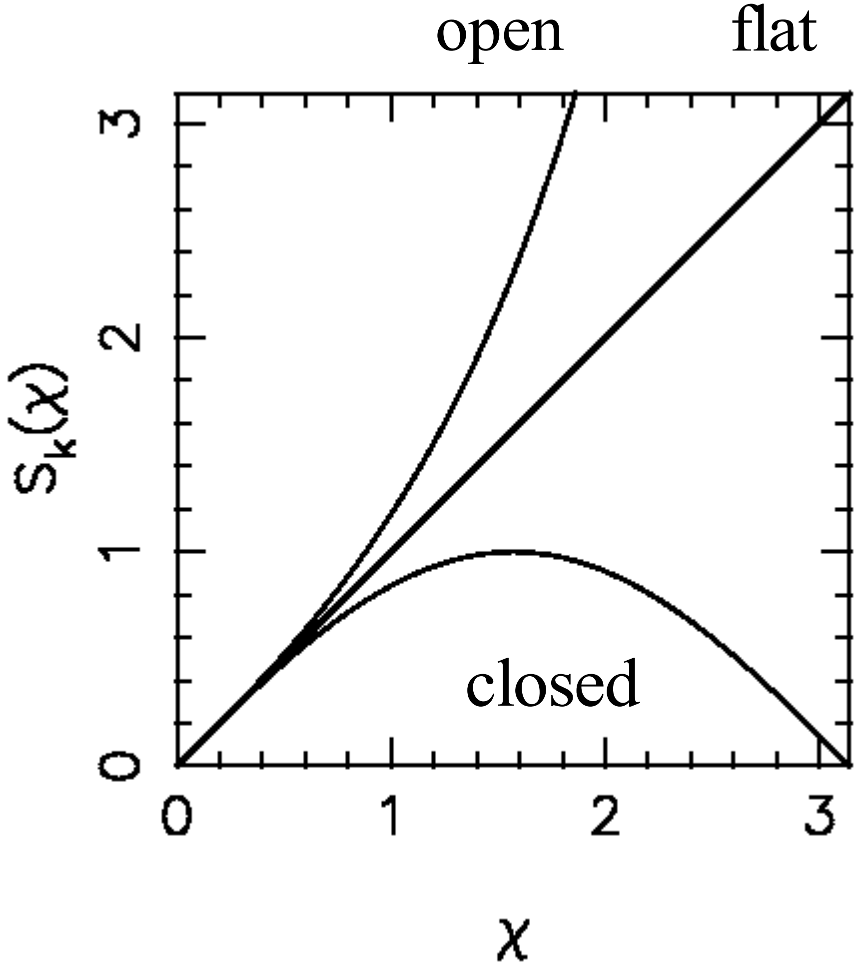
$$dl^2 = \frac{dr^2}{1 - k\,(r/R)^2} + r^2 d\psi^2$$

dimensionless radial coordinates :

$$u = r/R = S_k(\chi)$$

$$dl^2 = R^2 \left(\frac{du^2}{1 - k\,u^2} + u^2 d\psi^2 \right)$$

$$= R^2 \left(d\chi^2 + S_k^{\,2}(\chi)\, d\psi^2 \right)$$



$$S_{-1}(\chi) \equiv \sinh(\chi)\ , \quad S_0(\chi) \equiv \chi\ , \quad S_{+1}(\chi) \equiv \sin(\chi)$$

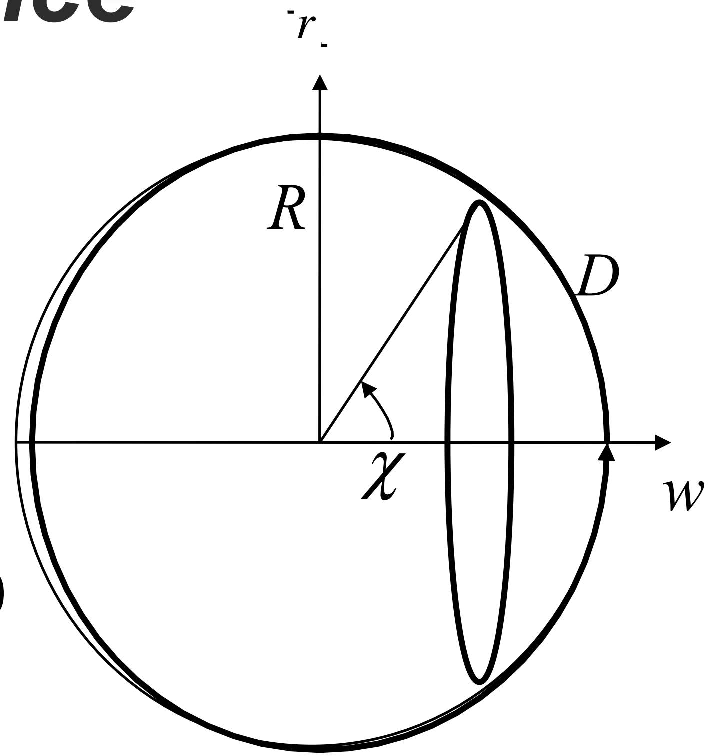
Circumference

metric :

$$dl^2 = \frac{dr^2}{1-k\,(r/R)^2} + r^2\, d\theta^2$$

radial distance (for $k=+1$) :

$$D = \int_0^r \frac{dr}{\sqrt{1-k\,(r/R)^2}} = R \sin^{-1}(r/R)$$



circumference :

$$C = \int_0^{2\pi} r\, d\theta = 2\pi\, r$$

"circumferencial" distance : $r \equiv \frac{C}{2\pi} = R\, S_k(D/R) = R\, S_k(\chi)$

If $k=+1$, coordinate r breaks down for $r > R$

Tutorial: Circumference

metric :

$$dl^2 = R^2\,(\,d\chi^2 + S_k^2(\chi)\,d\theta^2\,)$$

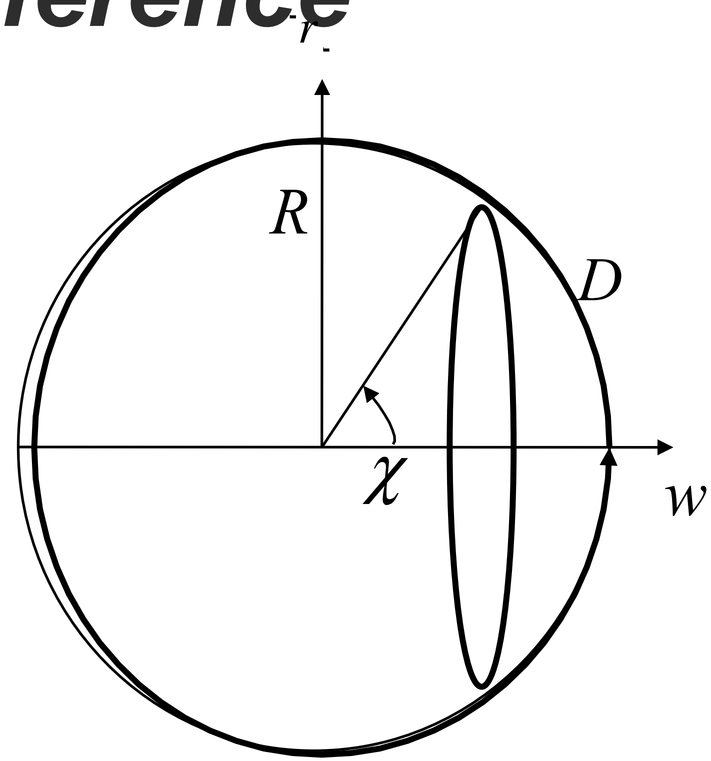
radial distance :

$$D = \int \sqrt{g_{\chi\chi}}\,d\chi = \int_0^{\chi} R\,d\chi = R\,\chi$$

circumference :

$$C = \oint \sqrt{g_{\theta\theta}}\,d\theta = \int_0^{2\pi} R\,S_k(\chi)\,d\theta = 2\pi\,R\,S_k(\chi)$$

$$= 2\pi\,D\,\frac{S_k(\chi)}{\chi}$$



Same result for any choice of coordinates.

Angular Diameter

metric :

$$dl^2 = R^2 \left(d\chi^2 + S_k^2(\chi)\, d\theta^2 \right)$$

radial distance :

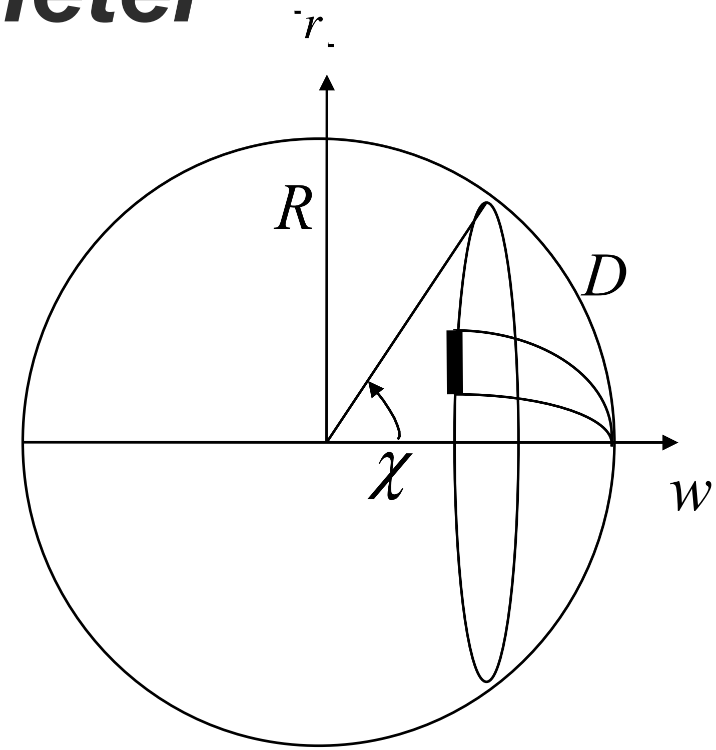
$$D = \int \sqrt{g_{\chi\chi}}\, d\chi = \int_0^{\chi} R\, d\chi = R\, \chi$$

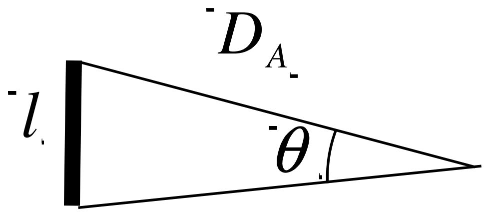
linear size : ($l << D$)

$$l = \int \sqrt{g_{\theta\theta}}\, d\theta = R\, S_k(\chi)\, \theta$$

angular size :

$$\theta = \frac{l}{D_A}$$





$D = R\ \chi =$ Radial Distance

$D_A = R\, S_k(\chi) =$ Angular Diameter Distance

Tutorial: Area of Spherical Shell

radial coordinate χ, angles θ, ϕ :

$$dl^2 = R^2 \left[d\chi^2 + S_k^2(\chi) \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]$$

area of shell :

$$A = \int \sqrt{g_{\theta\theta}} \, d\theta \sqrt{g_{\phi\phi}} \, d\phi$$

$$= R^2 \, S_k^2(\chi) \int_0^{\pi} d\theta \; \sin\theta \int_0^{2\pi} d\phi$$

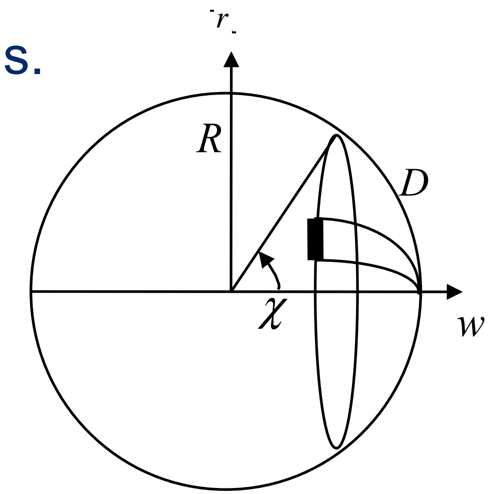
$$= 4\pi \, R^2 \, S_k^2(\chi)$$

flux :

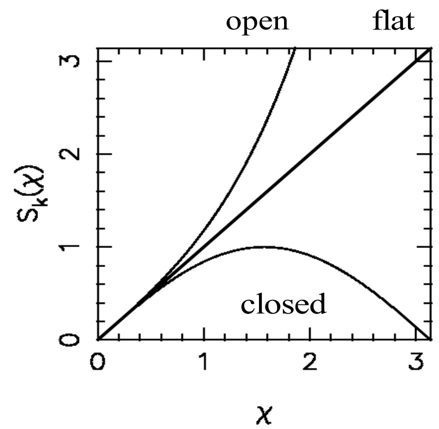
$$F = \frac{L}{A} = \frac{L}{4\pi \, D_L^2} \qquad D_L = R \, S_k(\chi) = \text{ Luminosity Distance}$$

Summary

- The **metric** converts coordinate steps to physical lengths.
- Use the metric to compute lengths, areas, volumes, …
- Radial distance: $$D \equiv \int \sqrt{g_{rr}}\, dr = R\chi$$



- "Circumferencial" distance
$$r \equiv \frac{C}{2\pi} = \left(\frac{A}{4\pi}\right)^{1/2} = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi = R\, S_k(\chi) = R\, S_k(D/R)$$
- "Observable" distances, defined in terms of local observables (angles, fluxes), give r, not D.
$$D_A \equiv \frac{l}{\theta} = r \qquad D_L \equiv \left(\frac{L}{4\pi F}\right)^{1/2} = r$$



- r can be smaller than D (positive curvature) or larger (negative curvature) or the same (flat).

Olber's Paradox

Why is the sky dark at night ?

Flux from all stars in the sky :

$$F = \int n_* \, F_* \, d(\mathrm{Vol}) = \int_0^{\chi_{\max}} n_* \left(\frac{L_*}{A(\chi)} \right) (\, A(\chi) \, R \, d\chi \,)$$

$$= n_* \, L_* \, R \, \chi_{\max}$$

$$\Rightarrow \infty \quad \text{for flat space, } R \to \infty.$$

A dark sky may imply :

(1) an edge (we don't observe one)

(2) a curved space (finite size)

(3) expansion ($R(t)$ => finite age, redshift)

Minkowski Spacetime Metric

$$ds^2 = -c^2 dt^2 + dl^2$$

$$d\tau^2 = dt^2 - \frac{dl^2}{c^2} = dt^2\left(1 - \frac{1}{c^2}\left(\frac{dl}{dt}\right)^2\right)$$

Time-like intervals:
$ds^2 < 0$, $d\tau^2 > 0$
Inside light cone.
Causally connected.

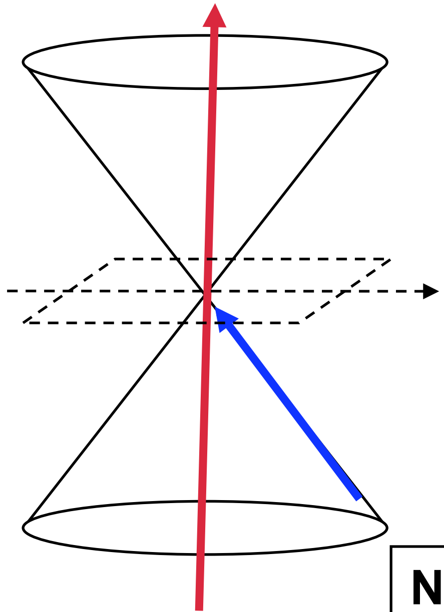
***Proper time** (moving clock):*

$$d\tau = \sqrt{-ds^2/c^2}$$

$$= dt\sqrt{1 - \frac{v^2}{c^2}} > 0$$

Space-like intervals:
$ds^2 > 0$, $d\tau^2 < 0$

Outside light cone.
Causally disconnected.



World line of massive particle at rest.

Null intervals light cone:
v = c, *$ds^2 = 0$*

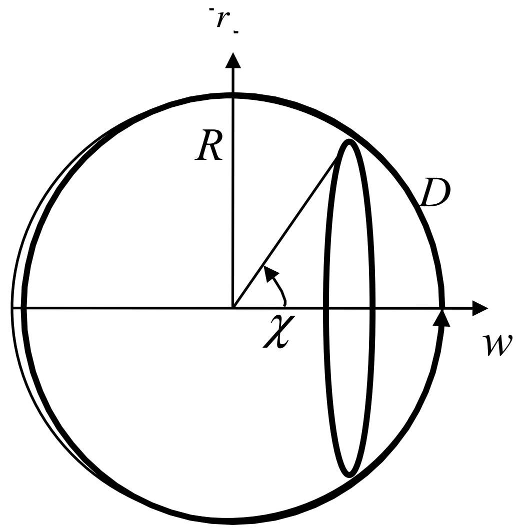
Photons arrive from our past light cone.

Robertson-Walker metric
uniformly curved, evolving spacetime

$$ds^2 = -c^2 dt^2 + R^2(t)\left(d\chi^2 + S_k^2(\chi)\, d\psi^2\right)$$

$$= -c^2 dt^2 + R^2(t)\left(\frac{du^2}{1-k\,u^2} + u^2\, d\psi^2\right)$$

$$= -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-k\,(r/R_0)^2} + r^2\, d\psi^2\right)$$



$$S_k(\chi) = \begin{cases} \sin\chi & (k=+1) \quad \text{closed} \\ \chi & (k=0) \quad \text{flat} \\ \sinh\chi & (k=-1) \quad \text{open} \end{cases}$$

$$d\psi^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$$

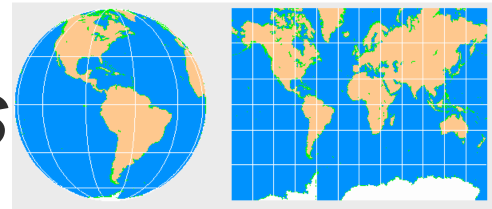
$$a(t) \equiv R(t)/R_0$$

$$R_0 \equiv R(t_0)$$

radial distance $= D(t) = R(t)\,\chi$

circumference $= 2\pi\, r(t)$ $\qquad r(t) = a(t)\, r = R(t)\, u = R(t)\, S_k(\chi)$

coordinate systems



Distance varies in time:

$$D(t)$$

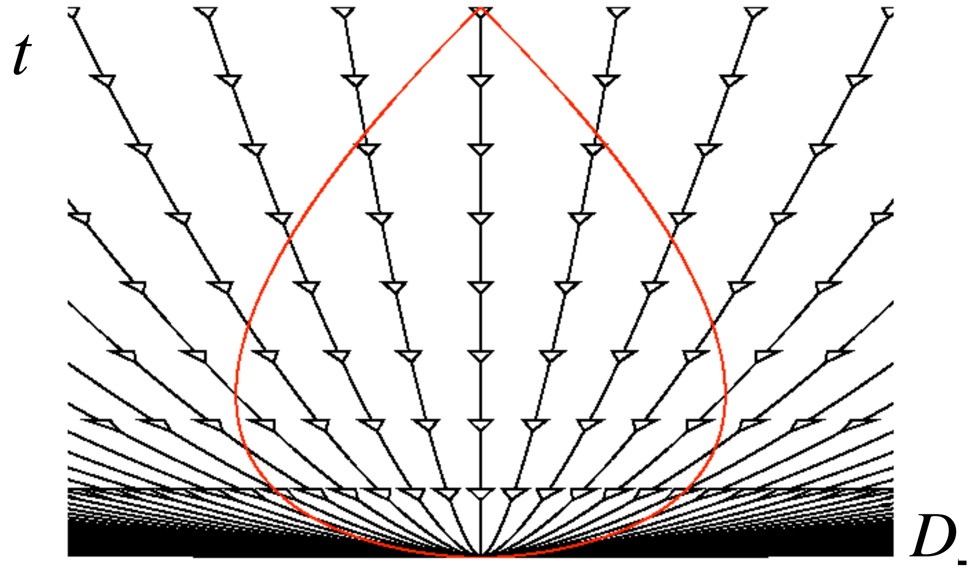
"Fiducial observers" (Fidos)

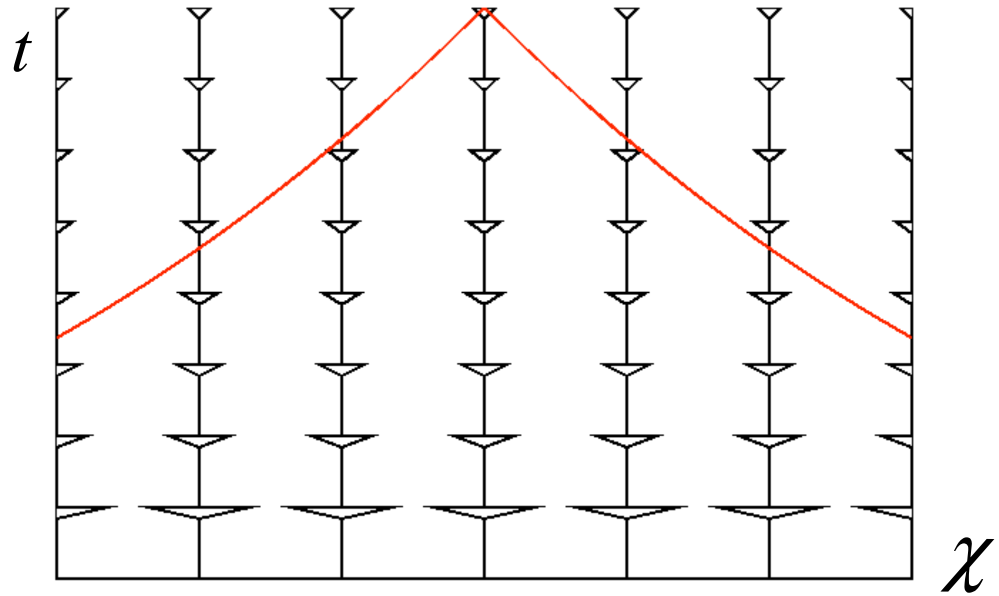
$$D(t) = R(t)\,\chi$$

"Co-moving" coordinates

$$\chi \quad \text{or} \quad D_0 \equiv R_0\,\chi$$

Labels the Fidos





Time and Distance vs Redshift

- We observe the **redshift :** $z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 \quad \lambda = \text{observed},$

$$\lambda_0 = \text{emitted (rest)}$$

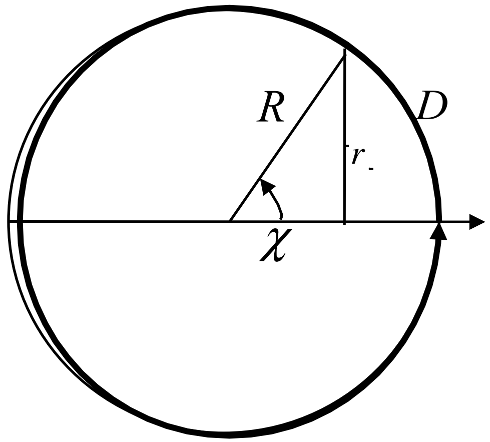
- Hence we know the **expansion factor:**

$$x \equiv 1 + z = \frac{\lambda}{\lambda_0} = \frac{\lambda(t_0)}{\lambda(t)} = \frac{R(t_0)}{R(t)} = \frac{R_0}{R(t)}$$

- When was the light emitted? $t(z) = ?$
- How far away was the source? $\chi(z) = ?$

$$D(t,\chi) = R(t)\,\chi \qquad D_A = r_0(\chi) / (1+z)$$

$$r(t,\chi) = R(t)\, S_k(\chi) \qquad D_L = r_0(\chi)\,(1+z)$$



- How do these depend on cosmological parameters?

$$H_0 \quad \Omega_M \quad \Omega_\Lambda$$

Tutorial: Time -- Redshift relation

$$x = 1 + z = \frac{R_0}{R}$$

$$\frac{dx}{dt} = -\frac{R_0}{R^2}\frac{dR}{dt}$$

$$= -\frac{R_0}{R}\frac{\dot{R}}{R}$$

$$= -x\ H(x)$$

$$\therefore\ dt = \frac{-dx}{x\ H(x)} = \frac{-dz}{(1+z)\ H(z)}$$

Memorise this derivation!

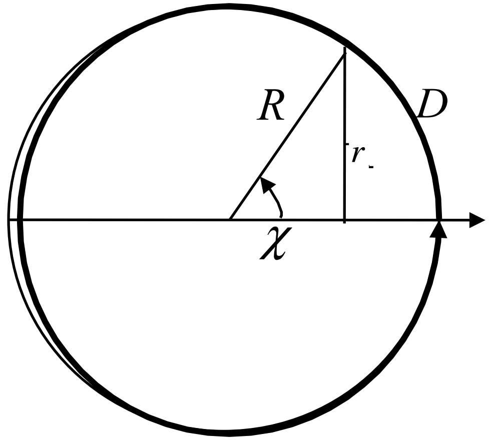
Hubble parameter : $H \equiv \frac{\dot{R}}{R}$

Time and Distance vs Redshift

$$\frac{d}{dt}\left(x = 1 + z = \frac{R_0}{R} \right) \rightarrow \quad dt = \frac{-dx}{x\,H(x)}$$

Look - back time :

$$t(z) = \int_t^{t_0} dt = \int_{1+z}^{1} \frac{-dx}{x\,H(x)} = \int_1^{1+z} \frac{dx}{x\,H(x)}$$



Age: $\quad t_0 = t(z \rightarrow \infty)$

Distance : $\quad D = R\,\chi \qquad r = R\,S_k(\chi)$

$$\chi(z) = \int d\chi = \int_t^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{R_0} \int_1^{1+z} \frac{R_0}{R(t)} \frac{dx}{x\,H(x)} = \frac{c}{R_0} \int_1^{1+z} \frac{dx}{H(x)}$$

Horizon : $\quad \chi_H = \chi(z \rightarrow \infty)$

Need to know *R(t)*, or R_0 and *H(x)*.

Einstein's General Relativity

- 1. *Spacetime geometry tells matter how to move*
  - gravity = effect of curved spacetime
  - free particles follow geodesic trajectories
    - $ds^2 < 0$ v < c time-like massive particles
    - $ds^2 = 0$ v = c null massless particles (photons)
    - $ds^2 > 0$ v > c space-like tachyons (not observed)

- 2. *Matter (+energy) tells spacetime how to curve*
  - Einstein field equations
    - nonlinear
    - second-order derivatives of metric

with respect to space/time coordinates

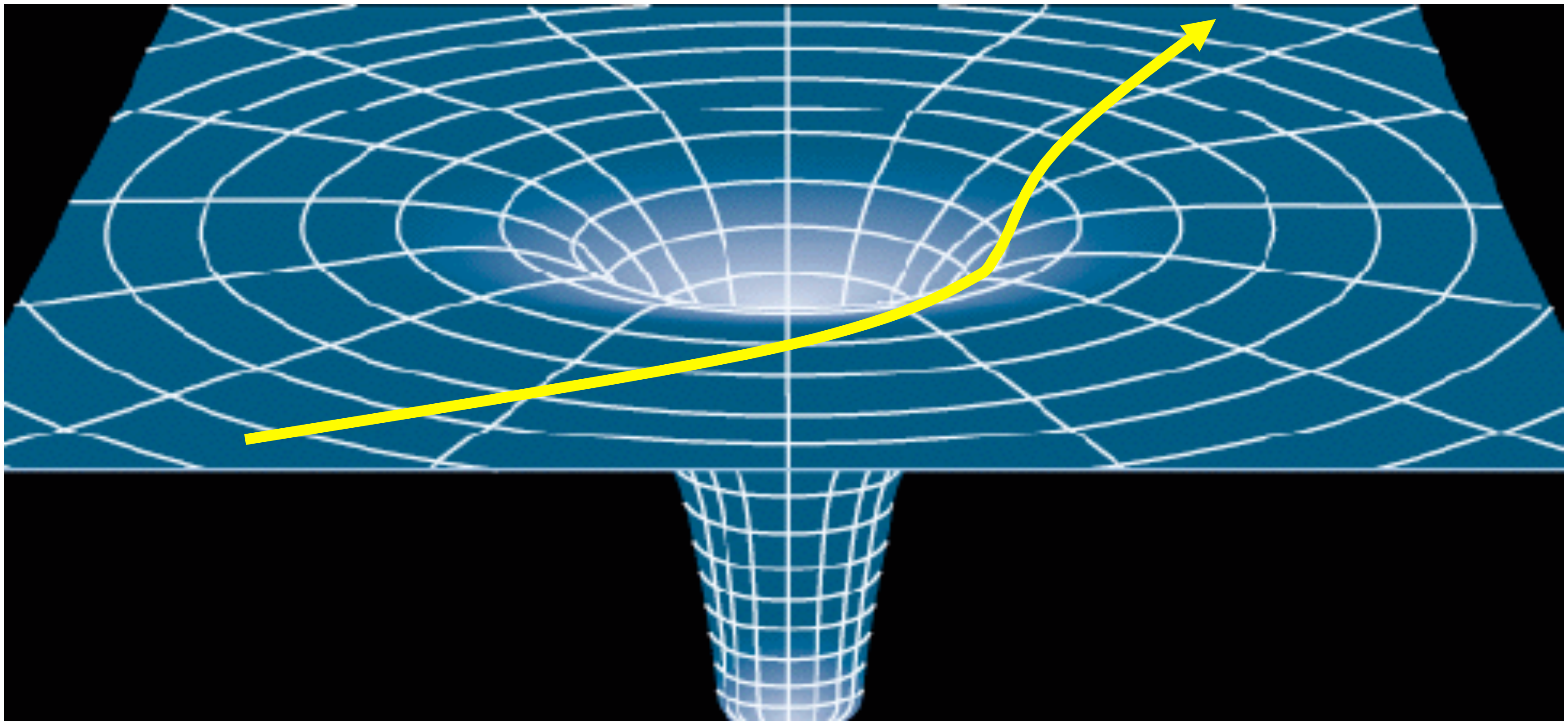
Geodesics

Gravity = curvature of space-time by matter/energy.

Freely-falling bodies follow **geodesic trajectories.**

Shortest possible path in curved space-time.

Local curvature replaces forces acting at distance.



Einstein Field Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R^{\alpha}{}_{\alpha}\, g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} - \Lambda\, g_{\mu\nu}$$

$g_{\mu\nu} =$ spacetime metric ($ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$)

$G_{\mu\nu} =$ Einstein tensor (spacetime curvature)

$R_{\mu\nu} =$ Ricci curvature tensor

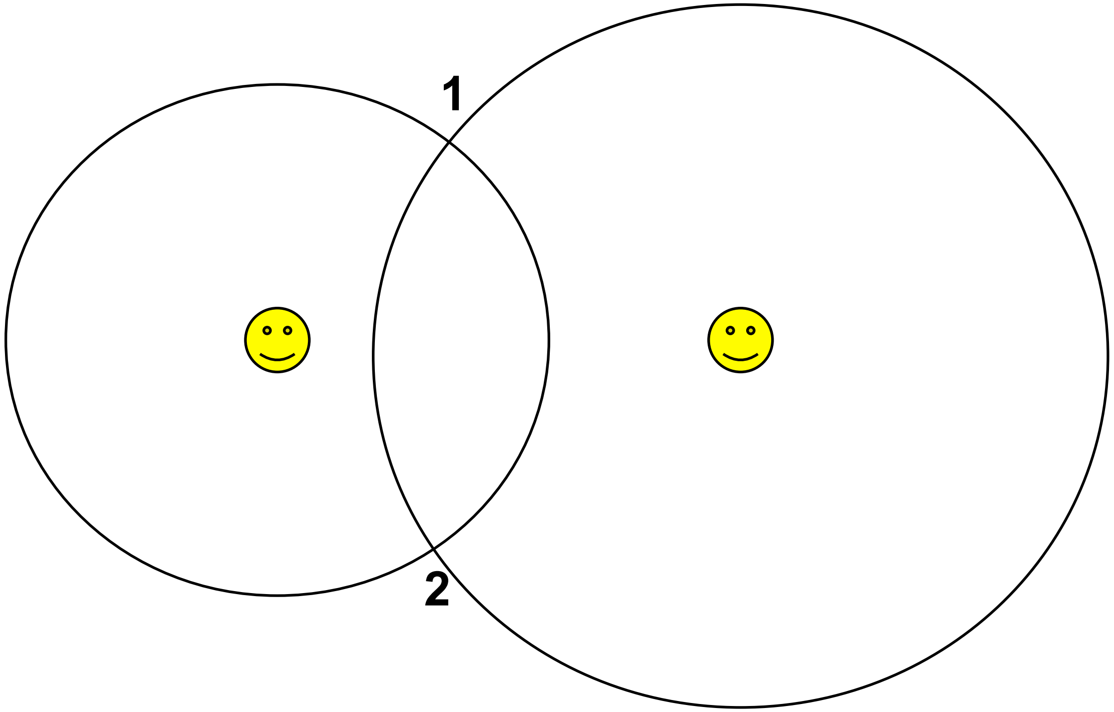
$R^{\alpha}{}_{\alpha} =$ Ricci curvature scalar

$G =$ Netwon's gravitational constant

$T_{\mu\nu} =$ energy - momentum tensor

$\Lambda =$ cosmological constant

Cosmological Principle (assumed) + Isotropy (observed) => Homogeneity



$$\rho_1 = \rho_2 \quad \text{otherise not isotropic for equidistant fidos}$$

Homogeneous perfect fluid

density ρ pressure p

Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^2}\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} - \Lambda \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

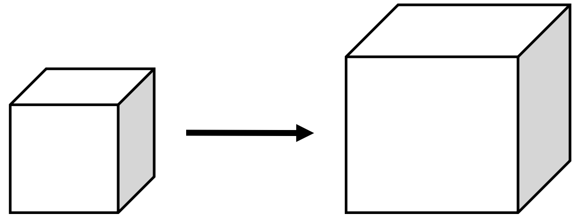
---> Friedmann equations :

$$\dot{R}^2 = \left(\frac{8\pi G \rho + \Lambda}{3} \right) R^2 - k\, c^2$$ **energy**

$$\ddot{R} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3p\right) R + \frac{\Lambda}{3} R$$ **momentum**

Note: energy density and pressure decelerate, Λ accelerates.

Tutorial: Local Conservation of Energy



$$d[energy] = work$$

$$d[\rho\, c^2 R^3] = -p\, d[R^3]$$

$$\dot{\rho}\, c^2 R^3 + \rho\, c^2\, (3\, R^2 \dot{R}) = -p\, (3\, R^2 \dot{R})$$

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} \qquad p = p(\rho) = \text{equation of state}$$

$$\text{Friedmann 1}: \dot{R}^2 = \frac{8\pi\, G}{3}\rho\, R^2 + \frac{\Lambda}{3} R^2 - k\, c^2$$

$$(2\, \dot{R}\, \ddot{R}) = \frac{8\pi\, G}{3}(\dot{\rho}\, R^2 + 2\, R\, \dot{R}\, \rho\,) + \frac{\Lambda}{3}(2\, R\, \dot{R})$$

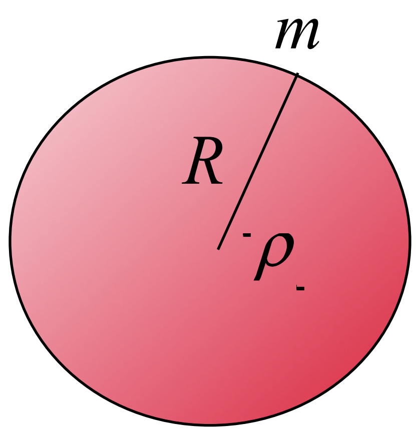
$$\ddot{R} = \frac{8\pi\, G}{3}\left(\frac{\dot{\rho}\, R^2}{2\, \dot{R}} + R\, \rho\right) + \frac{\Lambda}{3} R$$

$$\ddot{R} = -\frac{4\pi\, G}{3}\left(\rho + \frac{3\, p}{c^2}\right) R + \frac{\Lambda}{3} R = \text{Friedmann 2}$$

Newtonian Analogy

$$E = \frac{m}{2}\dot{R}^2 - \frac{G\,M\,m}{R} \qquad M = \frac{4\pi}{3} R^3 \rho$$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho\, R^2 + \frac{2E}{m}$$

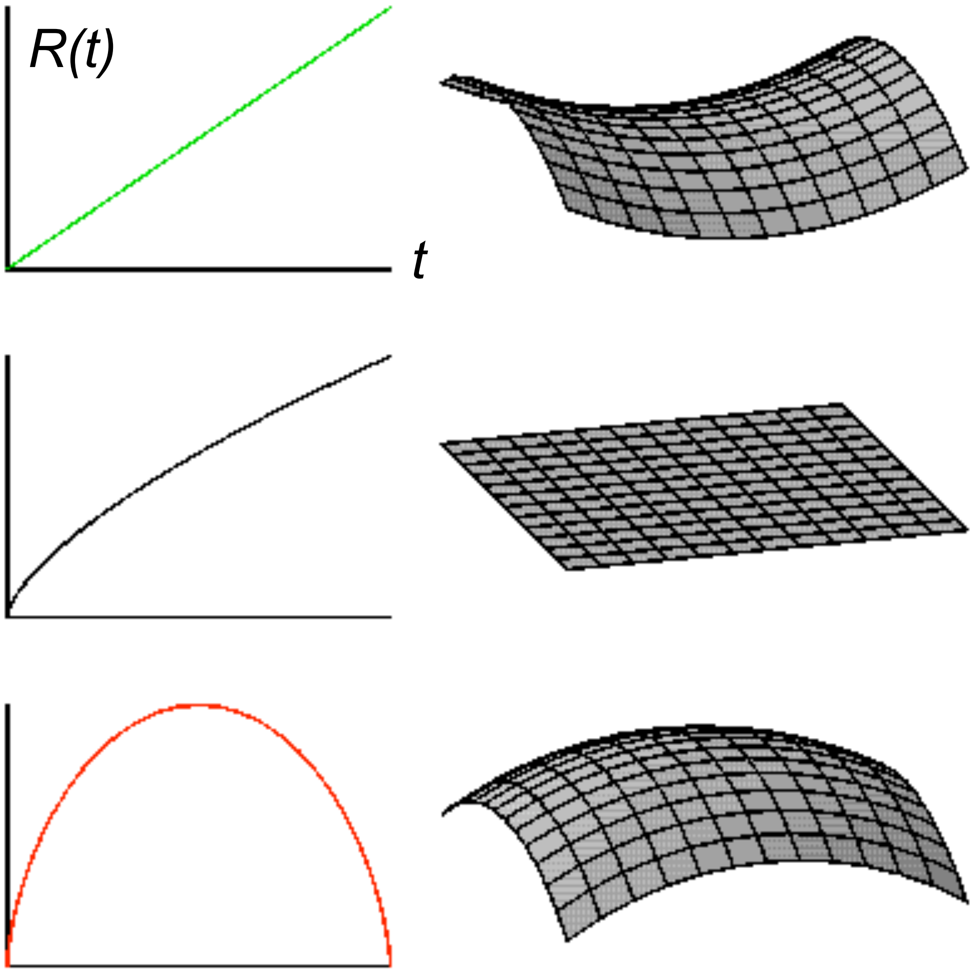


Friedmann equation:

$$\dot{R}^2 = \left(\frac{8\pi G\,\rho + \Lambda}{3}\right) R^2 - k\,c^2$$

same equation if $\rho \rightarrow \rho + \dfrac{\Lambda}{8\pi\,\mathrm{G}}$, $\dfrac{2E}{m} \rightarrow -k\,c^2$

Density - Evolution - Geometry



$\rho < \rho_c$ — **Open**, *k = -1*

$\rho = \rho_c$ — **Flat**, *k = 0*

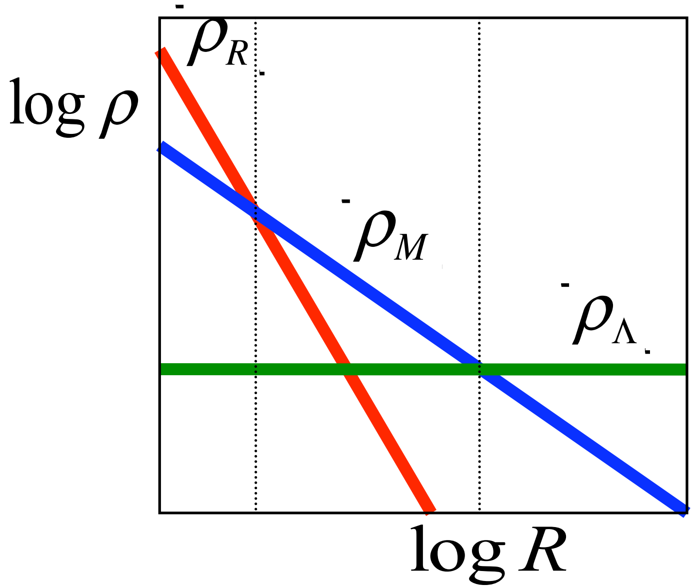
$\rho > \rho_c$ — **Closed**, *k = +1*

radiation -> matter -> vacuum

radiation : $\rho_R \propto R^{-4}$

matter : $\rho_M \propto R^{-3}$

vacuum: $\rho_\Lambda = \text{const}$

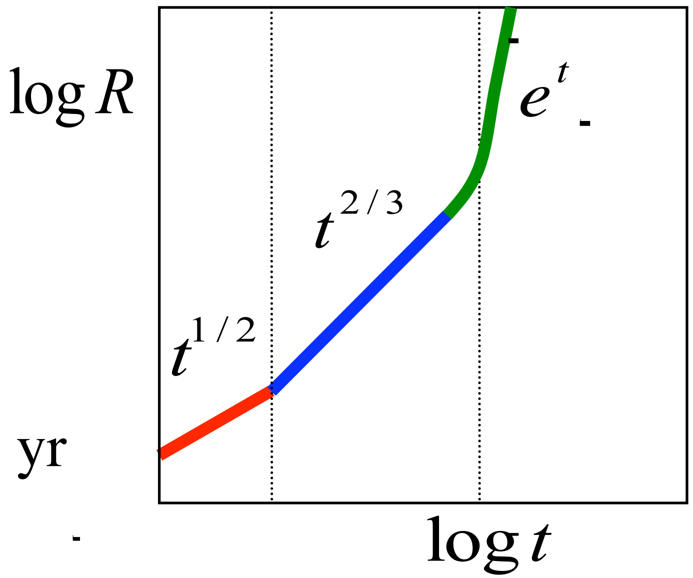


$$\rho(x) = \rho_R\, x^4 + \rho_M\, x^3 + \rho_\Lambda$$

$$x = 1 + z = \frac{R_0}{R} = \frac{1}{a}$$

$$\rho_R = \rho_M \quad \text{at} \quad x \sim \frac{\rho_M}{\rho_R} \sim 10^4 \qquad t \sim 10^4 \text{ yr}$$

$$\rho_M = \rho_\Lambda \quad \text{at} \quad x \sim \left(\frac{\rho_\Lambda}{\rho_M}\right)^{1/3} \quad z \sim 0.3 \qquad t \sim 10^{10} \text{ yr}$$

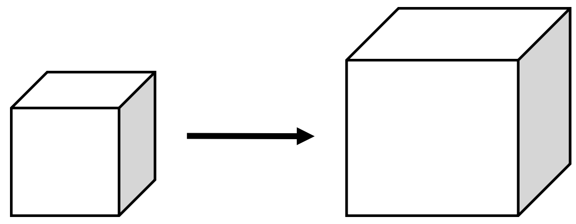


Equation of State ----- w

Equation of state :

$$\rho \propto R^{-n} \qquad n = 3(1+w)$$

$$w \equiv \frac{\text{pressure}}{\text{energy density}} = \frac{p}{\rho\, c^2} = \frac{n}{3} - 1$$



Radiation : $(n = 4, w = 1/3)$

$$p_R = \frac{1}{3} \rho_R \; c^2$$

Matter : $(n = 3, w = 0)$

$$p_M \sim \rho_M \; c_S^{\;2} << \rho_M \; c^2$$

Vacuum : $(n = 0, w = -1)$

$$p_\Lambda = -\rho_\Lambda \; c^2$$

Negative Pressure ! ?

$$d[energy] = work$$

$$d[\rho\, c^2 R^3] = -p\, d[R^3]$$

$$\rho\, c^2 \,(\, 3\, R^2 dR\,) + R^3\; c^2 d\rho = -p\,(\, 3\, R^2 dR\,)$$

$$1 + \frac{R\, d\rho}{3\, \rho\, dR} = -\frac{p}{\rho c^2} \equiv -w$$

$$w = -\frac{1}{3} \frac{d[\ln \rho]}{d[\ln R]} - 1$$

$$w = \frac{n}{3} - 1$$

Density Parameters

critical density :

$$\rho_c \equiv \frac{3\,H_0^{\ 2}}{8\pi\,G}$$

density parameters (today) :

$$\Omega_R \equiv \frac{\rho_R}{\rho_c} \quad \Omega_M \equiv \frac{\rho_M}{\rho_c} \quad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\,H_0^{\ 2}}$$

total density parameter today :

$$\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda$$

density at a past/future epoch in units of today's critical density :

$$\Omega \equiv \frac{\rho}{\rho_c} = \sum_w \Omega_w x^{3(1+w)} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda \qquad x \equiv 1 + z = R_0 / R$$

in units of critical density at the past/future epoch :

$$\Omega(x) \equiv \frac{8\pi G\rho}{3H^2} = \frac{H_0^{\ 2}}{H^2}\sum_w \Omega_w x^{3(1+w)} = \frac{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda}{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2}$$

Note: radiation dominates at high z, can be neglected at lower z.

Hubble Parameter Evolution -- H(z)

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k c^2}{R^2}$$

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda - \frac{k c^2}{H_0^{\,2} R_0^{\,2}} x^2$$

$$\text{evaluate at } x = 1 \quad \rightarrow \quad 1 = \Omega_0 - \frac{k\, c^2}{H_0^{\,2} R_0^{\,2}}$$

$$x = 1 + z = R_0 / R$$

$$\rho_c = \frac{3\, H_0^{\,2}}{8\pi G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}, \quad \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\, H_0^{\,2}}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$

Dimensionless Friedmann Equation:

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1 - \Omega_0)\, x^2$$

Curvature Radius today:

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}} \quad \rightarrow \begin{cases} k = +1 & \Omega_0 > 1 \\ k = \ 0 & \Omega_0 = 1 \\ k = -1 & \Omega_0 < 1 \end{cases}$$

Density determines Geometry

Possible Universes

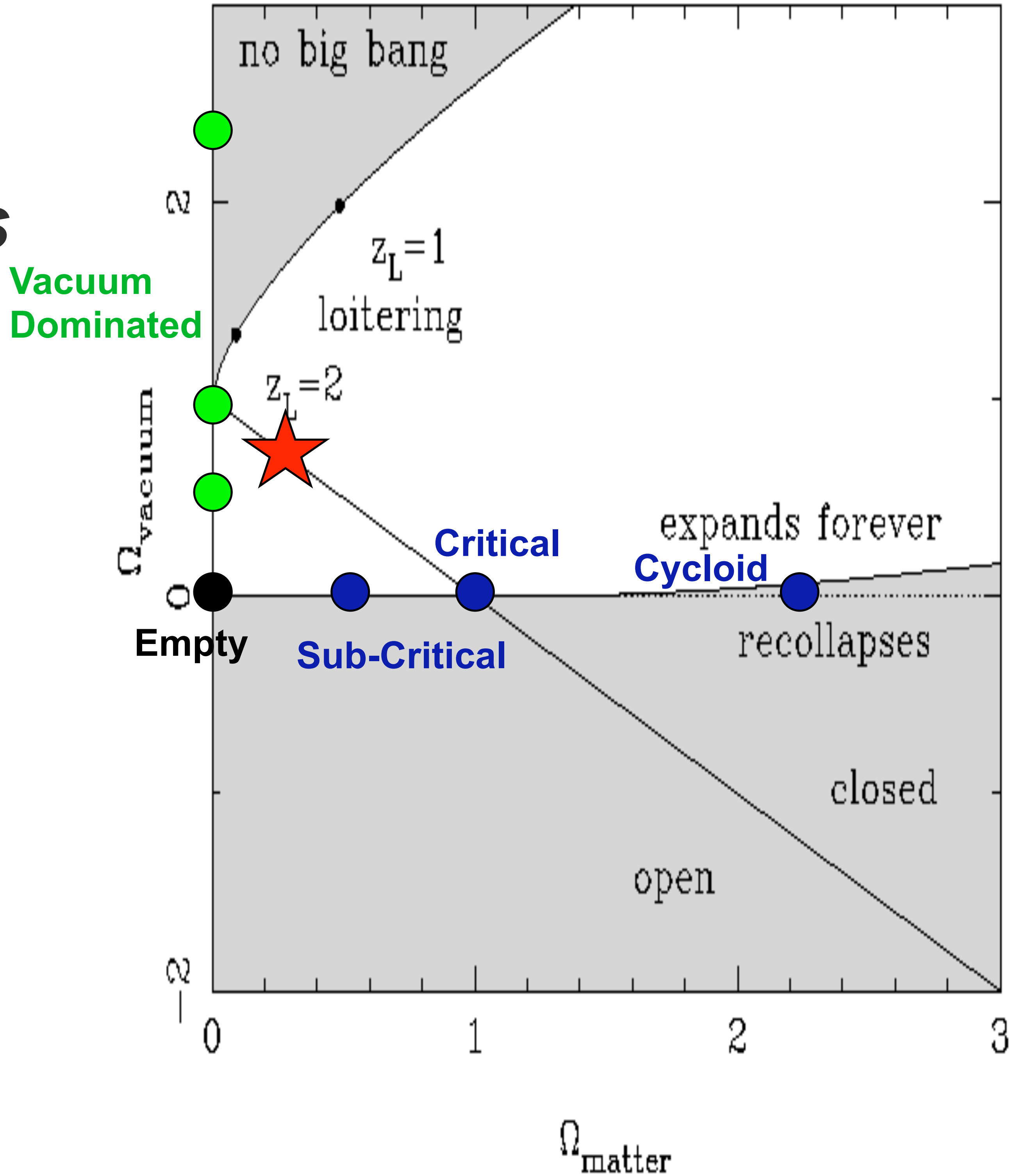
$$H_0 \approx 70 \frac{\text{km/s}}{\text{Mpc}}$$

$$\Omega_M \sim 0.3$$

$$\Omega_\Lambda \sim 0.7$$

$$\Omega_R \sim 8\times10^{-5}$$

$$\Omega = 1.0$$



Hubble Parameter Evolution -- H(z)

$$H^2 \equiv \left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k\,c^2}{R^2}$$

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda - \frac{k\,c^2}{H_0^{\,2} R_0^{\,2}} x^2$$

$$\text{evaluate at } x = 1 \quad \rightarrow \quad 1 = \Omega_0 - \frac{k\;c^2}{H_0^{\,2} R_0^{\,2}}$$

$$x = 1 + z = R_0 / R$$

$$\rho_c = \frac{3\,H_0^{\,2}}{8\pi\,G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}, \quad \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\,H_0^{\,2}}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$

Dimensionless Friedmann Equation:

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1 - \Omega_0)\, x^2$$

Curvature Radius today:

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}} \quad \rightarrow \begin{cases} k = +1 & \Omega_0 > 1 \\ k = \ 0 & \Omega_0 = 1 \\ k = -1 & \Omega_0 < 1 \end{cases}$$

Density determines Geometry

Possible Universes

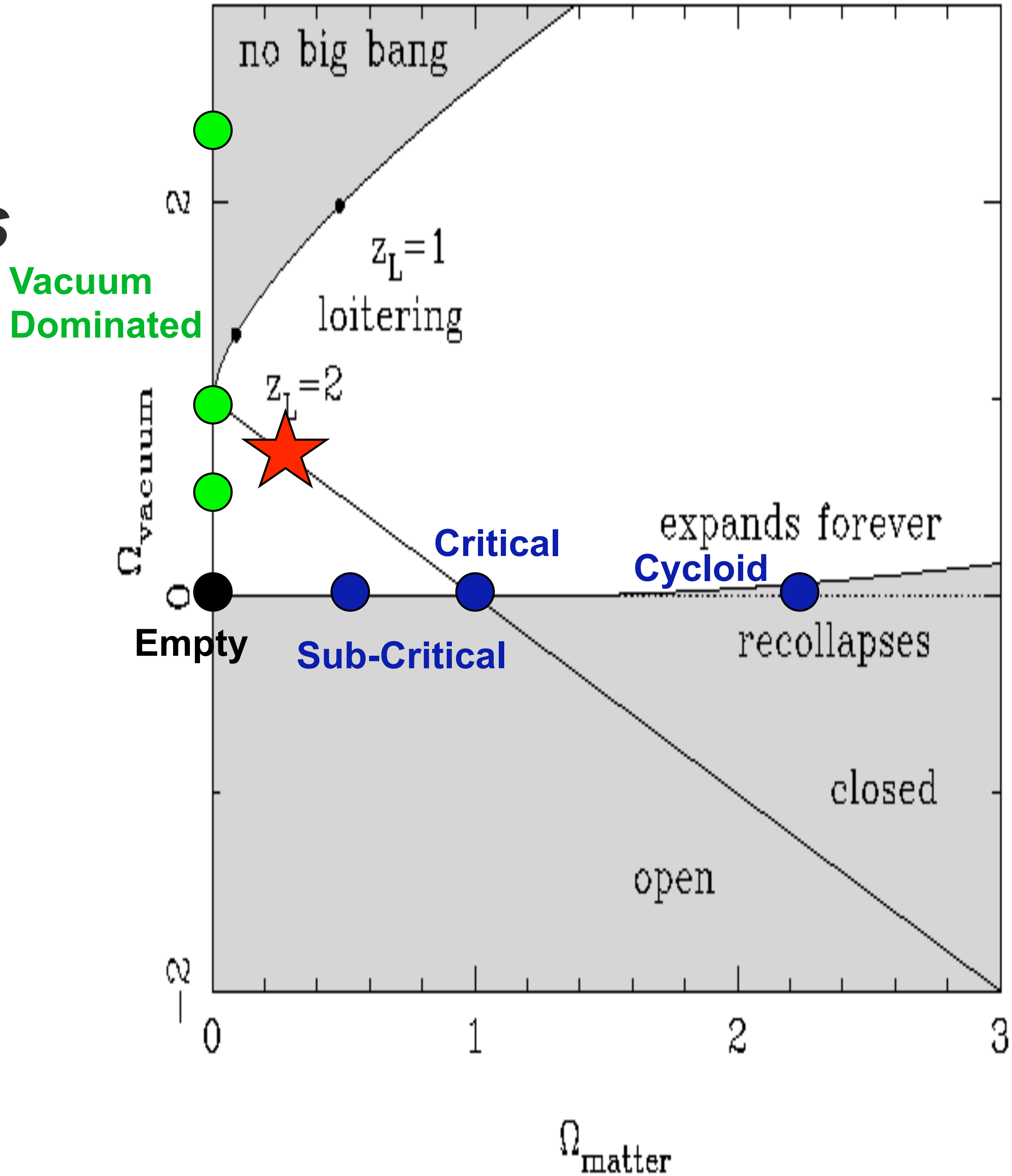
$$H_0 \approx 70 \frac{\text{km/s}}{\text{Mpc}}$$

$$\Omega_M \sim 0.3$$

$$\Omega_\Lambda \sim 0.7$$

$$\Omega_R \sim 8\times10^{-5}$$

$$\Omega = 1.0$$



Empty Universe (Milne)

$$\dot{R}^2 = \left(\frac{8\pi G \rho + \Lambda}{3} \right) R^2 - k\, c^2$$

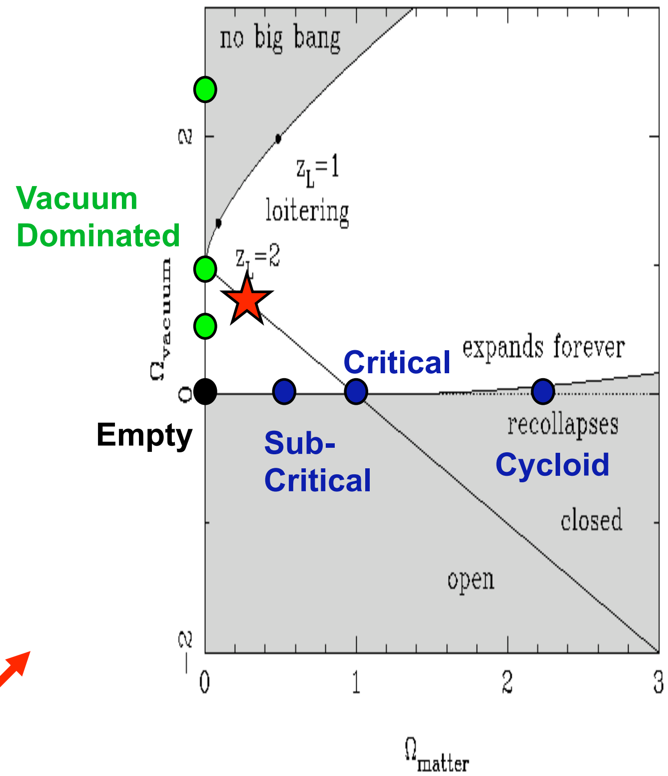
Set $\rho = 0, \quad \Lambda = 0.$ Then $\dot{R}^2 = -k\, c^2$

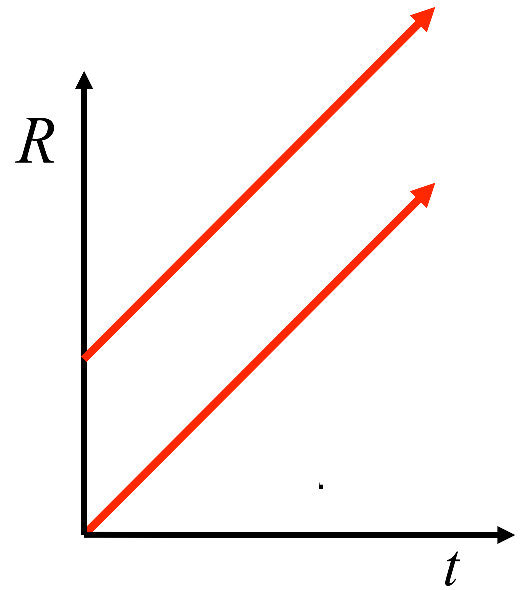
$\rightarrow \quad k = -1$ (negative curvature)

$\dot{R} = c, \quad R = c\, t$

$$H \equiv \frac{\dot{R}}{R} = \frac{1}{t}$$

$$\text{age:} \quad t_0 = \frac{R_0}{c} = \frac{1}{H_0}$$





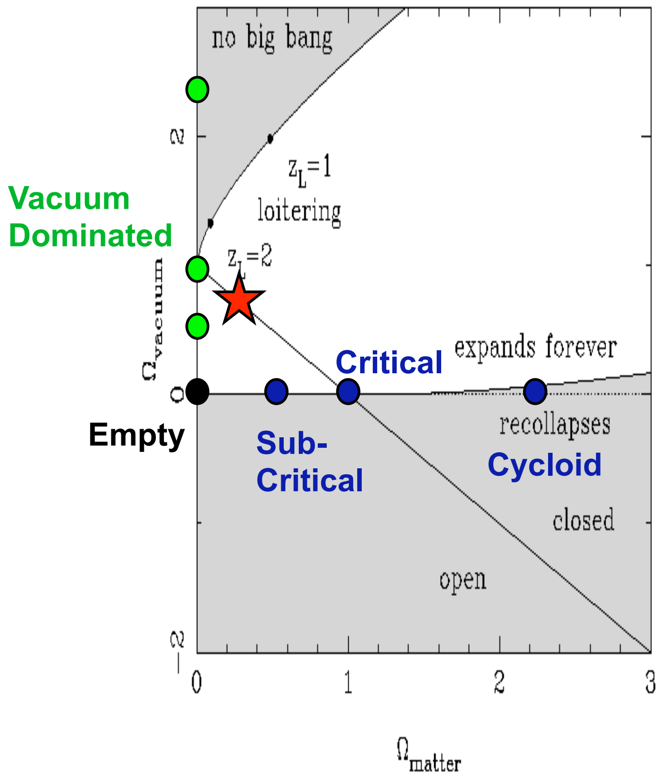
Negative curvature drives rapid expansion/flattening

Tutorial: Eternal Static Universe

- Einstein introduced Λ to enable an eternal static universe.

$$\dot{R}^2 = \left(\frac{8\pi G \rho + \Lambda}{3} \right) R^2 - k c^2$$

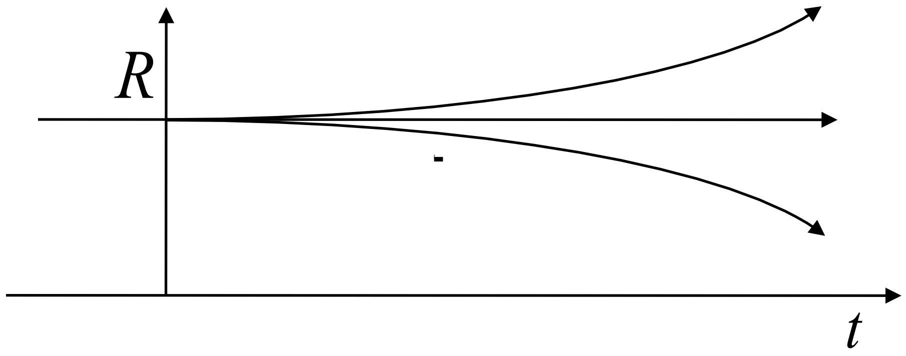
$$\dot{R} = 0 \quad \rightarrow \quad \Lambda = \frac{3 k c^2}{R^2} - 8\pi G \rho$$



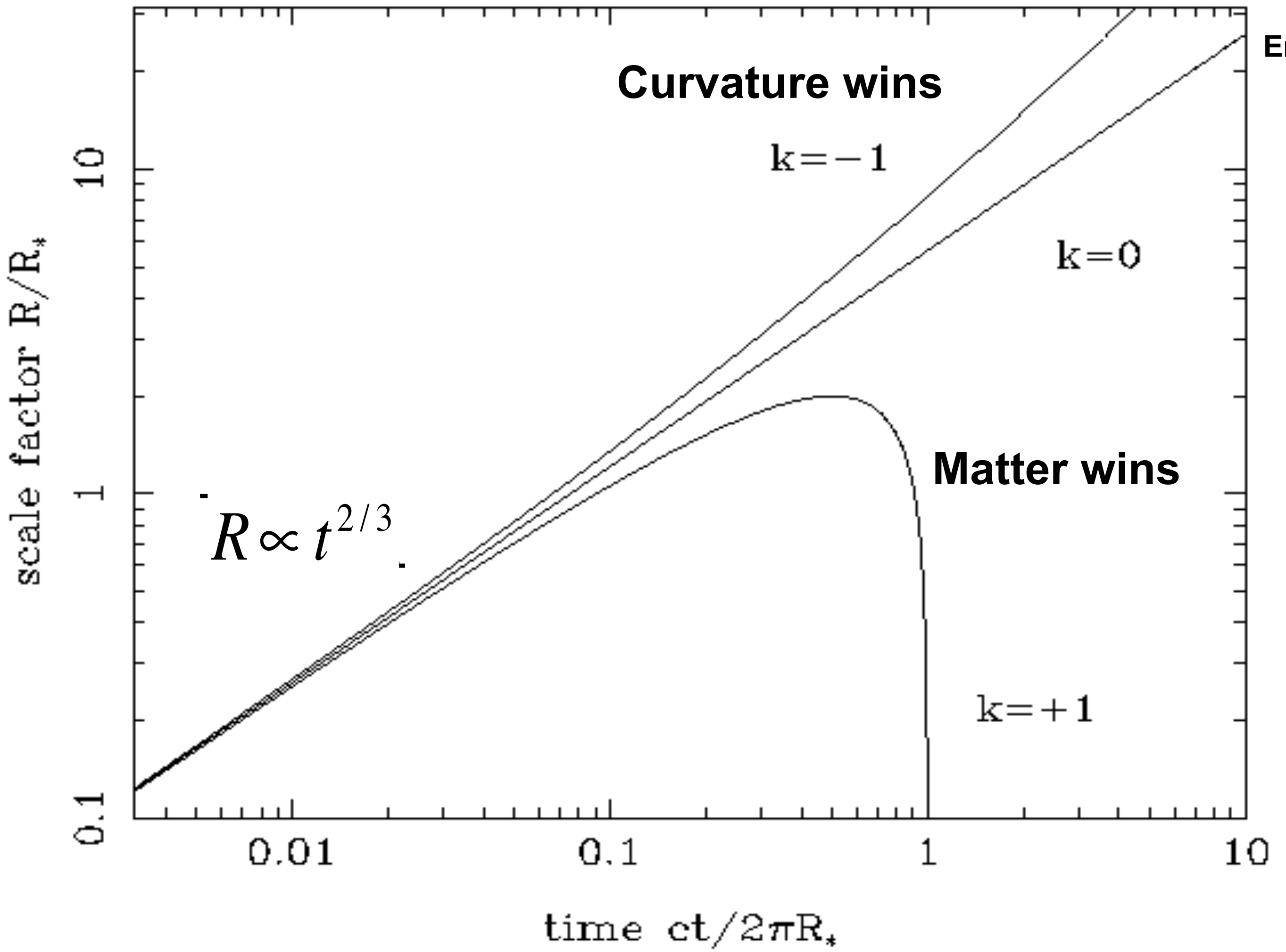
Einstein's biggest blunder. (Or, maybe not.)

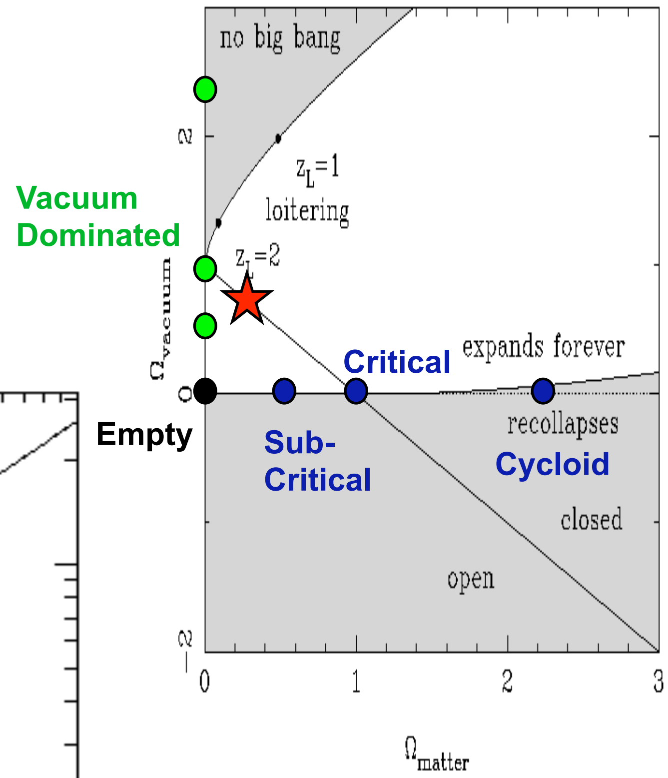
Static models unstable.

Fine tuning.



Tutorial: Matter-dominated Universes





All evolve as
$R \sim t^{2/3}$
at early times.

Tutorial: Critical Universe (Einstein - de Sitter)

$$\Omega_M \equiv \frac{\rho_M}{\rho_c} = 1$$

$$\Omega_R = \Omega_\Lambda = 0 \quad \rightarrow \quad k = 0 \;\; (\text{ flat })$$

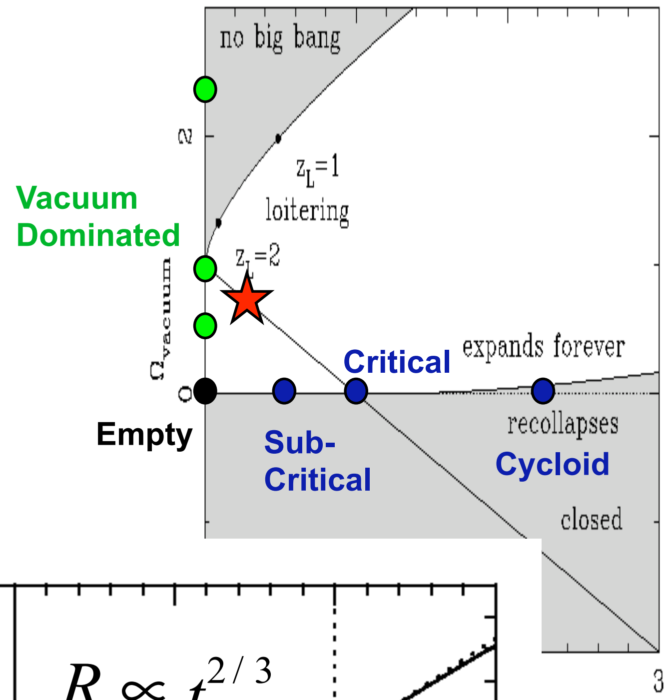
$$\rho = \frac{3\,H_0^{\;2}}{8\pi\,G}\left(\frac{R_0}{R}\right)^3$$

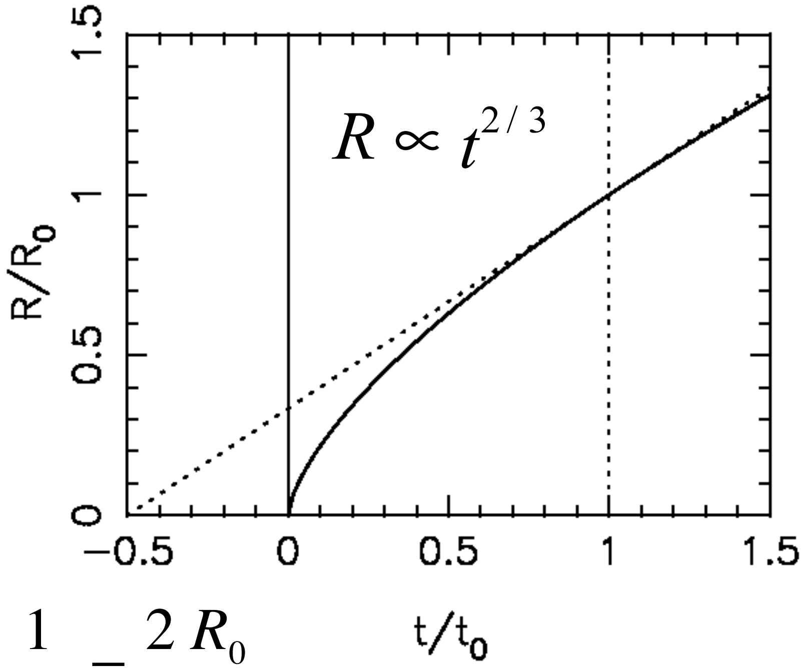
$$\dot{R}^2 = \frac{8\pi\,G}{3}\rho\,R^2 = \frac{H_0^{\;2}\;R_0^{\;3}}{R}$$

$$dR\;R^{1/2} = H_0\;R_0^{\;3/2}\;dt$$

$$\frac{2}{3}R^{3/2} = H_0\;R_0^{\;3/2}\;t$$

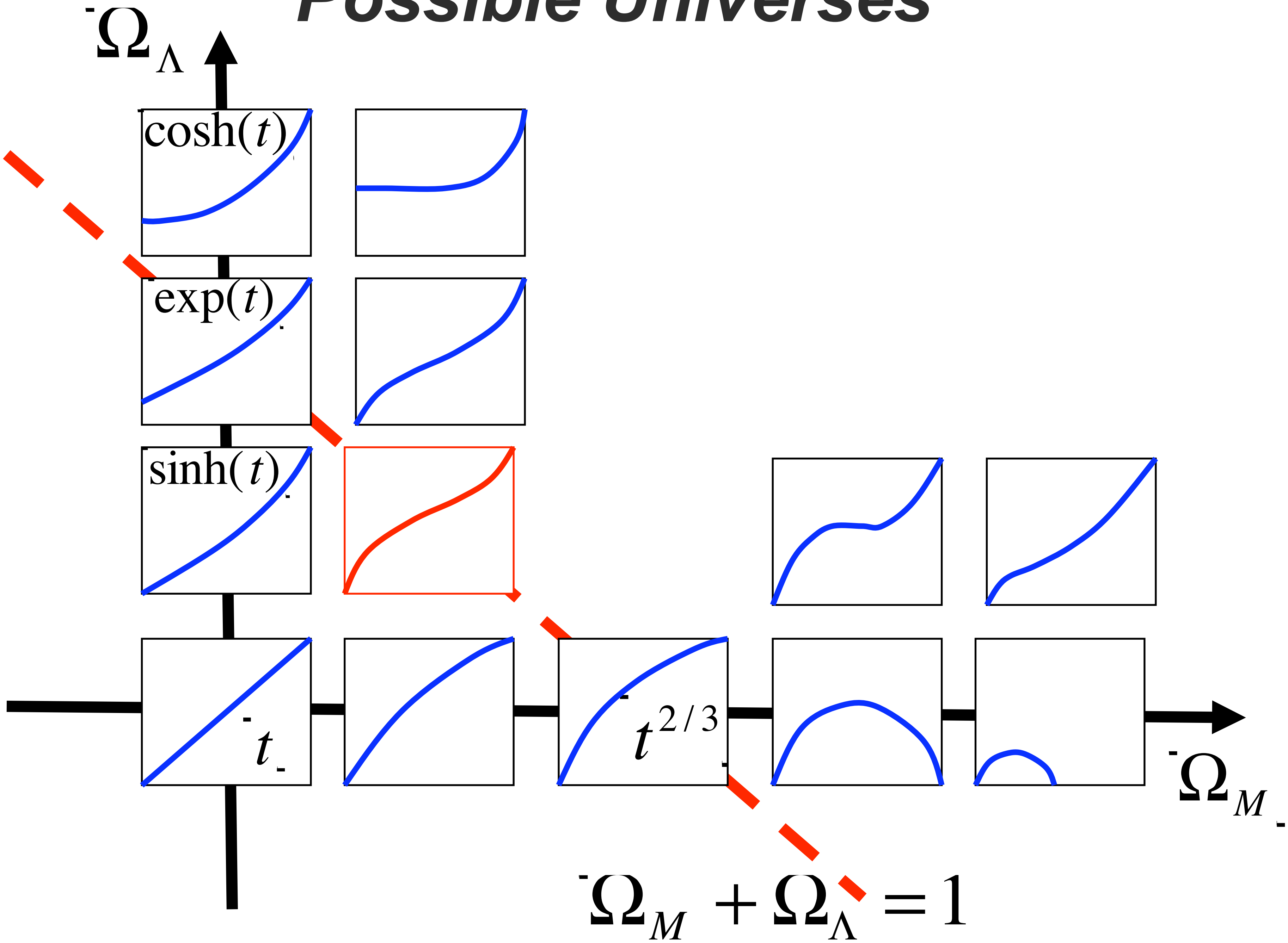
$$\frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{2/3}, \qquad \text{age :} \quad t_0 = \frac{2}{3}\frac{1}{H_0} = \frac{2}{3}\frac{R_0}{c}$$





Matter decelerates expansion.

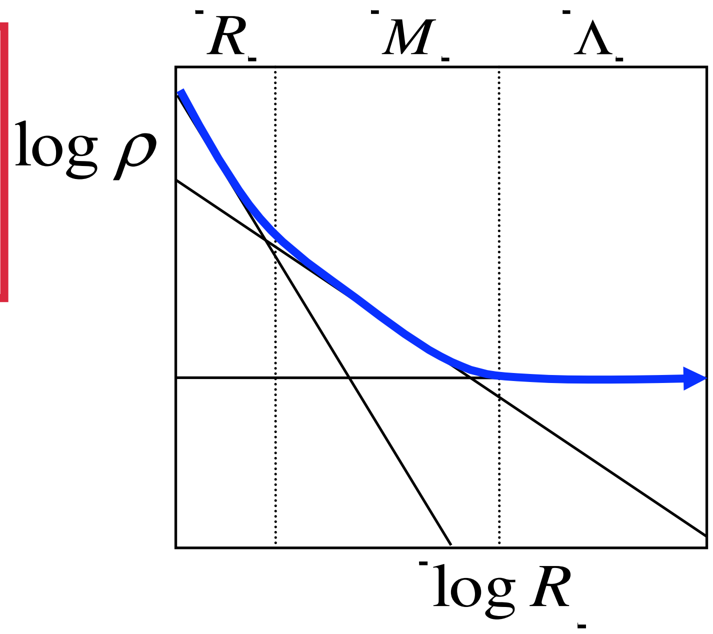
Possible Universes



Tutorial: Rad. => Matter =>Vacu.

$$\Omega(z) = \Omega_R\, x^4 + \Omega_M\, x^3 + \Omega_\Lambda = \sum_i \Omega_i\, x^{3(w_i+1)}$$

$$x \equiv 1 + z = R_0 / R \equiv a^{-1} \qquad w \equiv p / \rho c^2$$

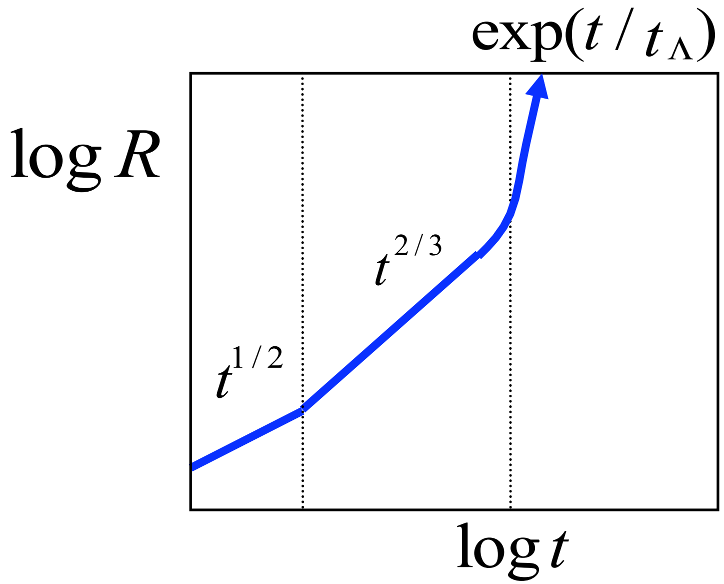


$$\Omega_R x^4 = \Omega_M x^3 \quad \rightarrow \quad x = \Omega_M / \Omega_R$$

$$\rightarrow \quad z = \left(\frac{\Omega_M}{\Omega_R} \right) - 1 = \frac{0.3}{8.4 \times 10^{-5}} \approx 3600$$

$$\Omega_M x^3 = \Omega_\Lambda \rightarrow \quad x^3 = \Omega_\Lambda / \Omega_M$$

$$\rightarrow \quad z = \left(\frac{\Omega_\Lambda}{\Omega_M} \right)^{\frac{1}{3}} - 1 = \left(\frac{0.7}{0.3} \right)^{\frac{1}{3}} - 1 \approx 0.33$$



Tutorial: calculate H(z) given densities

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k c^2}{R^2}$$

$$\left(\frac{H(x)}{H_0}\right)^2 = \Omega_R \, x^4 + \Omega_M \, x^3 + \Omega_\Lambda - \frac{k c^2}{H_0^{\,2} R_0^{\,2}} x^2$$

$$\text{evaluate at } x = 1 \quad \rightarrow \quad 1 = \Omega_0 - \frac{k c^2}{H_0^{\,2} R_0^{\,2}}$$

$$\left(\frac{H(x)}{H_0}\right)^2 = \Omega(x) + (1 - \Omega_0)\, x^2$$

$$H(x) = H_0 \sqrt{\Omega_R \, x^4 + \Omega_M \, x^3 + \Omega_\Lambda + (1 - \Omega_0)\, x^2}$$

$$x = 1 + z = R_0 / R$$

$$\rho_c = \frac{3 H_0^{\,2}}{8\pi G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}, \quad \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3 H_0^{\,2}}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$

Tutorial: What observations justify the "Concordance" Parameters?

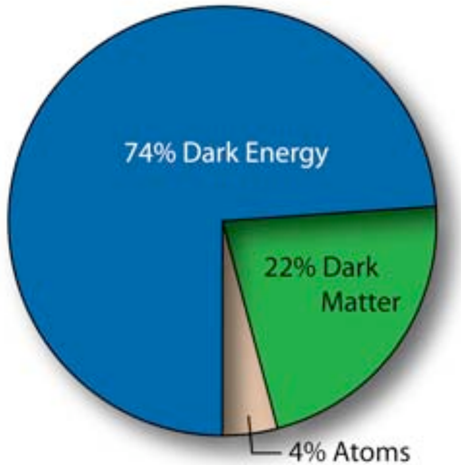
$$H_0 \equiv 100\, h\, \frac{\text{km/s}}{\text{Mpc}} \approx 70 \frac{\text{km/s}}{\text{Mpc}} \qquad h \approx 0.7$$

$$\Omega_R \approx 4.2 \times 10^{-5} h^{-2} \approx 8.4 \times 10^{-5} \; (CMB\ \ photons + neutrinos)$$

$$\Omega_B \sim 0.02\, h^{-2} \sim 0.04 \quad (baryons)$$

$$\Omega_M \sim 0.3 \quad (Dark\ Matter)$$

$$\Omega_\Lambda \sim 0.7 \quad (Dark\ Energy)$$



$$\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda = 1.0 \quad \rightarrow \quad Flat\ Geometry$$

"Concordance" Model

Three main constraints:

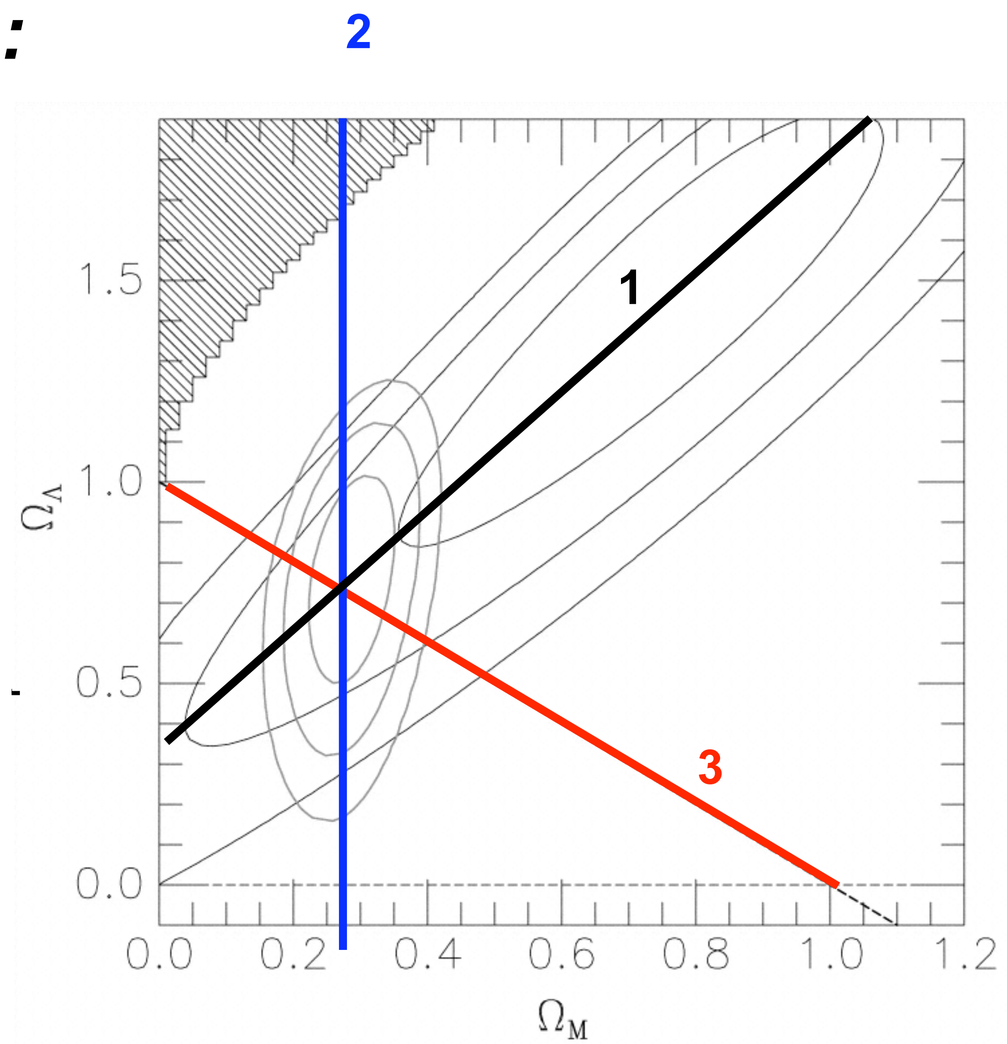
1. Supernova Hubble Diagram
2. Galaxy Counts + M/L ratios

   $\Omega_M \sim 0.3$

3. Flat Geometry

   (inflation, CMB fluctuations)

   $\Omega_0 = \Omega_M + \Omega_\Lambda = 1$



concordance model

$$H_0 \approx 72 \quad \Omega_M \approx 0.3 \quad \Omega_\Lambda \approx 0.7$$

Beyond H_0

- Globular cluster ages: $t < t_0$ --> acceleration
- Radio jet lengths: $D_A(z)$ --> deceleration
- Hi-Redshift Supernovae: $D_L(z)$ --> acceleration

- Dark Matter estimates ---> $\Omega_M \sim 0.3$

- Inflation ---> Flat Geometry $\Omega_0 \approx 1.0$
- CMB power spectra

- "Concordance Model" $\Omega_M \sim 0.3 \quad \Omega_\Lambda \sim 0.7$

$$\Omega_0 = \Omega_M + \Omega_\Lambda \approx 1.0$$

Deceleration parameter

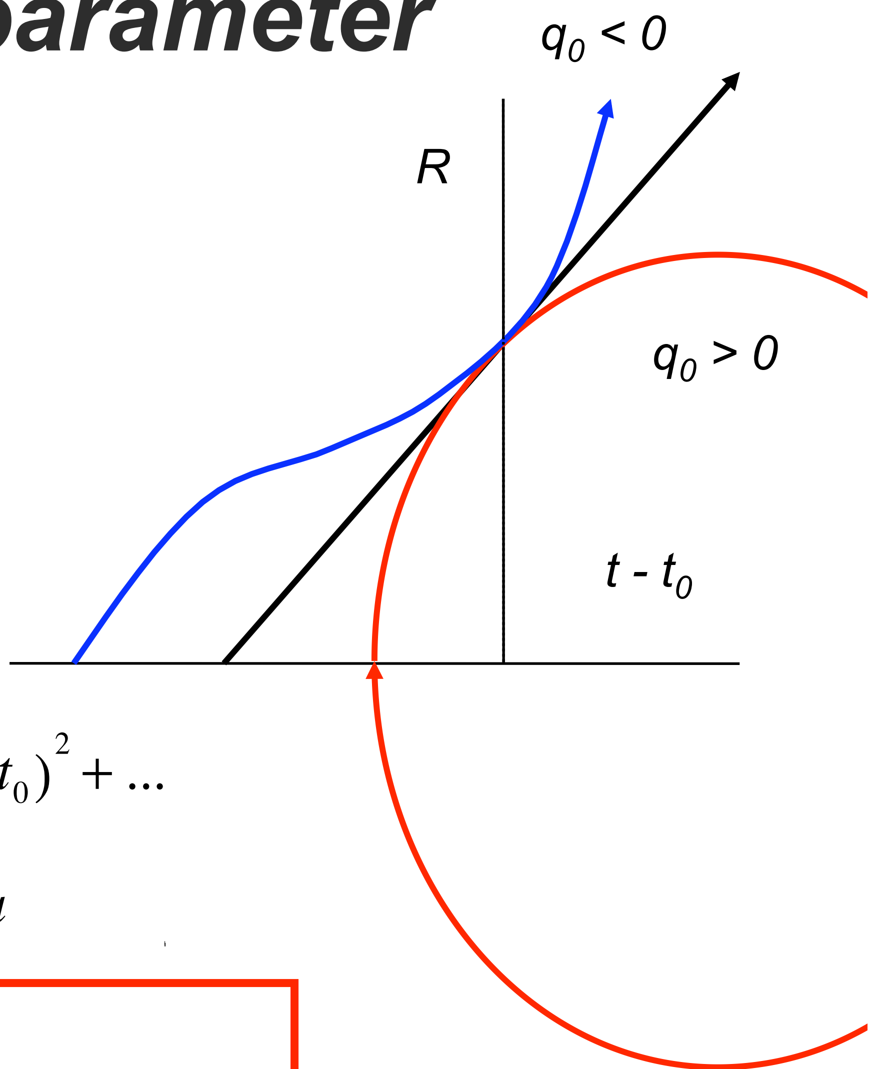
Dimensionless measure of the *deceleration* of the Universe

$$q \equiv -\frac{\ddot{R}\, R}{\dot{R}^2} = -\frac{\ddot{R}}{R\, H^2}$$

$$q_0 \equiv -\left(\frac{\ddot{R}\, R}{\dot{R}^2}\right)_0 = -\left(\frac{\ddot{R}}{R\, H^2}\right)_0$$

$$a(t) \equiv \frac{R(t)}{R_0} = 1 + H_0\,(t - t_0) - \frac{q_0}{2} H_0^{\,2}\,(t - t_0)^2 + \ldots$$

$$\dot{a} = H\, a \qquad \ddot{a} = -q\, H^2 a$$



$q_0 > 0$ => **deceleration**

$q_0 = 0$ => *coasting at constant velocity*

$q_0 < 0$ => **acceleration**

Deceleration parameter

$$q \equiv -\frac{\ddot{R}\,R}{\dot{R}^2} = -\frac{\ddot{R}}{R\,H^2} \qquad q_0 \equiv -\left(\frac{\ddot{R}\,R}{\dot{R}^2}\right)_0 = -\left(\frac{\ddot{R}}{R\,H^2}\right)_0$$

Friedmann momentum equation :

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda}{3}R$$

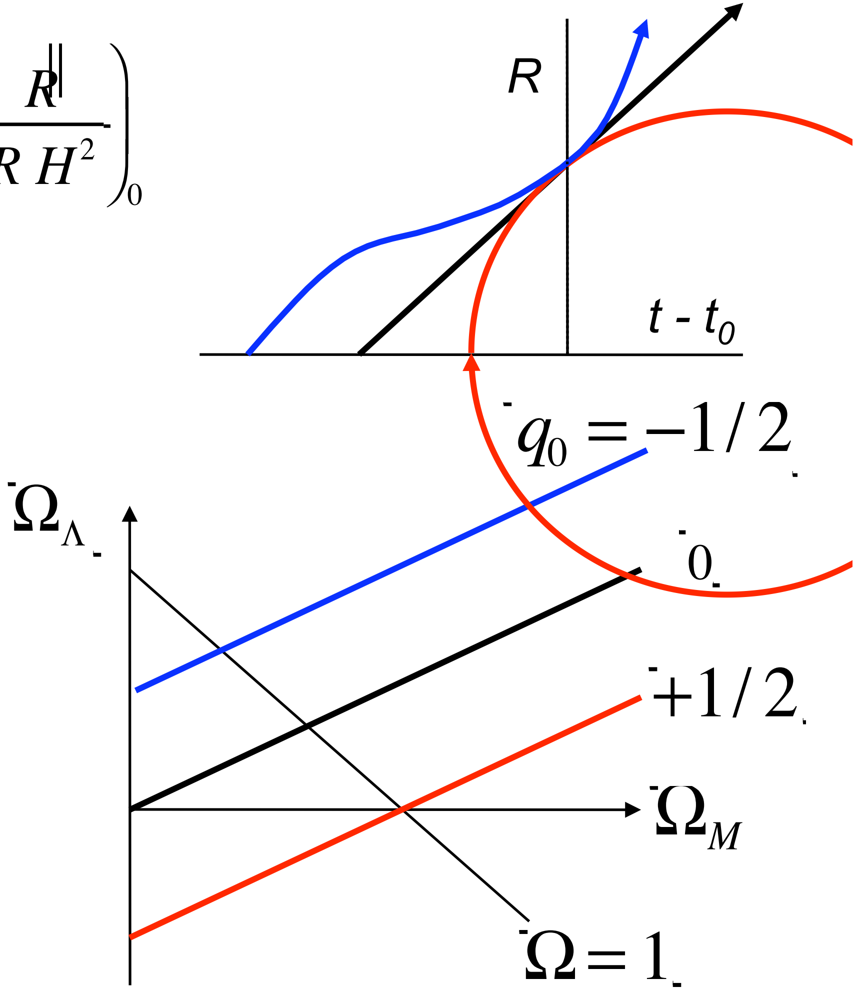
$$\frac{\ddot{R}}{H_0^{\,2}R} = -\frac{4\pi G}{3H_0^{\,2}}\rho(1+3w) + \frac{\Lambda}{3H_0^{\,2}}$$

$\rho,\, p > 0$ decelerate, $\Lambda > 0$ accelerates

Equation of state : $p = \sum_i w_i\,\rho_i\,c^2$

$$w_R = \frac{1}{3} \quad w_M = 0 \quad w_\Lambda = -1$$

$$q_0 = -\left(\frac{\ddot{R}}{R\,H^2}\right)_0 = \sum_i \left(\frac{1+3\,w_i}{2}\right)\Omega_i = \Omega_R + \frac{\Omega_M}{2} - \Omega_\Lambda$$



Deceleration Parameter

$$q_0 \equiv -\left(\frac{\ddot{R}R}{\dot{R}^2}\right)_0 = \frac{\Omega_M}{2} - \Omega_\Lambda$$

Matter decelerates

Vacuum (Dark) Energy accelerates

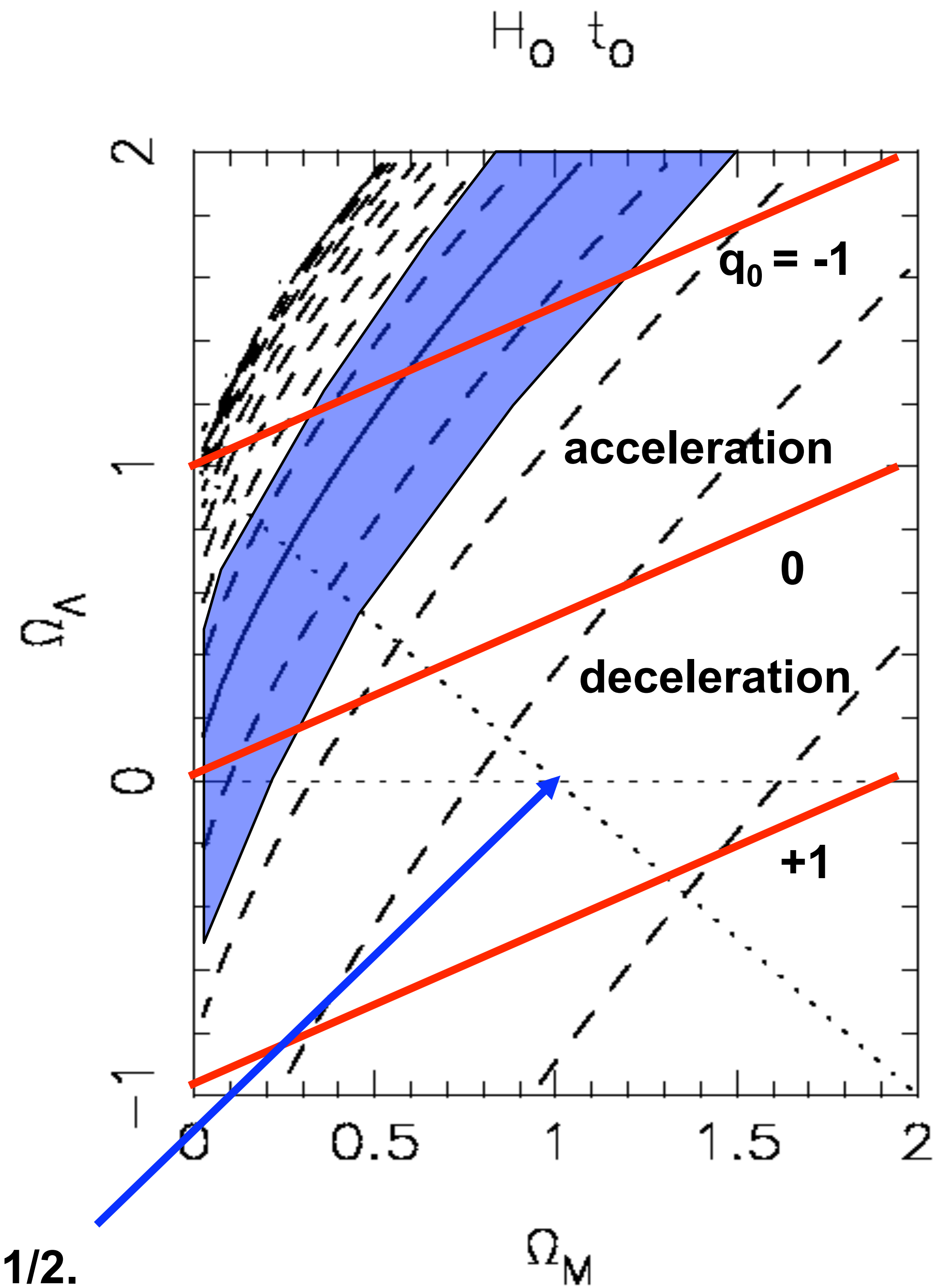
Measure q_0 via :

1 . $D_A(z)$

(e.g. radio jet lengths)

2. $D_L(z)$

(curvature of Hubble Diagram)



Critical density matter-only --> q_0=1/2.

Observable Distances

angular diameter distance :

$$\theta = \frac{l}{D_A} \qquad D_A = \frac{r_0}{(1+z)} = \frac{c\,z}{H_0}\left(1 - \frac{q_0+3}{2}\,z + ...\right)$$

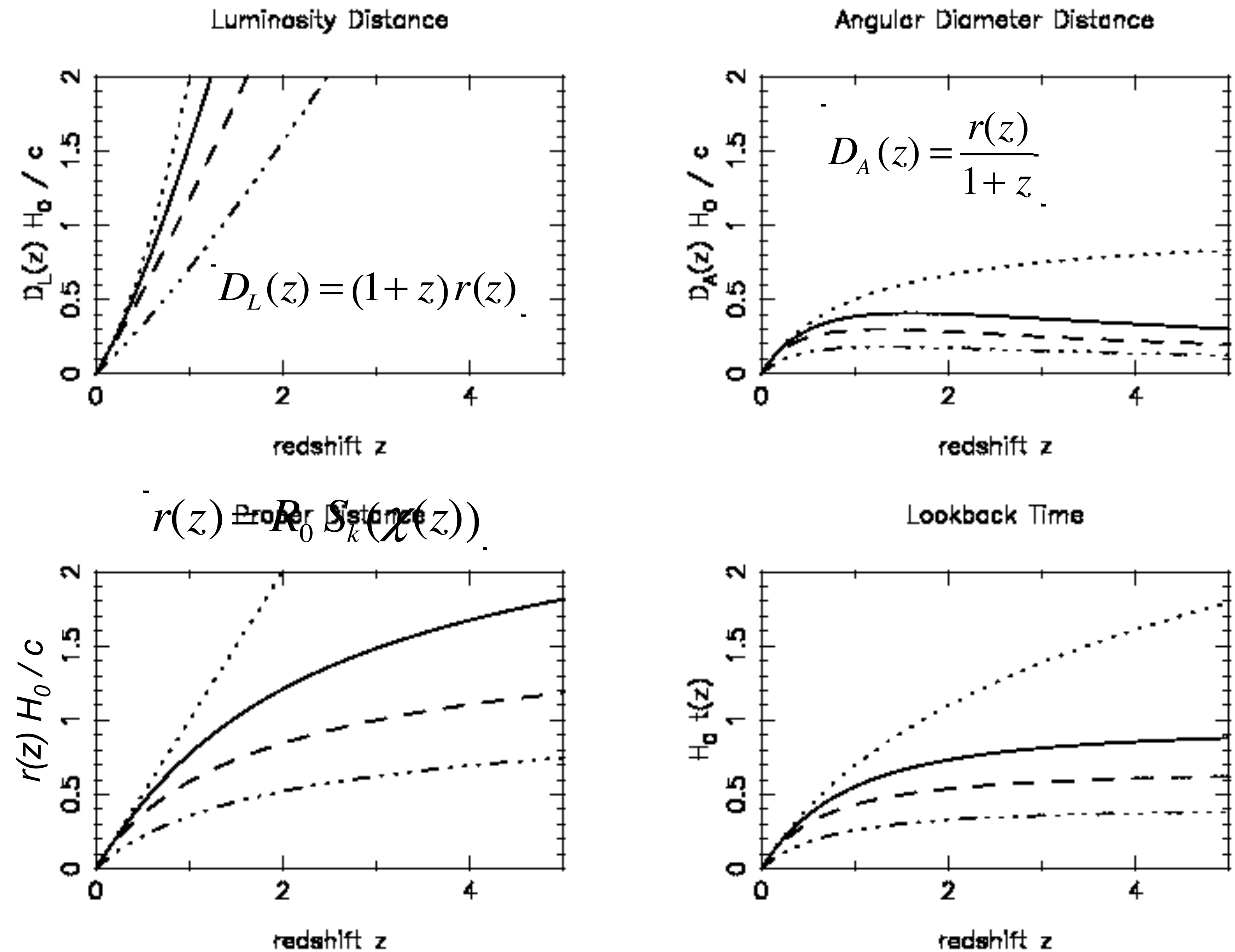
luminosity distance :

$$F = \frac{L}{4\pi\,D_L^{\;2}} \qquad D_L = r_0\,(1+z) = \frac{c\,z}{H_0}\left(1 + \frac{1-q_0}{2}\,z + ...\right)$$

deceleration parameter :

$$q_0 = \frac{\Omega_M}{2} - \Omega_\Lambda$$

Verify these low-z expansions.


Luminosity Distance
$D_L(z) = (1+z)\, r(z)$
$D_L(z)\, H_0 / c$
redshift z
Angular Diameter Distance
$D_A(z) = \frac{r(z)}{1+z}$
$D_A(z)\, H_0 / c$
redshift z
Proper Distance
$r(z) = R_0\, S_k(\chi(z))$
$r(z)\, H_0 / c$
redshift z
Lookback Time
$H_0\, t(z)$
redshift z

Hubble Diagram

$$m = M + 5\log\left(\frac{D_L(z)}{\text{Mpc}}\right) + 25 + A + K(z)$$

m = apparent mag

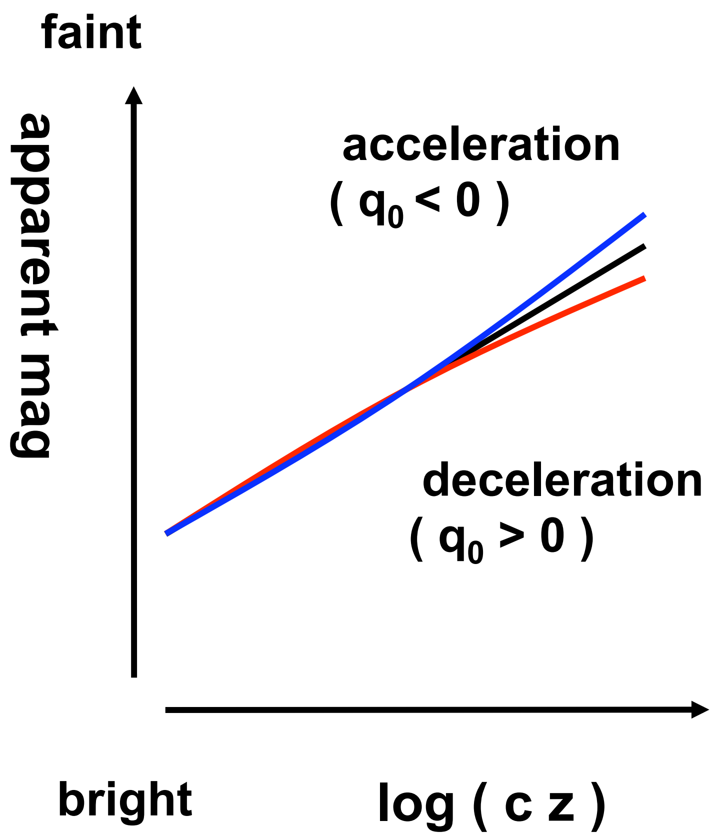
M = absolute mag

A = extinction (dust in galaxies)

$K(z)$ = K correction

(accounts for redshift of spectra relative to observed bandpass)

$$D_L(z) = \frac{c\,z}{H_0}\left(1 + \frac{1-q_0}{2}z + \ldots\right)$$



slope = +5

vertical shift --> H_0

curvature --> q_0

Calibrating "Standard Bombs"

1. Brighter ones decline more slowly.

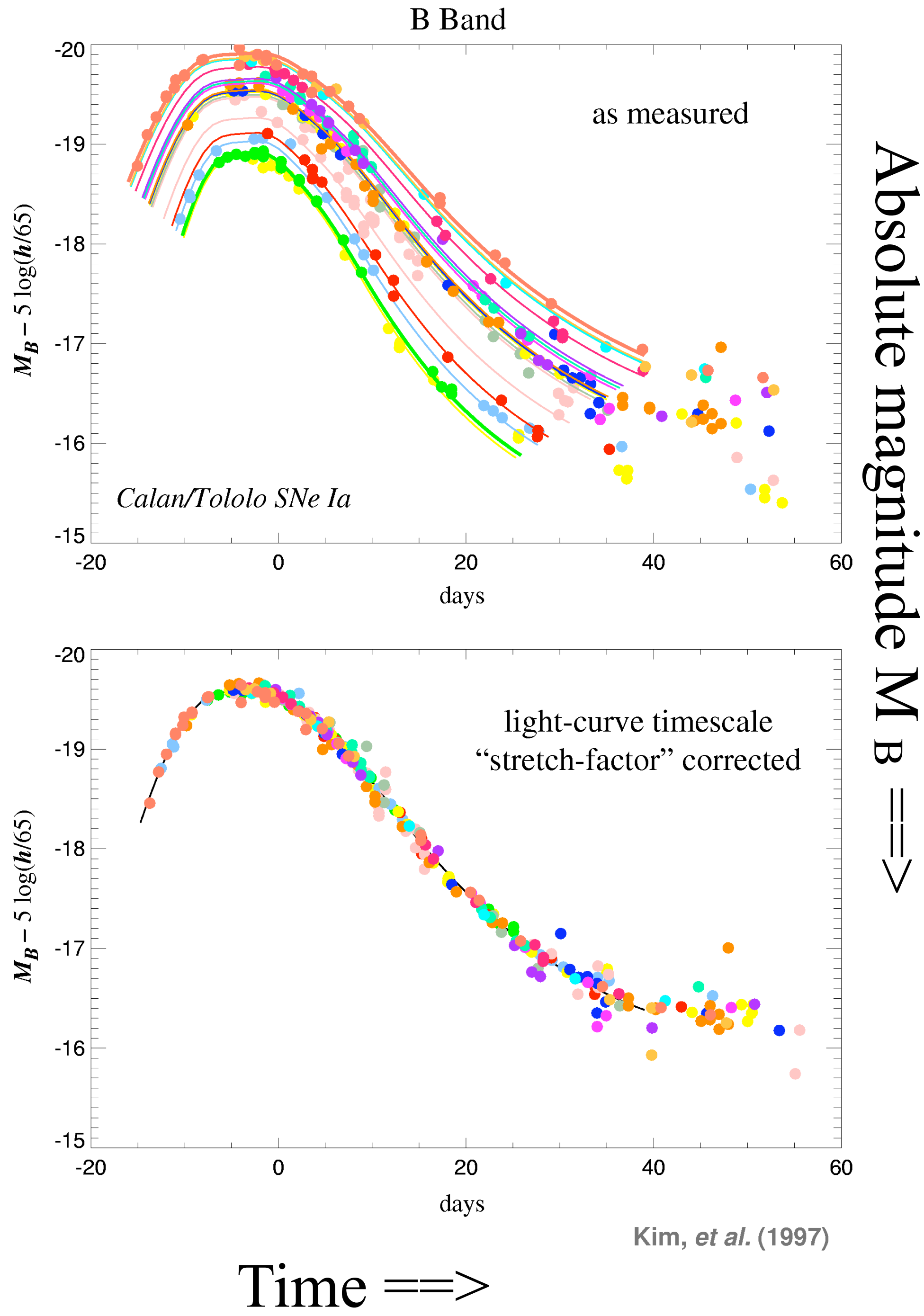
2. Time runs slower by factor (1+z).

AFTER correcting: Constant peak brightness $M_B = -19.7$

Observed peak magnitude:

$$m = M + 5 \log(d/\mathrm{Mpc}) + 25$$

gives the distance!



Kim, *et al.* (1997)

SN Ia at z ~ 0.8 are ~25% fainter than expected

Acceleration (! ?)

1. Bad Observations?

-- 2 independent teams agree

1. Dust ?

-- corrected using reddening

2. Stellar populations ?

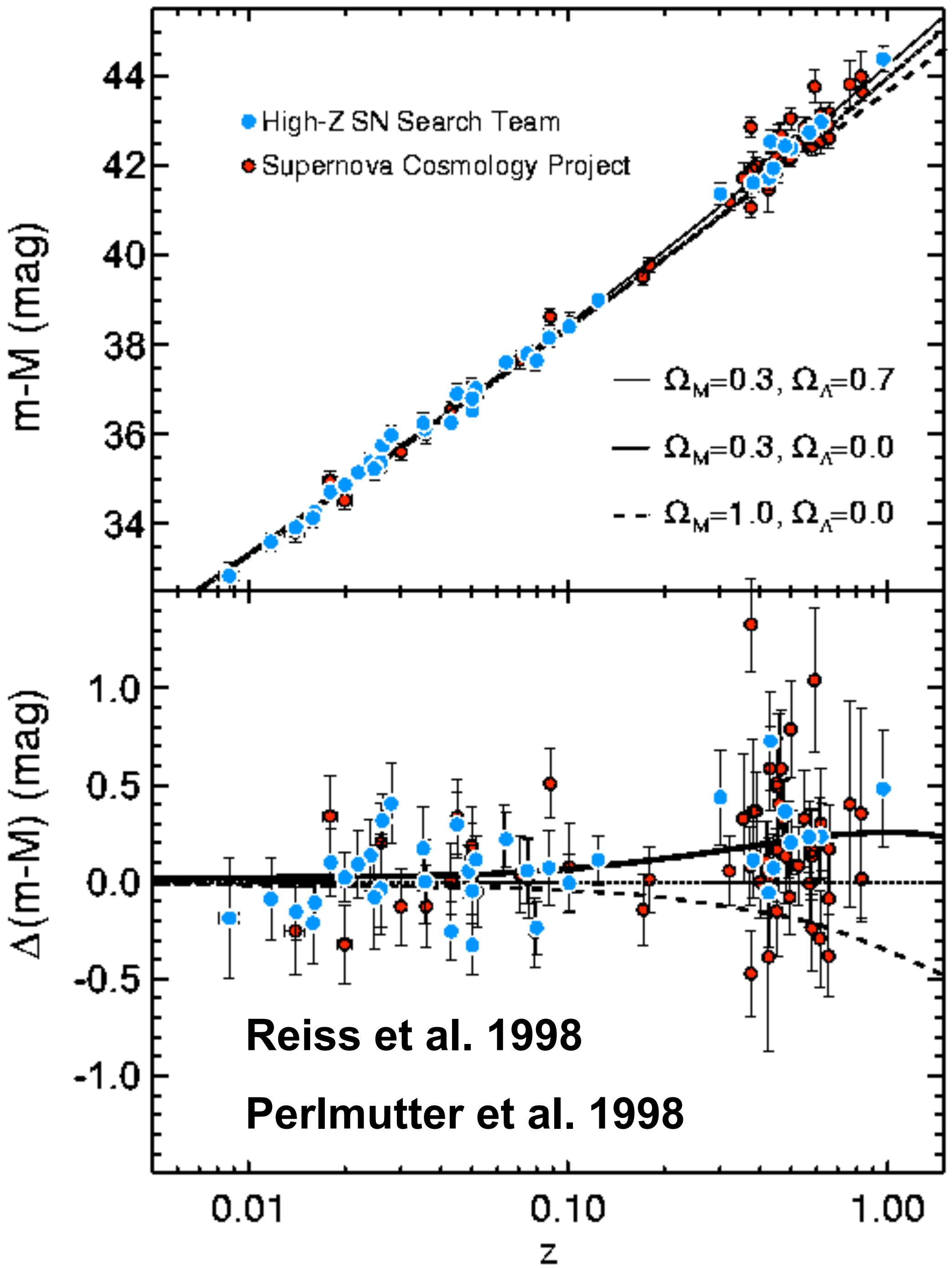
-- earlier generation of stars

-- lower metalicity

3. Lensing?

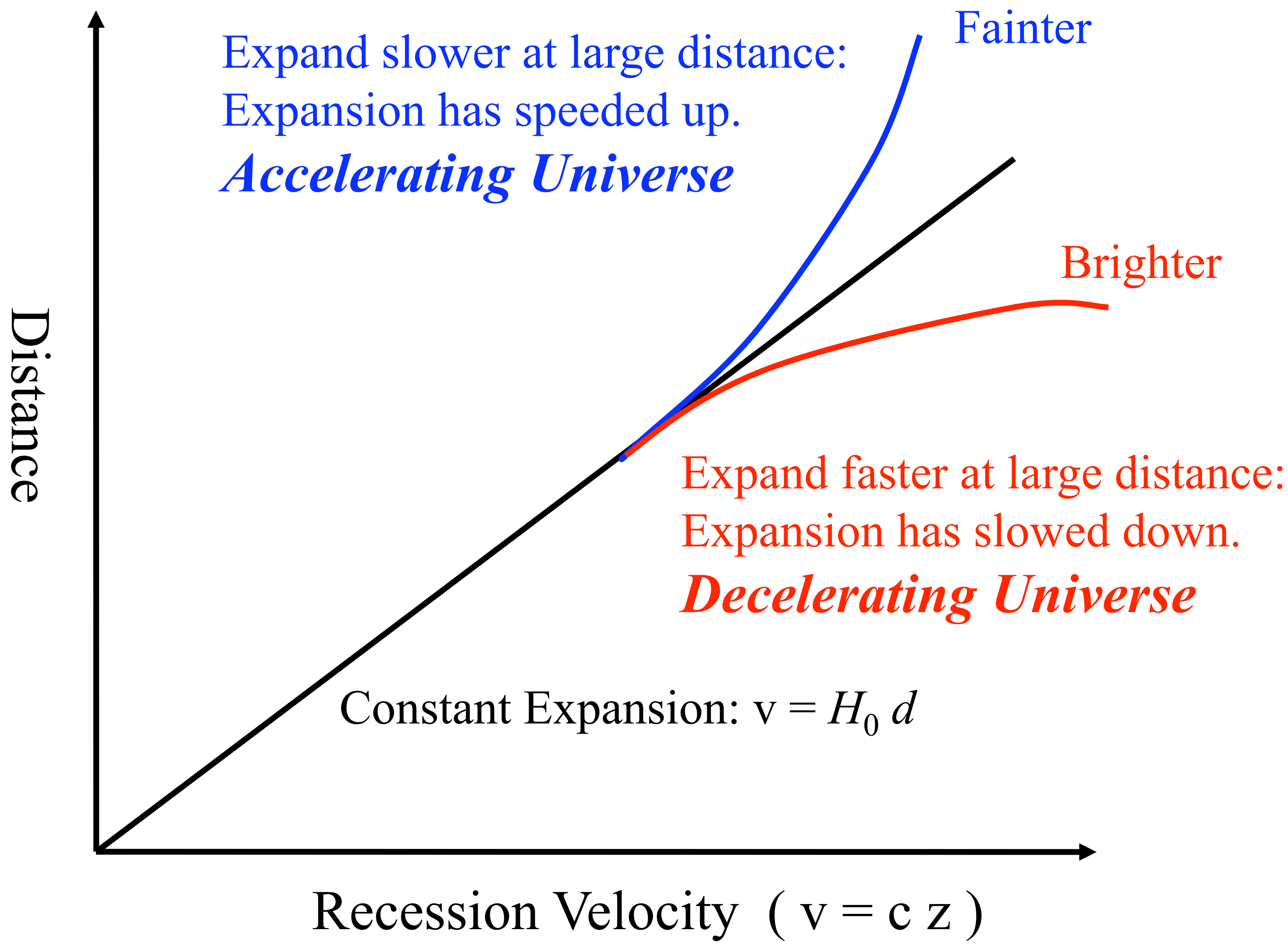
-- some brighter, some fainter

-- effect small at z ~ 0.8



Reiss et al. 1998

Perlmutter et al. 1998



Expand slower at large distance:
Expansion has speeded up.
Accelerating Universe
Fainter
Brighter
Distance
Expand faster at large distance:
Expansion has slowed down.
Decelerating Universe
Constant Expansion: $v = H_0 d$
Recession Velocity ($v = c z$)

"Concordance" Model

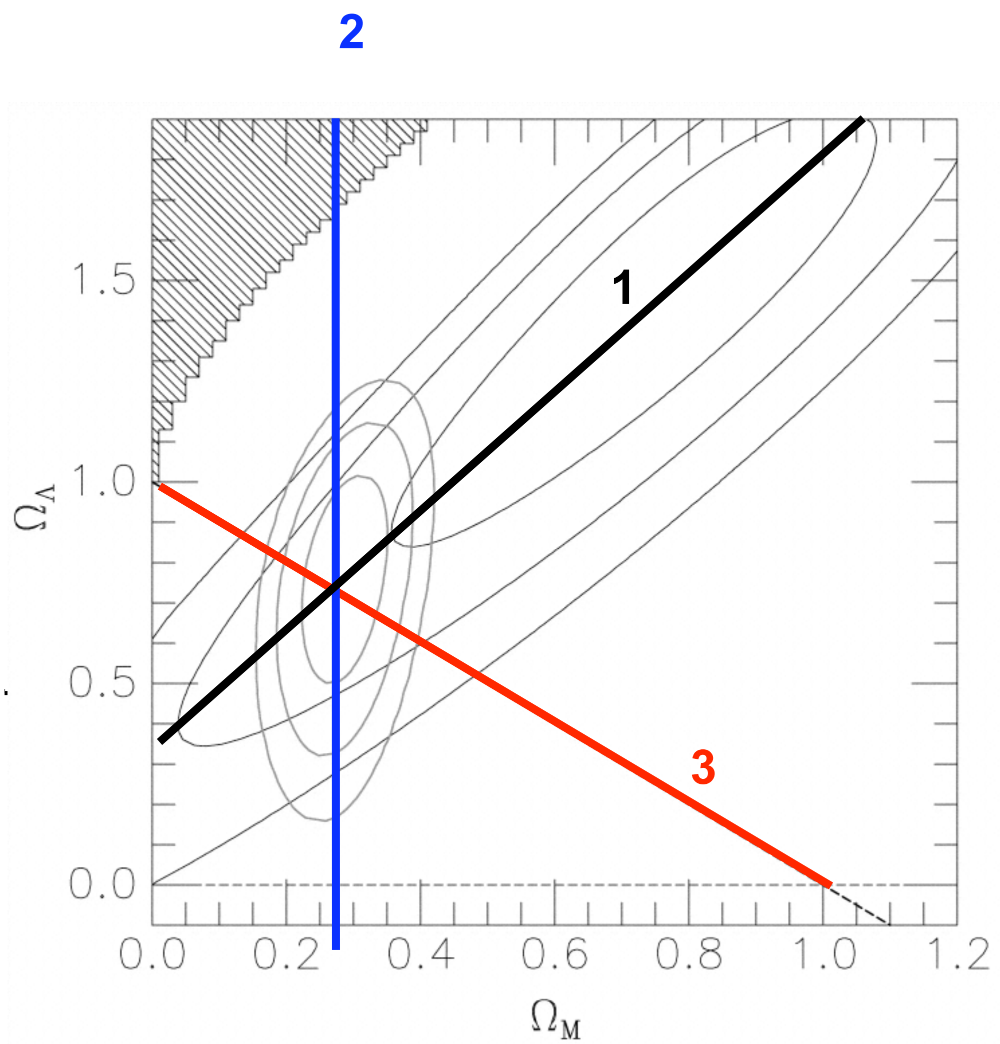
1. Supernova Hubble Diagram
2. Galaxy Counts + M/L ratios

   $\Omega_M \sim 0.3$

3. Flat Geometry

   (inflation, CMB fluctuations)

   $\Omega_0 = \Omega_M + \Omega_\Lambda = 1$



concordance model

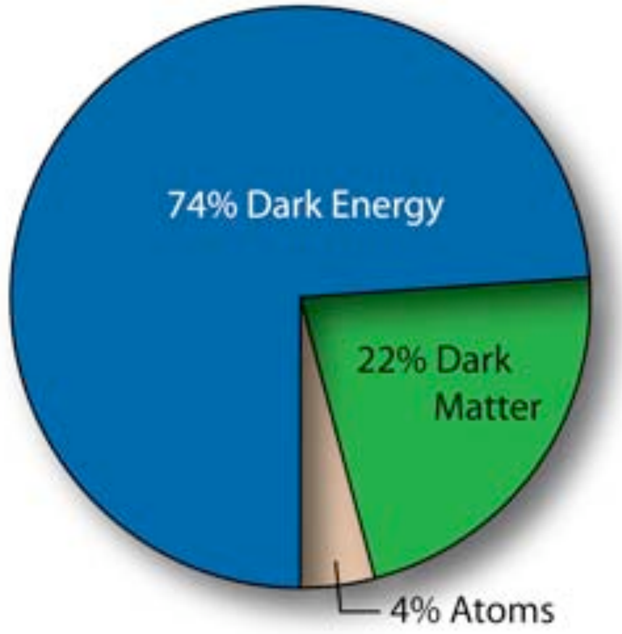
$$H_0 \approx 72 \quad \Omega_M \approx 0.3 \quad \Omega_\Lambda \approx 0.7$$

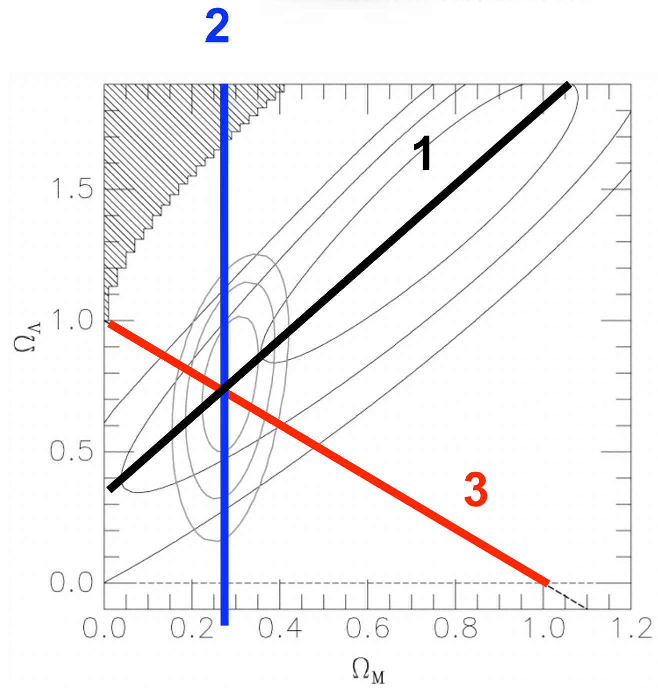
Dark Matter

Galaxy Counts
Redshift Surveys
Galaxy Rotation Curves
Cluster Dynamics
Gravitational Lenses

$$\Omega_M \sim 0.3$$

$$\Omega_b \approx 0.04$$





Mass Density by Direct Counting

- **Add up the mass of all the galaxies per unit volume**
  - **Volume calculation as in Tutorial problem.**
- **Need representative volume > 100 Mpc.**
- **Can't see faintest galaxies at large distance.**
  Use local Luminosity Functions to include fainter ones.
- **Mass/Light ratio depends on type of galaxy.**
- **Dark Matter needed to bind Galaxies and Galaxy Clusters dominates the normal matter (baryons).**
- **Hot x-ray gas dominates the baryon mass of Galaxy Clusters.**

Schechter Luminosity Function

3 Schechter parameters :

$\alpha \quad L^* \quad \Phi^*$

luminosity of a typical big galaxy

$L^* \approx 10^{11}\ \mathrm{L}_{\mathrm{sun}}$

luminosity of any galaxy :

$L = x\,L^* \qquad x \equiv \frac{L}{L^*}$

number of galaxies per unit luminosity

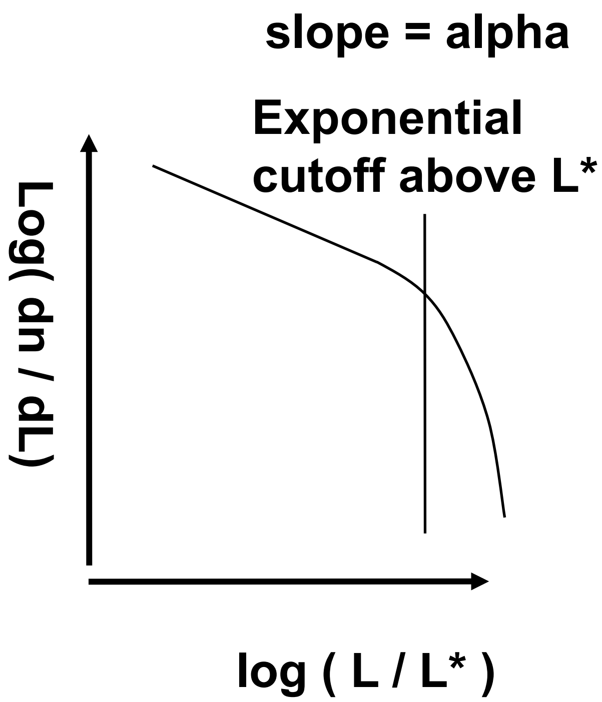
$$\Phi(x) \equiv \frac{dn}{dx} = \Phi^*\ x^{\alpha}\ e^{-x}$$

add up the luminosities

$$\rho_L = \int_0^\infty L\,\frac{dn}{dx}\,dx = L^*\Phi^* \int_0^\infty x^{\alpha+1}\ e^{-x}\ dx$$

add up the mass (need mass/light ratio)

$$\rho_M = \int_0^\infty \frac{M}{L}\,L\,\frac{dn}{dx}\,dx = \left\langle \frac{M}{L} \right\rangle \rho_L$$



Measure Schechter parameters using:

galaxy clusters

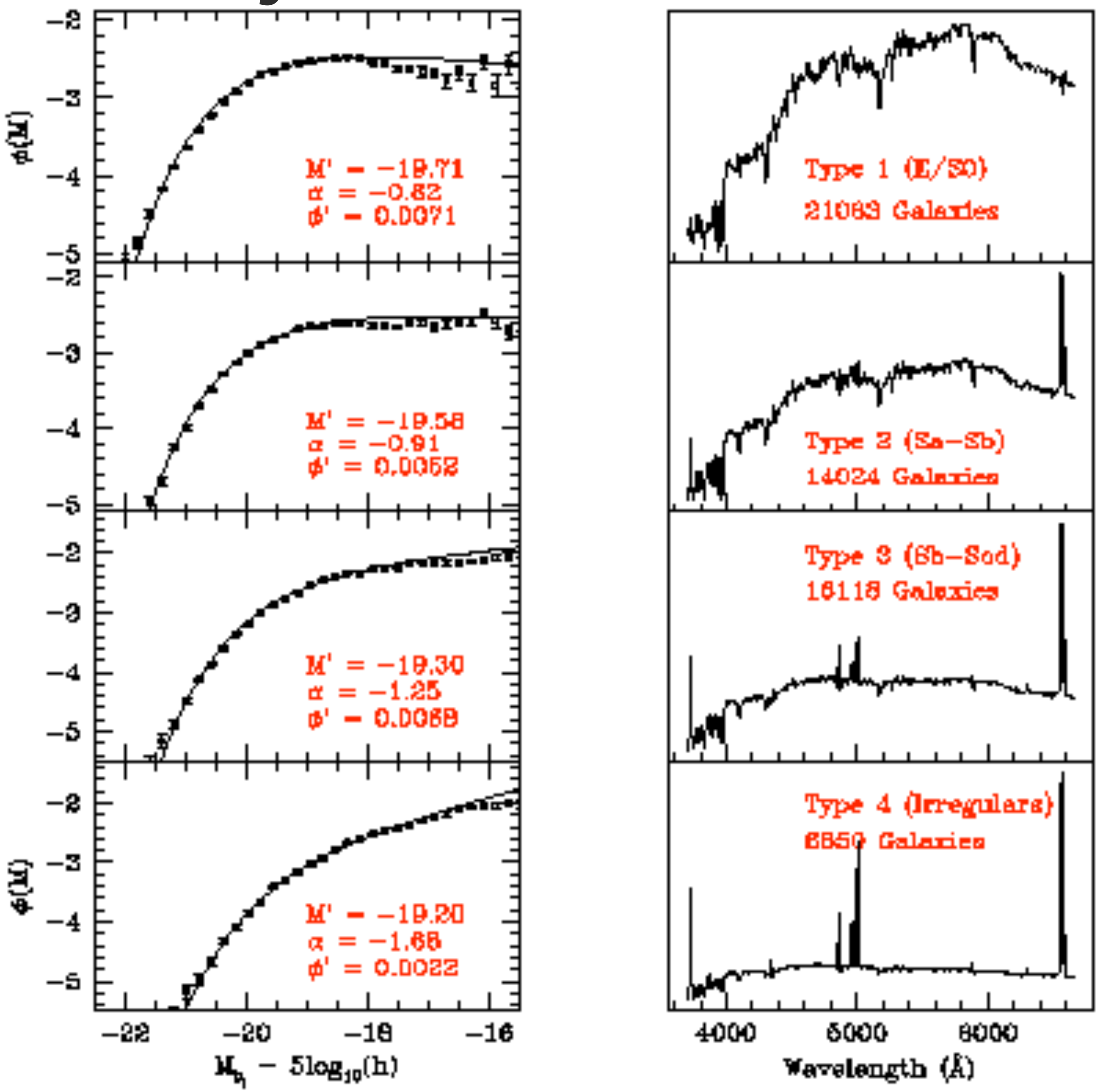
galaxy redshift surveys

Measure M/L for :

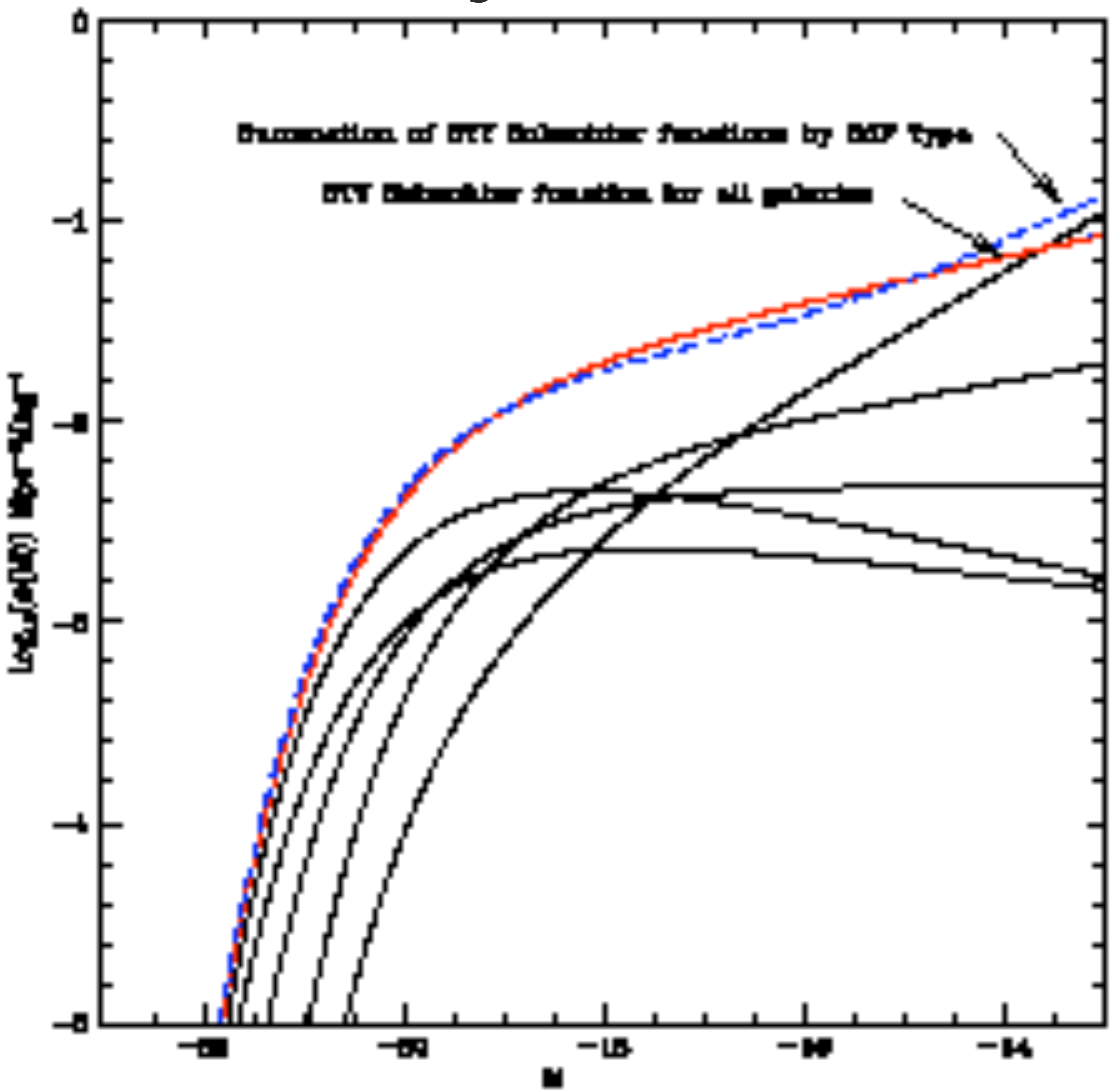
Nearby galaxies, galaxy clusters

Galaxy Luminosity Function

Schechter parameters depend on galaxy type.



Galaxy Luminosity Function



Schechter function also fits sum of all galaxy types.

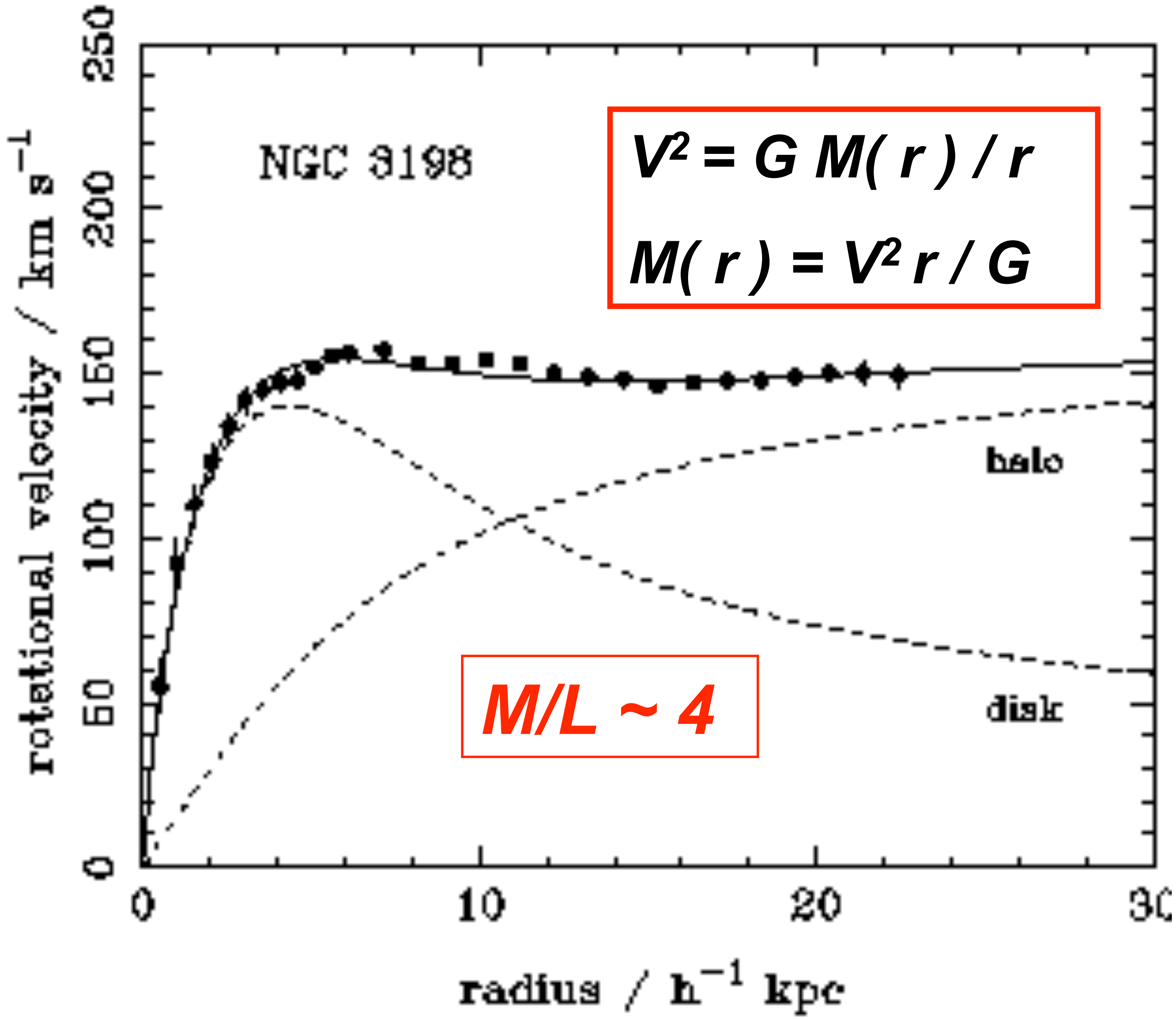
But each type has a different M/L.

Galaxy Rotation Curves

HI velocities

Flat rotation curves

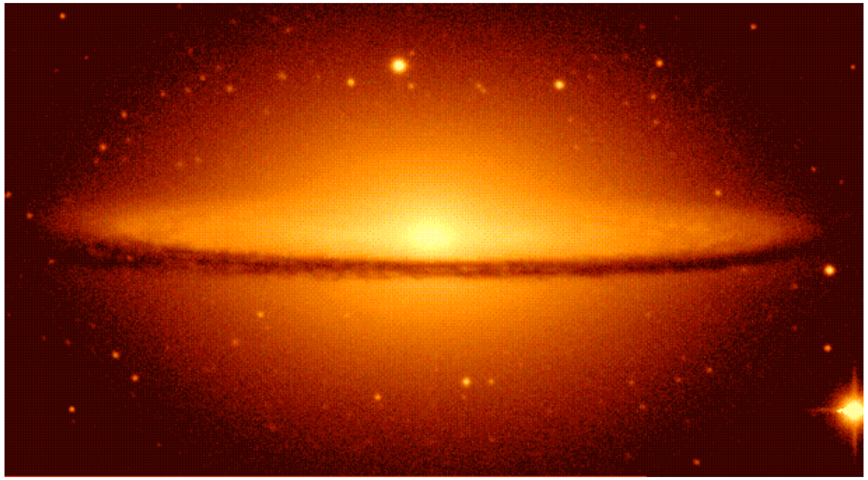
Dark Matter Halos

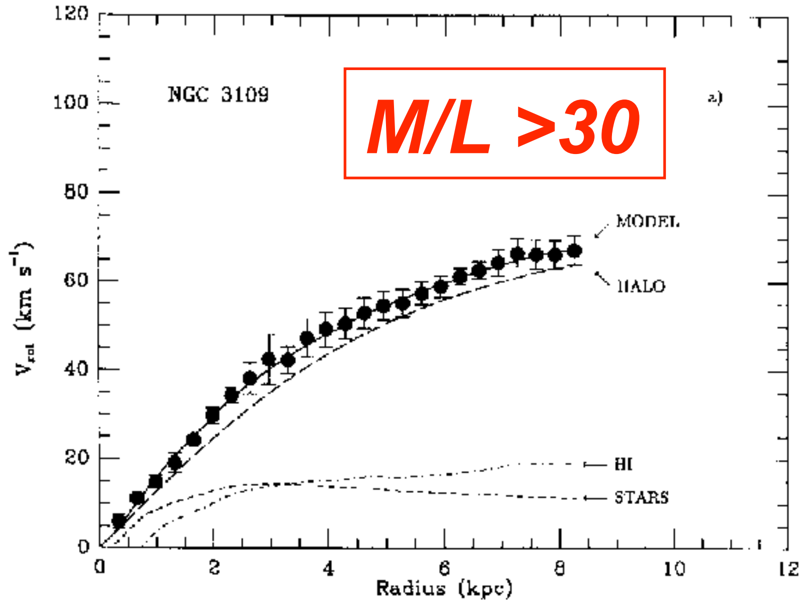


Spirals, Ellipticals:
M/L~4-10

Some dwarf galaxies:
M/L ~ 100.

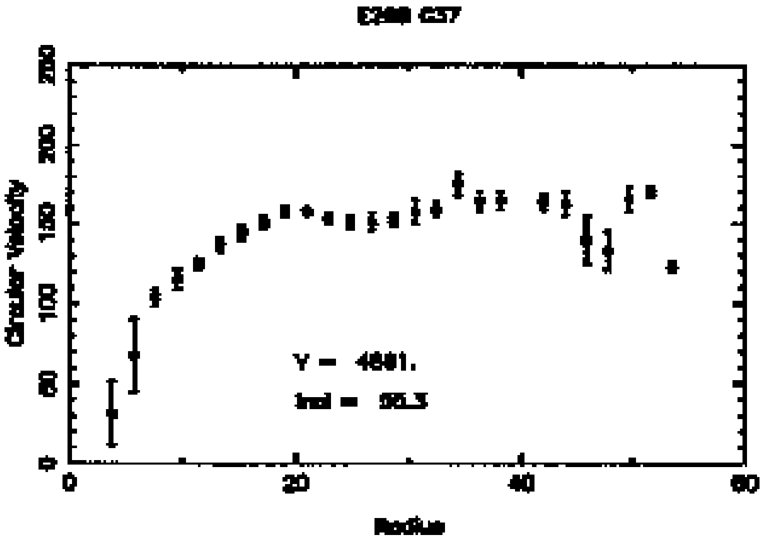
Galaxy Rotation Curves

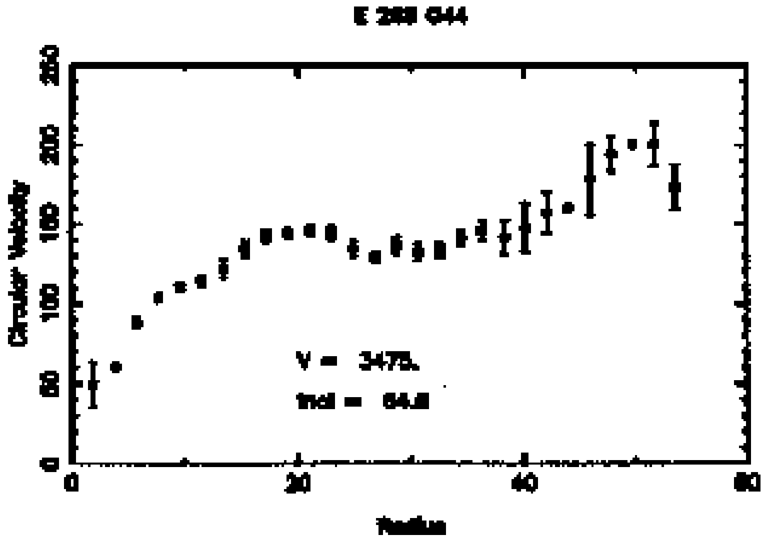




Small galaxies : V(r) rises

Large galaxies : V(r) flat





Mass / Light ratios

galaxy luminosity distribution

$$\frac{dn}{dL} = \Phi(L) = \Phi^* \left(\frac{L}{L^*}\right)^\alpha \exp\left(-\frac{L}{L^*}\right)$$

luminosity density $\rho_L = \int L\,\Phi(L)\,dL$

e.g. blue light $\approx 2 \pm 0.7 \times 10^8 h\ L_{sun}\ \mathrm{Mpc}^{-3}$

mass density $\rho_M = \int \left(\frac{M}{L}\right) L\,\Phi(L)\,dL$

$$= \Omega_M\ \rho_{\mathrm{crit}} = 2.8 \times 10^{11}\ \Omega_M\ h^2\ M_{sun}\ \mathrm{Mpc}^{-3}$$

Universe: $M/L = 1400\,\Omega_M\ h^2 \sim 200\ (\Omega_M/0.3)(h/0.7)^2$

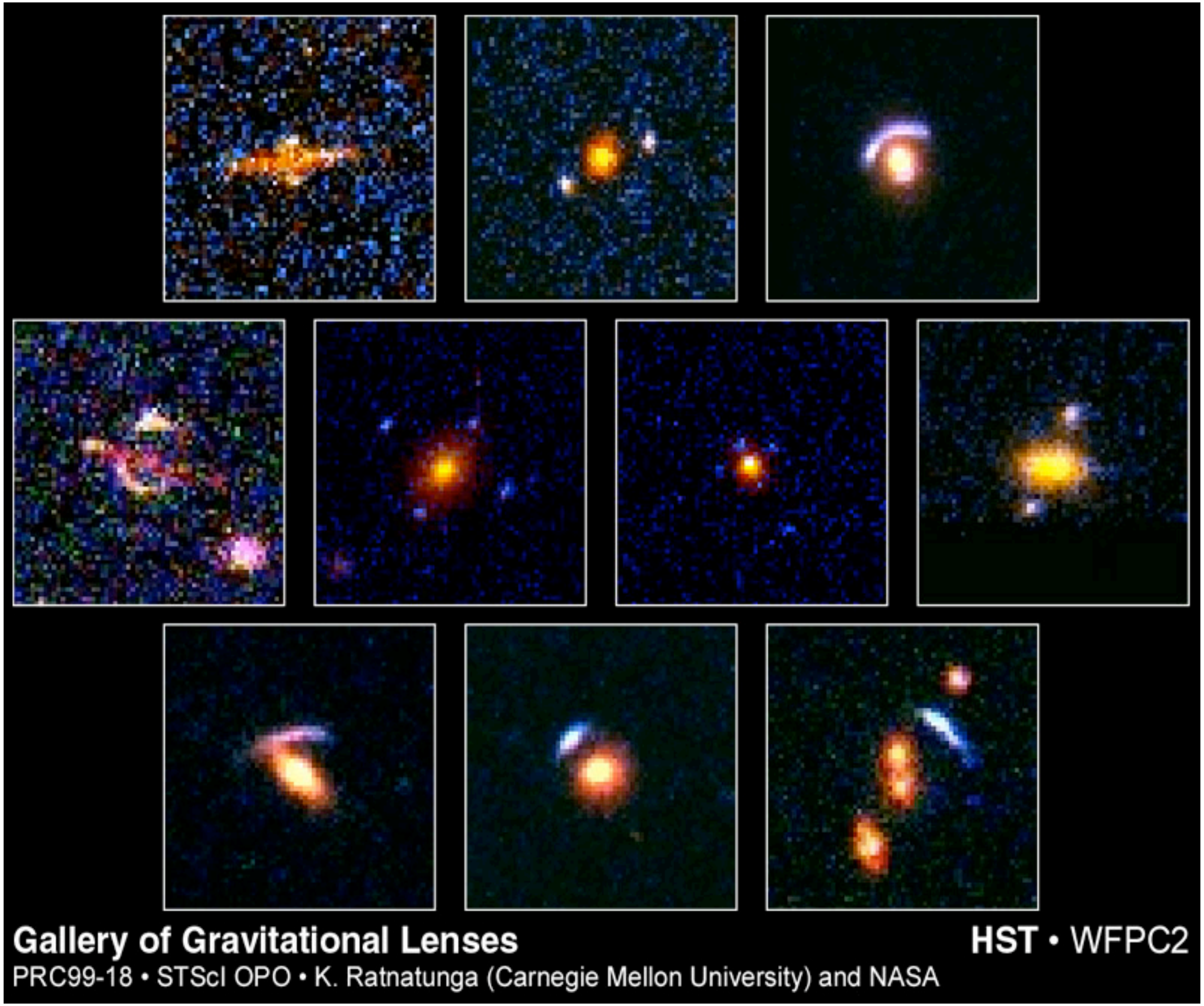
Sun: $M/L = 1$ (by definition)

main sequence stars: $M/L \propto M^{-3}$ (since $L \propto M^4$)

comets, planets: $M/L \sim 10^{9-12}$

Is our Dark Matter halo filled with MACHOs ?
NO. Gravitational Lensing results rule them out.

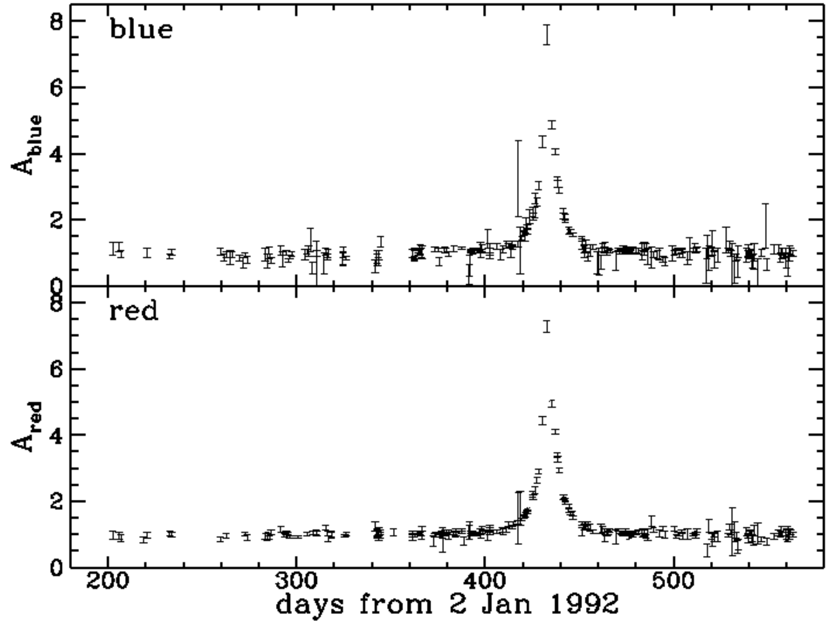
Quasars Lensed by Galaxies

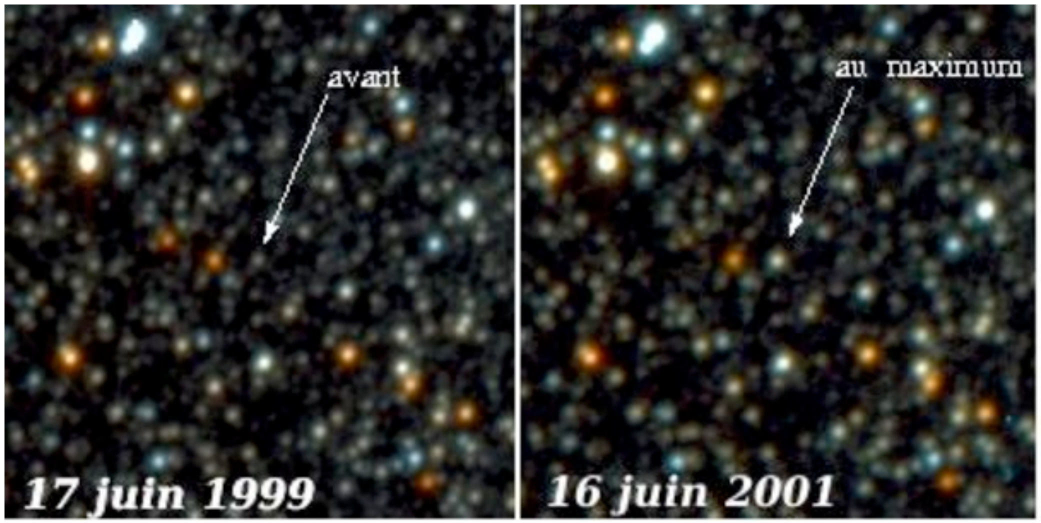


Dark Matter Candidates

- **MACHOS = Massive Compact Halo Objects**
  - Black holes
  - Brown Dwarfs
  - Loose planets
  - Ruled out by gravitational lensing experiments.
- **WIMPS = Weakly Interacting Massive Particles**
  - Massive neutrinos
  - Supersymmetry partners
  - Might be found soon by Large Hadron Collider in CERN

Microlensing in the LMC



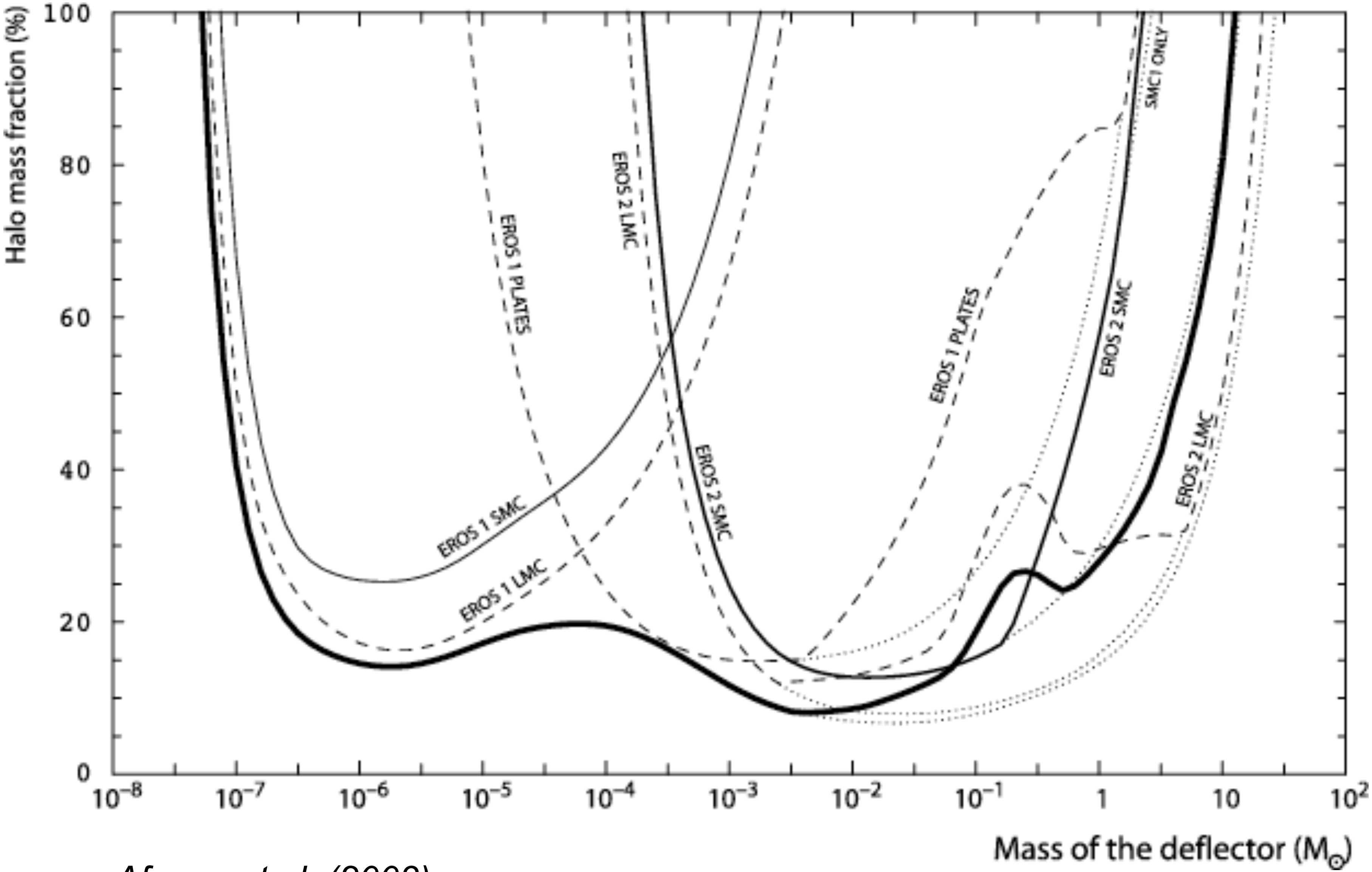


Massive Compace Halo Objects (MACHOS) would magnify LMC stars dozens of times each year. Only a few are seen.

Long events -> high mass

Short events -> low mass

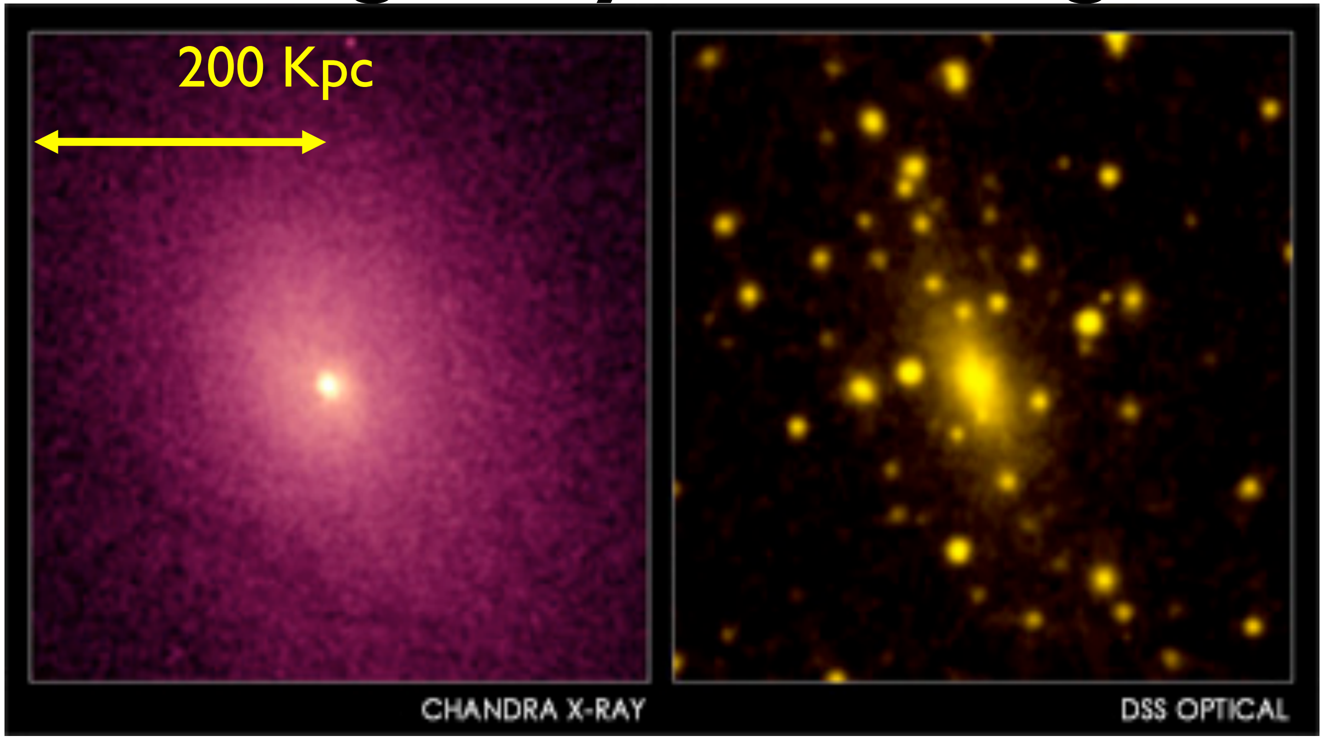
LMC Microlensing says NO to MACHOs



Afonso et al. (2003)

Dark Matter in Galaxy Clusters

Probes gravity on 10x larger scales



$$z = 0.0767$$

$$d \approx \frac{c\,z}{H_0}$$

$$= 320 \text{ Mpc}$$

Chandra X-ray Image of Abell 2029

The galaxy cluster Abell 2029 is composed of thousands of galaxies enveloped in a gigantic cloud of hot gas, and an amount of **dark matter** equivalent to more than **a hundred trillion Suns**. At the center of this cluster is an enormous, elliptically shaped galaxy that is thought to have been formed from the mergers of many smaller galaxies.

Cluster Masses from X-ray Gas

hydrostatic equilibrium :

$$\frac{dP}{dr} = -\rho\, g = -\rho\, \frac{G\, M(< r)}{r^2}$$

gas law :

$$P = \frac{\rho\, k\, T}{\mu\, m_H}$$

X - ray emission from gas gives : $T(r),\, n_e(r) \rightarrow \rho(r), P(r)$

$$M(< r) = -\frac{r^2}{G\, \rho(r)} \frac{dP}{dr}$$

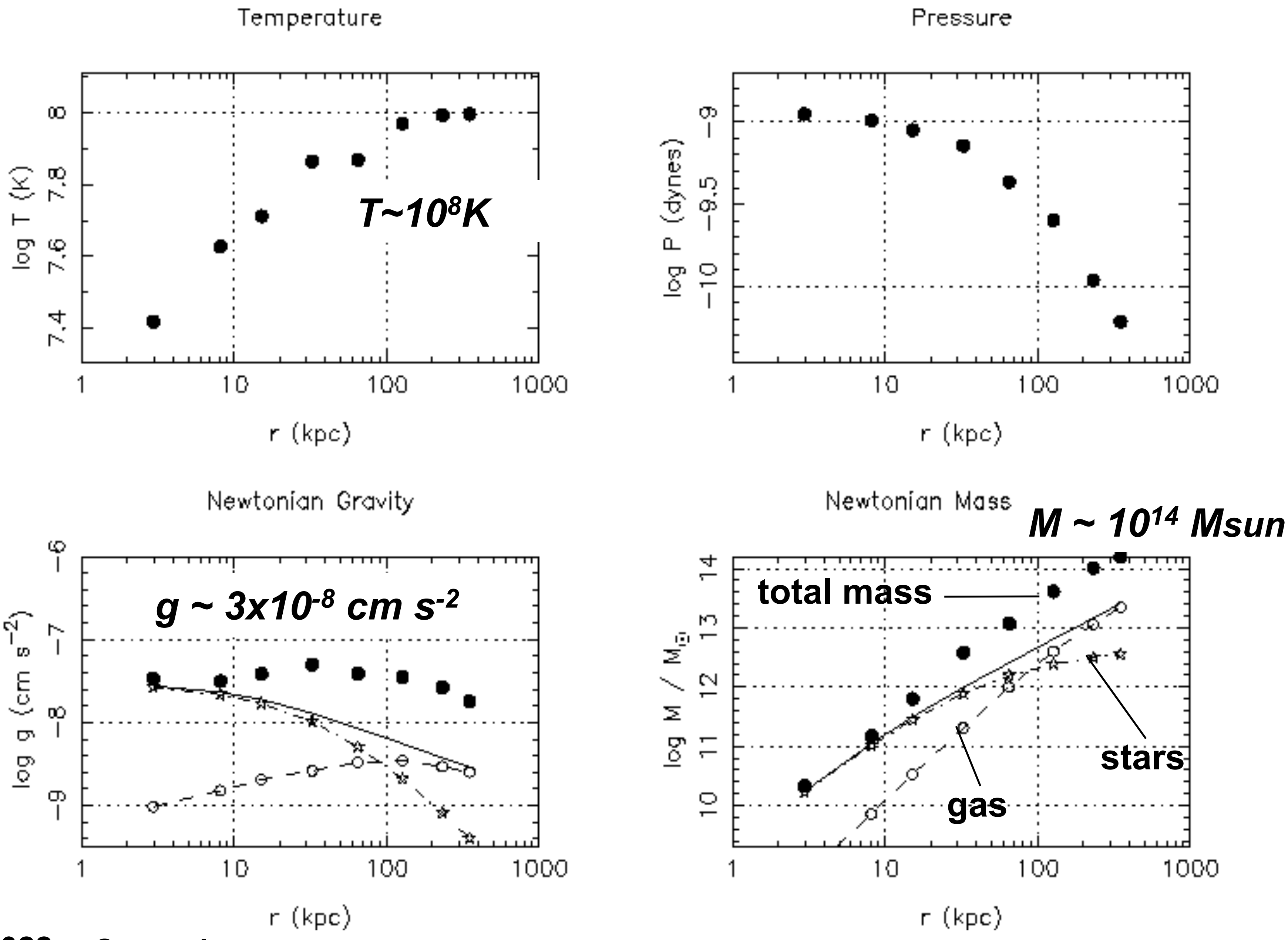
Coma Cluster:

M(gas)~M(stars)~3x10^{13} Msun

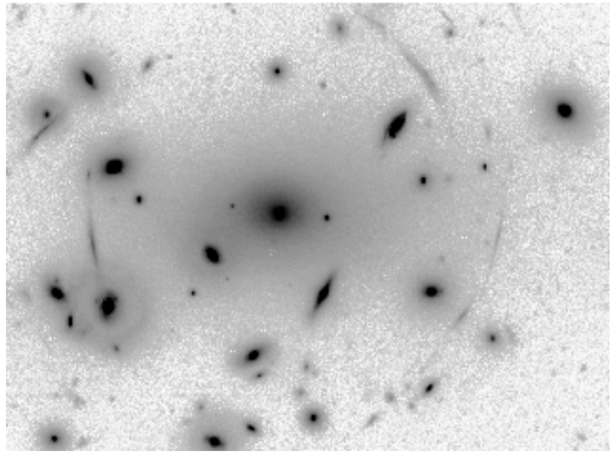
often M(gas) > M(stars)

M/L~100-200

Cluster Masses from X-ray Gas



Masses from Gravitational Lensing



$$\theta_E = \frac{R_E}{D_L} = \left(\frac{4\,G\,M}{c^2} \frac{D_{LS}}{D_L D_S} \right)^{1/2}$$

$$\frac{M}{10^{11}\,M_{sun}} = \frac{D_L D_S / D_{LS}}{\mathrm{Gpc}} \left(\frac{\theta_E}{\mathrm{arcsec}} \right)^2$$

Use redshifts, z_L, z_S,

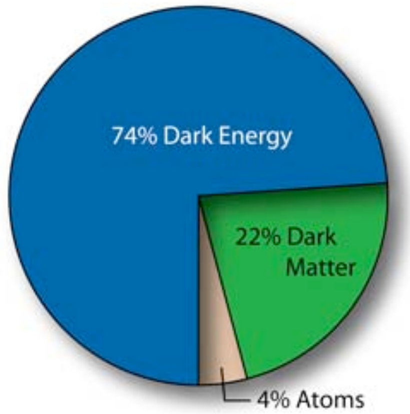
for the angular diameter distances.

General agreement with Virial Masses.

Evidence for Dark Matter ?

Galaxies: ($r \sim 20$ Kpc)

Flat Rotation Curves $V \sim 200$ km/s

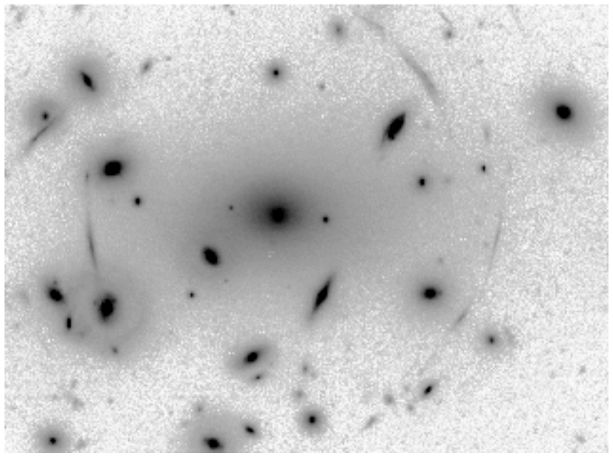


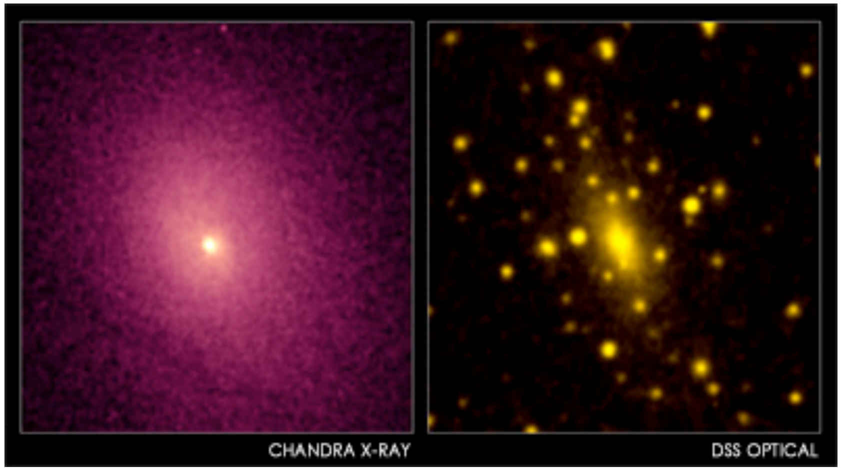
Galaxy Clusters: ($r \sim 200$ Kpc)

Galaxy velocities $V \sim 1000$ km/s

X-ray Gas $T \sim 10^8$ K

Giant Arcs



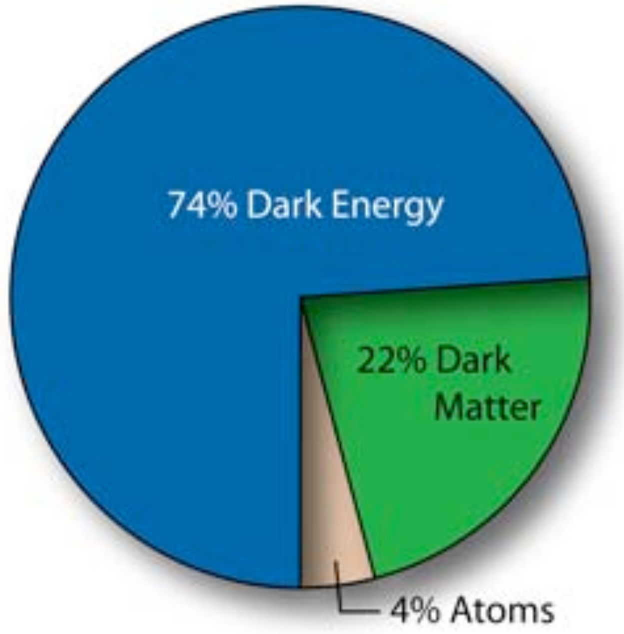


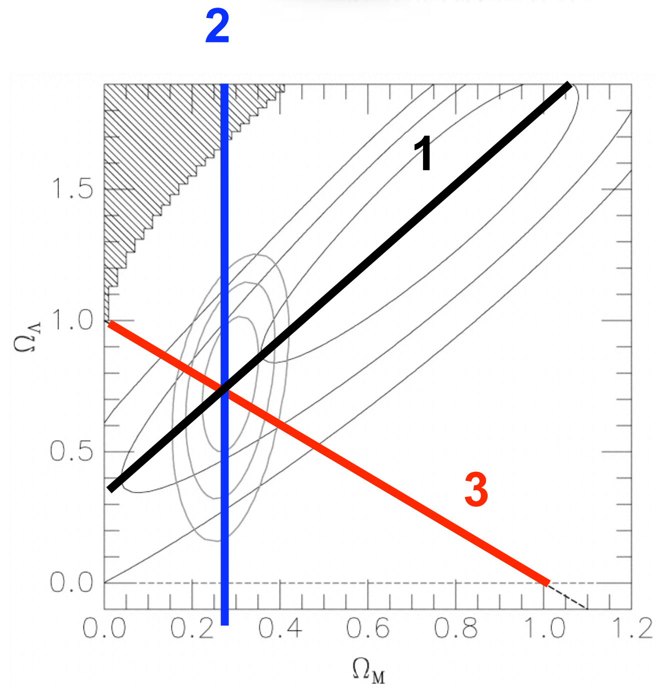
X-ray **Optical**

Cosmic Microwave Background

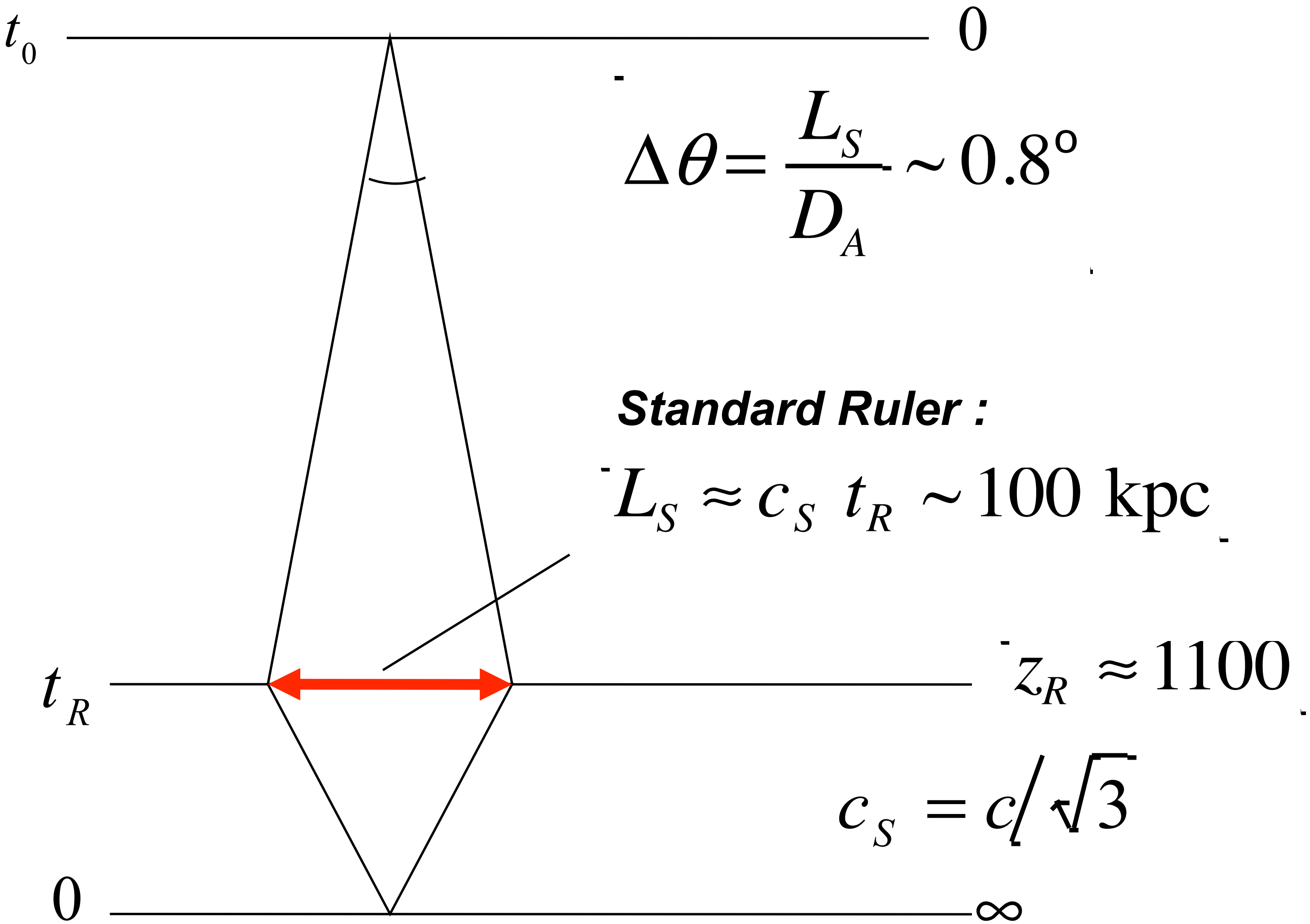
Flat Geometry

$$\Omega_0 = \Omega_M + \Omega_\Lambda \approx 1.0$$





Sound Horizon at z = 1100



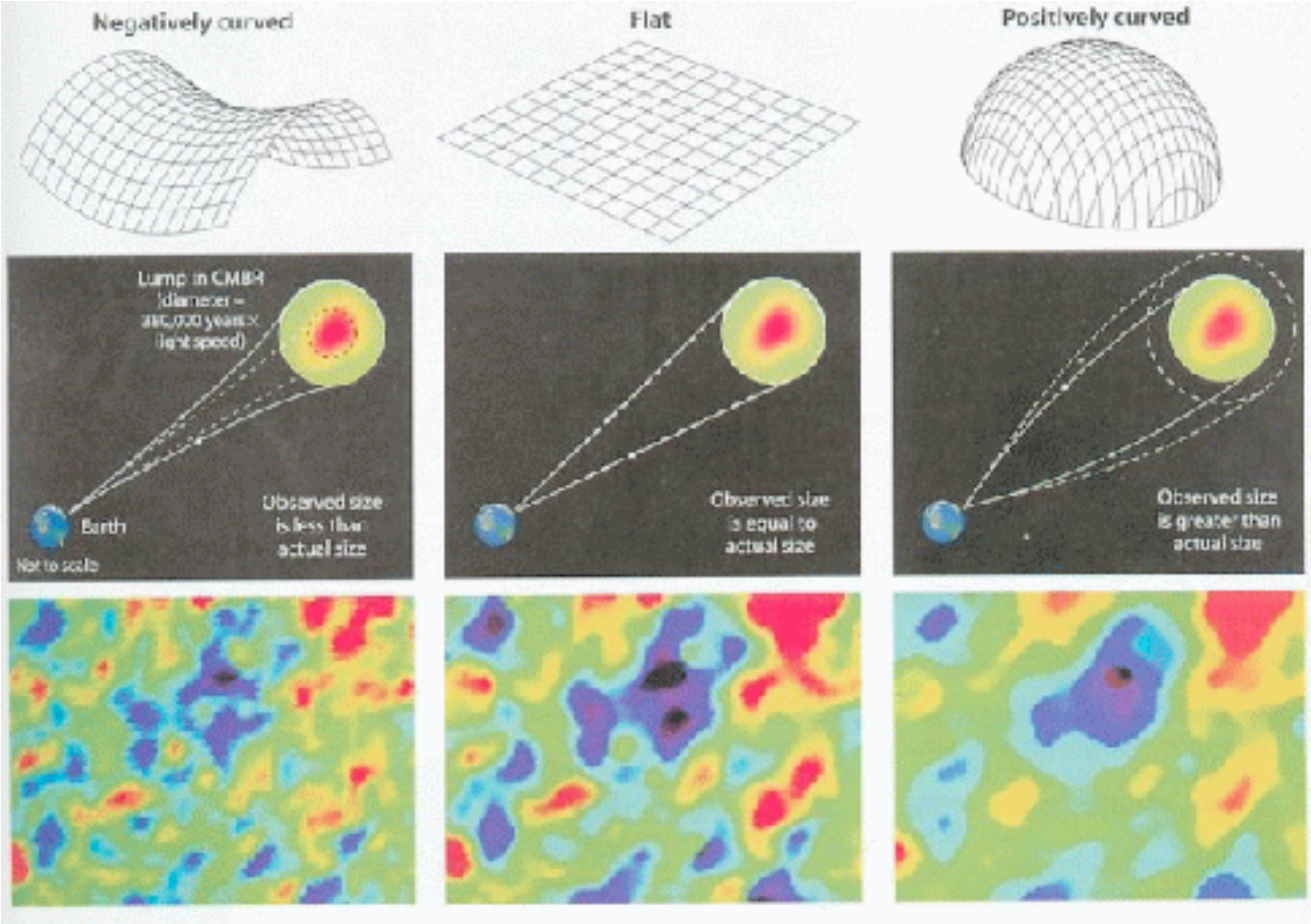
$$\Delta\theta = \frac{L_S}{D_A} \sim 0.8^\circ$$

Standard Ruler :

$$L_S \approx c_S \; t_R \sim 100 \text{ kpc}$$

$$c_S = c/\sqrt{3}$$

Angular scale --> Geometry



Sound Horizon at z = 1100

distance travelled by a sound wave

$$c_S \, dt$$

recombination
at ***z = 1100***

$$x \equiv 1+z = \frac{R_0}{R(t)}$$

expand each step by factor $R(t_R)/R(t)$:

$$dt = \frac{-dx}{x\,H(x)}$$

sound speed

$$c_S \approx \frac{c}{\sqrt{3}}$$

$$L_S(t_R) = R(t_R) \int_0^{t_R} \frac{c_S\,dt}{R(t)}$$

dt = - dx / x H(x)
R(t) = R₀ / x

$$= \frac{R_0}{1+z} \int_{1+z}^{\infty} \frac{x}{R_0} \frac{c_S\,dx}{x\,H(x)}$$

$$= \frac{c_S}{(1+z)} \int_{1+z}^{\infty} \frac{dx}{H(x)}$$

H(x) **from Friedmann Eqn.**

$$= \frac{c_S}{(1+z)\,H_0} \int_{1+z}^{\infty} \frac{dx}{\sqrt{x^4\,\Omega_R + x^3\,\Omega_M + \Omega_\Lambda + (1-\Omega_0)\,x^2}}$$

$$\approx \frac{c_S}{(1+z)\,H_0} \int_{1+z}^{\infty} \frac{dx}{\sqrt{x^4\,\Omega_R + x^3\,\Omega_M}}$$

keep 2 largest terms.

Sound Horizon at z = 1100

$$L_S(t_R) = \frac{c_S}{(1+z)} \int_{1+z}^{\infty} \frac{dx}{H(x)} \approx \frac{c_S}{(1+z)\, H_0} \int_{1+z}^{\infty} \frac{dx}{\sqrt{x^4\, \Omega_R + x^3\, \Omega_M}}$$

$$= \frac{c_S}{(1+z)\, H_0 \sqrt{\Omega_R}} \int_{1+z}^{\infty} \frac{dx}{\sqrt{x^3 (x + x_0)}} \qquad x_0 \equiv \frac{\Omega_M}{\Omega_R} \approx 3500 \left(\frac{\Omega_M}{0.3} \right)$$

$$= \frac{c_S}{(1+z)\, H_0 \sqrt{\Omega_R}} \left(-\frac{2}{x_0} \sqrt{1 + \frac{x_0}{x}} \right)_{1+z}^{\infty}$$

$$= \frac{2 c_S}{(1+z)\, H_0 \sqrt{\Omega_M\, x_0}} \left(\sqrt{1 + \frac{x_0}{1+z}} - 1 \right) \qquad c_S = \frac{c}{\sqrt{3}}$$

$$= \frac{c}{H_0} \frac{2\left(\sqrt{4.6} - 1\right)}{1100 \sqrt{3 \times 0.3 \times 3500}}$$

$$= 3.4 \times 10^{-5} \frac{c}{H_0} \approx 110 \left(\frac{0.7}{h} \right) \left(\frac{0.3}{\Omega_M} \right)^{1/2} \text{kpc}$$

Expands by factor
1 + z = 1100
to ~120 Mpc today.

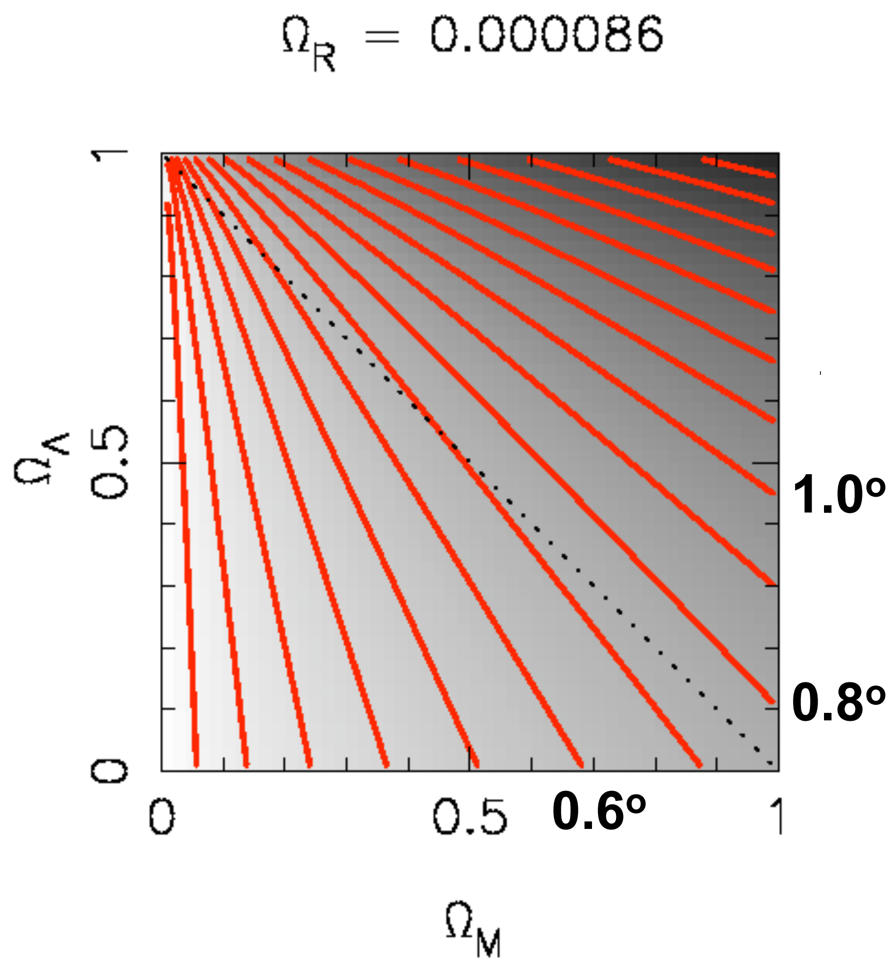
Angular Scale measures Ω_0

sound horizon : angular diameter distance :

$$L_S(z) = \frac{1}{1+z} \int_{1+z}^{\infty} \frac{c_S\, dx}{H(x)} \qquad D_A(z) = \frac{R_0\, S_K(\chi)}{1+z} \qquad \chi = \int_t^{t_0} \frac{c\, dt}{R(t)} = \frac{c}{R_0} \int_1^{1+z} \frac{dx}{H(x)}$$

angular scale

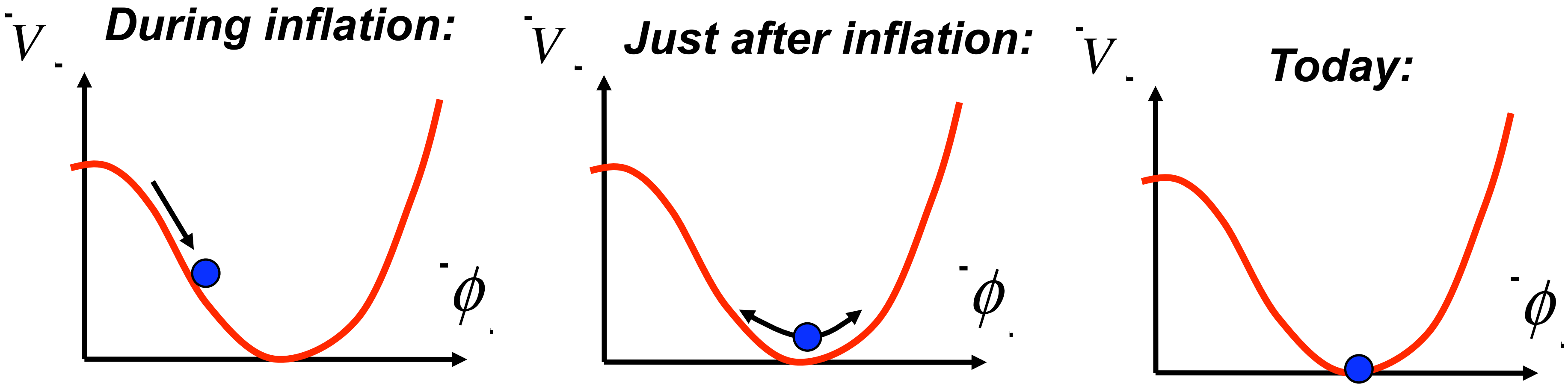
$$\theta = \frac{L_S(z)}{D_A(z)} = \frac{\displaystyle\int_{1+z}^{\infty} \frac{c_S\, dx}{H(x)}}{R_0\, S_k\left(\dfrac{c}{R_0} \displaystyle\int_1^{1+z} \frac{dx}{H(x)} \right)}$$

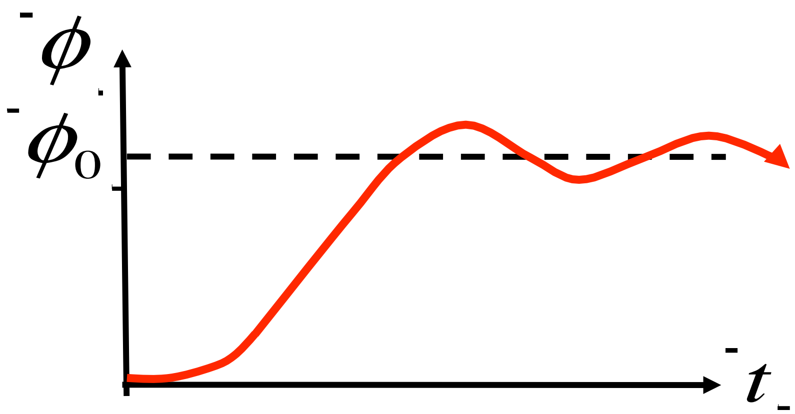


Angular scale depends mainly on the curvature.

Gives ⌧⌧⌧~ 0.8° for flat geometry, $\Omega_0 = \Omega_M + \Omega_\Lambda = 1$

Scalar Field Dynamics





Kinetic energy of the oscillations is damped.
Re-heats the Universe, creating all types of particle-antiparticle pairs, launching the Hot Big Bang.

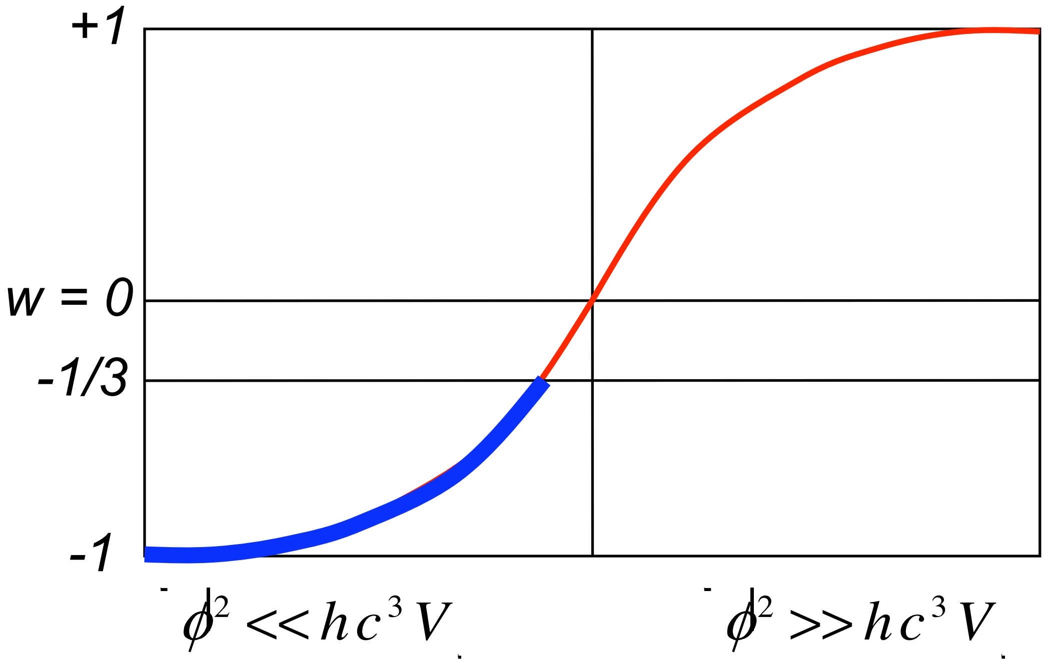
Scalar Field Equation of State

Equation of State: $w = p / \varepsilon$

Uniform field: $c^2 (\nabla \phi)^2 \ll \dot{\phi}^2$

$$p_\phi = \frac{1}{2} \frac{1}{\hbar c^3} (\dot{\phi}^2 + c^2 \nabla \varphi^2) - V(\phi)$$

$$\varepsilon_\phi = \frac{1}{2} \frac{1}{h c^3} (\dot{\phi}^2 + c^2 \nabla \varphi^2) + V(\phi)$$



Required for inflation:

$$w = \frac{p}{\varepsilon} = \frac{\dot{\phi}^2 - 2 h c^3 V}{\dot{\phi}^2 + 2 \hbar c^3 V} < -\frac{1}{3}$$

$$\varepsilon + 3p = 4 (\dot{\phi}^2 - \hbar c^3 V) < 0$$

Terminal velocity:

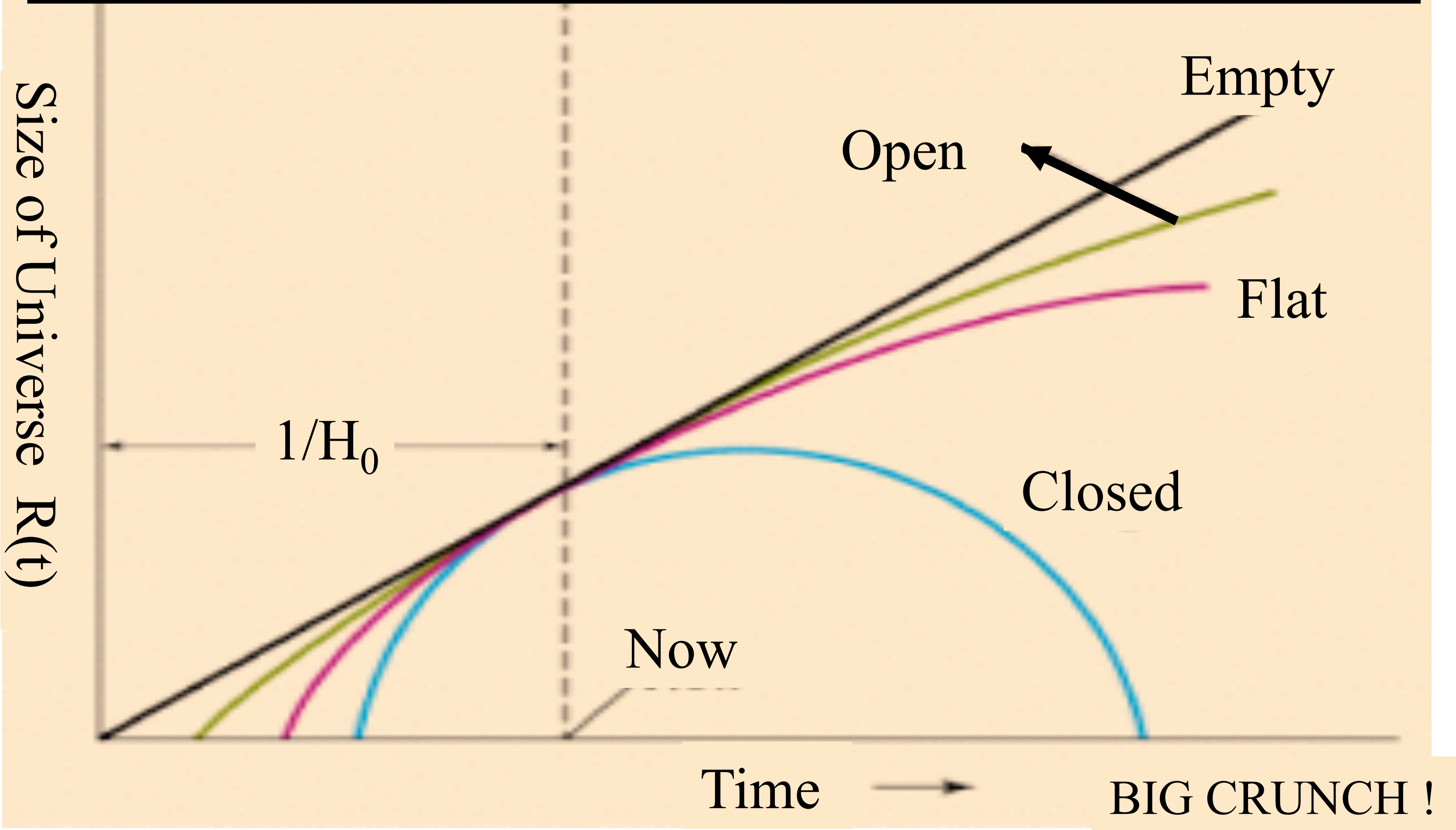
$$\dot{\phi} \Rightarrow \frac{h c^3}{3H} \frac{\partial V}{\partial \phi}$$

$$\left(\frac{\partial V}{\partial \phi} \right)^2 < \frac{9H^2}{\hbar c^3} V$$

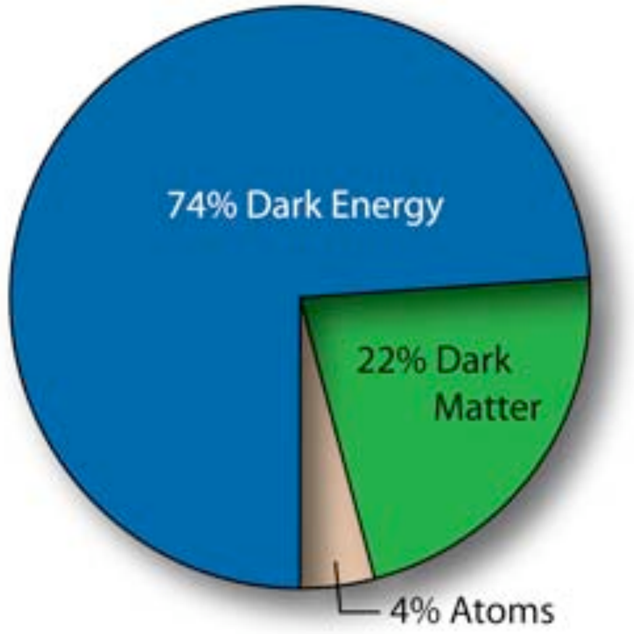
Need a long slow roll over a nearly flat potential.

Re-collapse or Eternal Expansion ?

Inflation => expect FLAT GEOMETRY
CRITICAL DENSITY



Or …. Has General Relativity Failed ?



Can an *Alternative Gravity* Model fit all the data without Dark Matter and Dark Energy ?

No luck yet, but people are trying.