4th Concept: The Energy density of Universe

The Universe is made up of three things:

- **VACUUM**
- **MATTER**
- **PHOTONS (radiation fields)**

The total energy density of the universe is made up of the sum of the energy density of these three components.

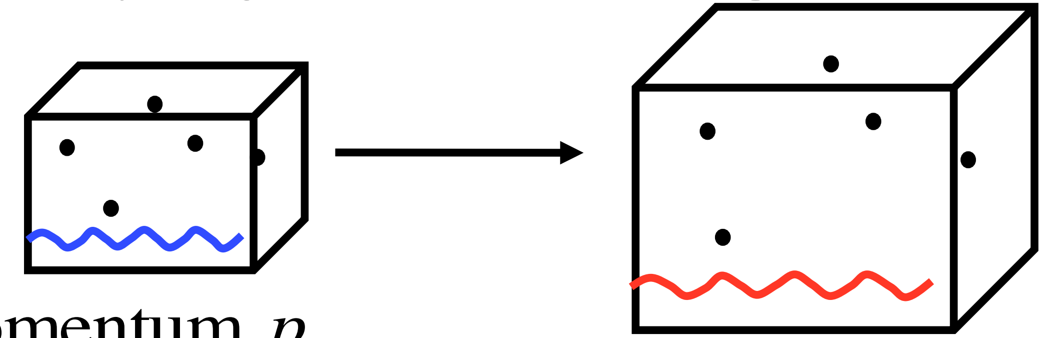
$$\varepsilon(t) = \varepsilon_{vac} + \varepsilon_{matter} + \varepsilon_{rad}$$

From t=0 to t=10^9 years the universe has expanded by R(t).

Energy Density of expanding box

volume R^3

N particles



particle mass m momentum p

energy $E = h\nu = \sqrt{m^2c^4 + p^2c^2} = m c^2 + \dfrac{p^2}{2m} + \ldots$

Cold gas or Cold DM ($p << mc$)

$$E \approx m c^2 = \text{const}$$

$$\varepsilon_M \approx \frac{N m c^2}{R^3} \propto R^{-3}$$

Radiation: ($m = 0$)

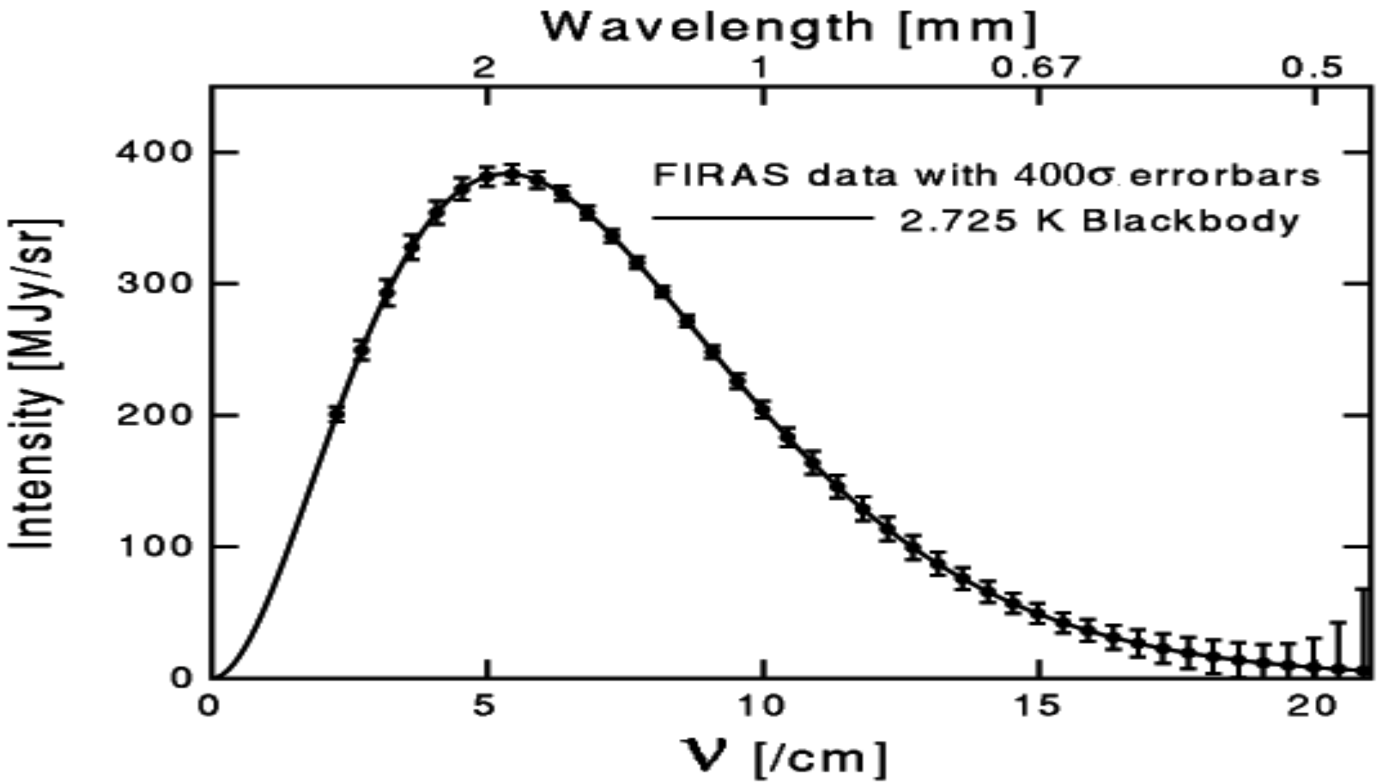
Hot neutrino: ($p >> mc > 0$)

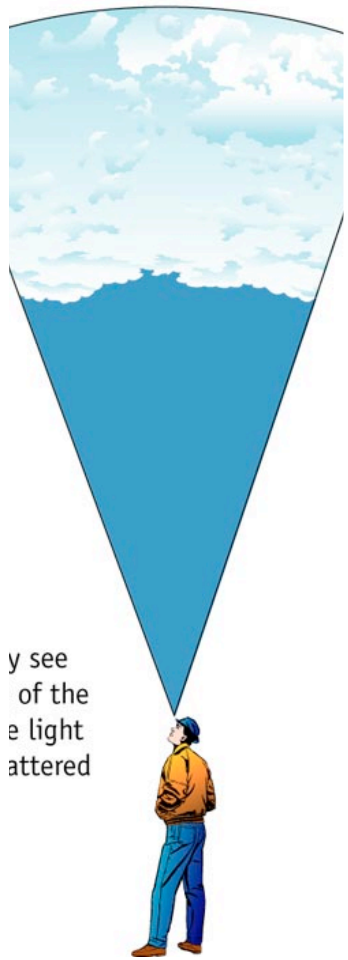
$\lambda \propto R$ (wavelengths stretch) :

$$E = h\nu = \frac{hc}{\lambda} \propto R^{-1}$$

$$\varepsilon_R = \frac{N h \nu}{R^3} \propto R^{-4}$$

COBE spectrum of CMB





A perfect Blackbody !

No spectral lines -- strong test of Big Bang.
Expansion preserves the blackbody spectrum.

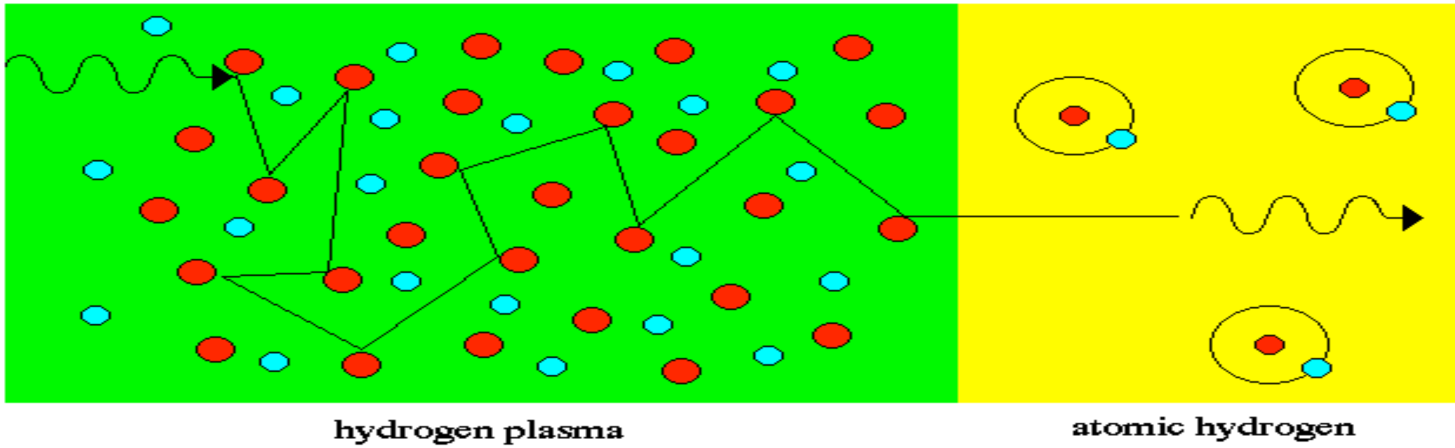
$$T(z) = T_0\,(1+z) \qquad T_0 \sim 3000\ K \quad z \sim 1100$$

Acronyms in Cosmology

- **Cosmic Background Radiation (CBR)**
  - **Or CMB (microwave because of present temperature 3K)**
    - Tutorial: Argue **about 10^5 photons fit in a 10cmx10cmx10cm microwave oven. [Hint: $3kT = h\, c / \lambda$]**

Last Scattering Epoch

As the Universe cooled, the free electrons and protons could finally bond togther to form hydrogen atoms. At the same time, the Universe went from a rich plasma to a gas of neutral hydrogen.



In a plasma, the mean free path of a photon is very short. In a gas of atomic hydrogen, the mean free path is very long, as long as the size of the Universe. Thus, the transition from the early plasma to atomic hydrogen is the epoch of last scattering, the point in time when the photons became free to travel without hindrance.

Cosmic Neutrino Background:

- neutrinos (Hot DM) decouple from electrons (due to very weak interaction) while still hot (relativistic 0.5 Mev ~ kT >mc^2 ~ 0.02-2 eV)
- Presently there are 3 x 113 neutrinos and 452 CMB photons per cm^3 . Details depend on
  - Neutrinos have 3 species of spin-1/2 fermions while photons are 1 species of spin-1 bosons
  - Neutrinos are a wee bit colder, 1.95K vs. 2.7K for photons [during freeze-out of electron-positions, more photons created]
- Initially mass doesn't matter in hot universe
  - relativistic (comparable to photon number density ~ R^{-3} ~ T^3),
  - frequent collisions with other species to be in thermal equilibrium and cools with photon bath.
  - Photon numbers (approximately) conserved, so is the number of relativistic massive particles

Concept: Particle-Freeze-Out?

Freeze-out of equilibrium means NO LONGER in thermal equilibrium.

Freeze-out temperature means a species of particles have the SAME TEMPERATURE as radiation up to this point, then they bifurcate.

Decouple = switch off the reaction chain

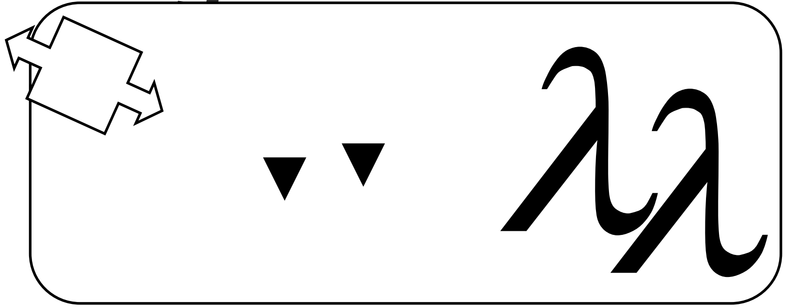
= insulation = Freeze-out

a massive particle

CDM/WIMPs: Cold Dark Matter, weakly-interact massive particles

- **If DM decoupled from photons at kT ~ 10^{14}K ~ 0.04 mc^2**
  - Then **that dark particles were**
    - **non-relativistic (v/c << 1), hence "cold".**
    - **And massive (m >> m_{proton} =1 GeV)**

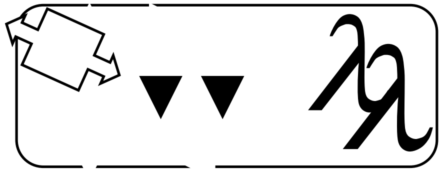
Eq. of State for Expansion & analogy of baking bread



Vacuum~air holes in bread

Matter ~nuts in bread

Photons ~words painted



Verify expansion doesn't change N_{hole}, N_{proton}, N_{photon}

- **No Change with rest energy of a proton, changes energy of a photon**

$$\varepsilon(t) = \rho_{eff}(t)c^2$$

$$\frac{\varepsilon(t)}{c^2} = \rho_{eff}(t)$$

VACUUM ENERGY: $\rho = \text{constant} \quad \Rightarrow \; \mathrm{E}_{vac} \propto R^3$

MATTER:

$$\rho R^3 = \text{constant}, \quad \Rightarrow \; m \approx \text{constant}$$

RADIATION: **number of photons N_{ph} = constant**

$$\text{Wavelength stretches}: \lambda \sim R$$

$$\Rightarrow n_{ph} \approx \frac{N_{ph}}{R^3}$$

$$\text{Photons}: \mathrm{E} = \mathrm{h}\upsilon = \frac{\mathrm{hc}}{\lambda} \sim \frac{1}{R}$$

$$\Rightarrow \varepsilon_{ph} \sim n_{ph} \times \frac{hc}{\lambda} \sim \frac{1}{R^4}$$

Total Energy Density *rho* c² =epsilon

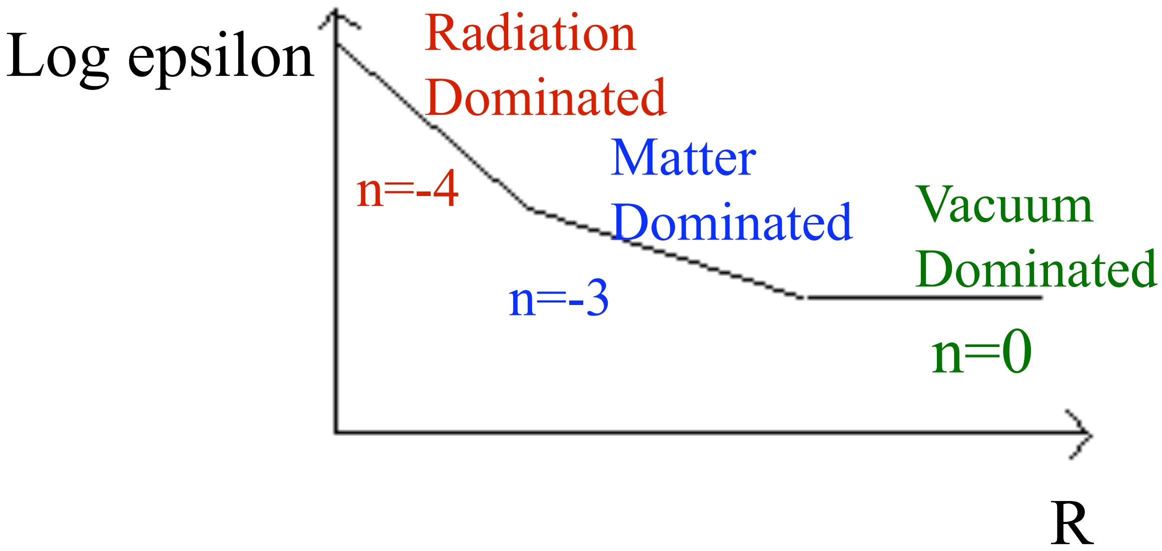
is given by:

$$\varepsilon \propto \varepsilon_{vac} + \varepsilon_{matter} + \varepsilon_{ph}$$

$\varepsilon_{vac} \propto R^0$

$\varepsilon_{matter} \propto R^{-3}$

$\varepsilon_{ph} \propto R^{-4}$



Tutorial: Typical scaling of expansion

$H^2=(dR/dt)^2/R^2=8\pi G\ (\rho_{cur}+ \rho_m + \rho_r + \rho_v)/3$

Assume domination by a component $\rho \sim R^{-n}$

Show Typical Solutions Are

$$\rho \propto R^{-n} \propto t^{-2}$$

$$n = 2(\text{curvature constant dominate})$$

$$n = 3(\textit{matter}\ \text{dominate})$$

$$n = 4(\textit{radiation}\ \text{dominate})$$

$$n \sim 0(\text{vaccum dominate}) : \ln(R) \sim t$$

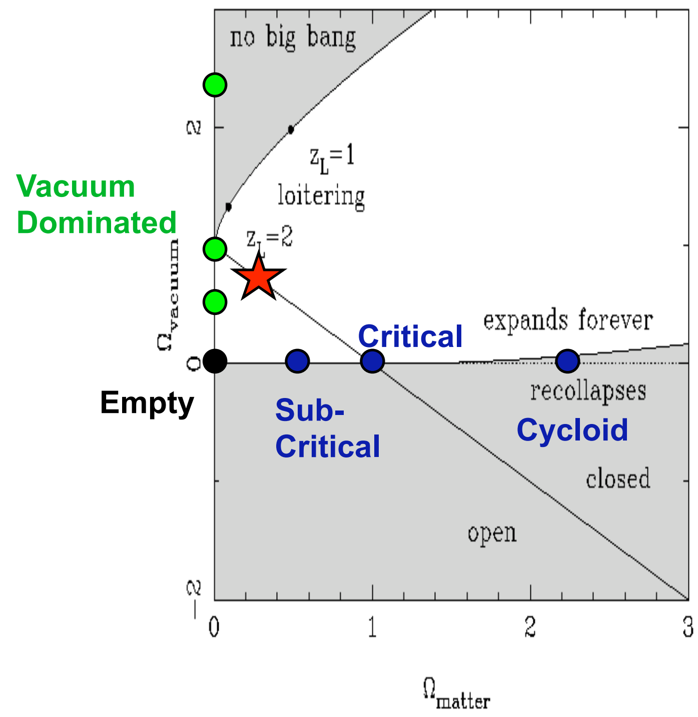
Argue also $H = (2/n)\ t^{-1} \sim t^{-1}$. Important thing is scaling!

Tutorial: Eternal Static (R=cst) and flat (k=0) Universe

- Einstein introduced Λ to enable an eternal static universe.

$$\dot{R}^2 = \left(\frac{8\pi G\,\rho + \Lambda}{3}\right) R^2 - k\,c^2$$

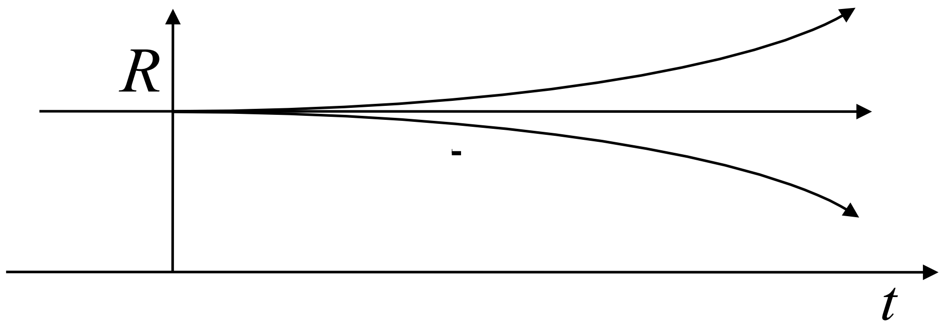
$$\dot{R} = 0 \quad \rightarrow \quad \Lambda = \frac{3\,k\,c^2}{R^2} - 8\pi G\,\rho$$



Einstein's biggest blunder. (Or, maybe not.)

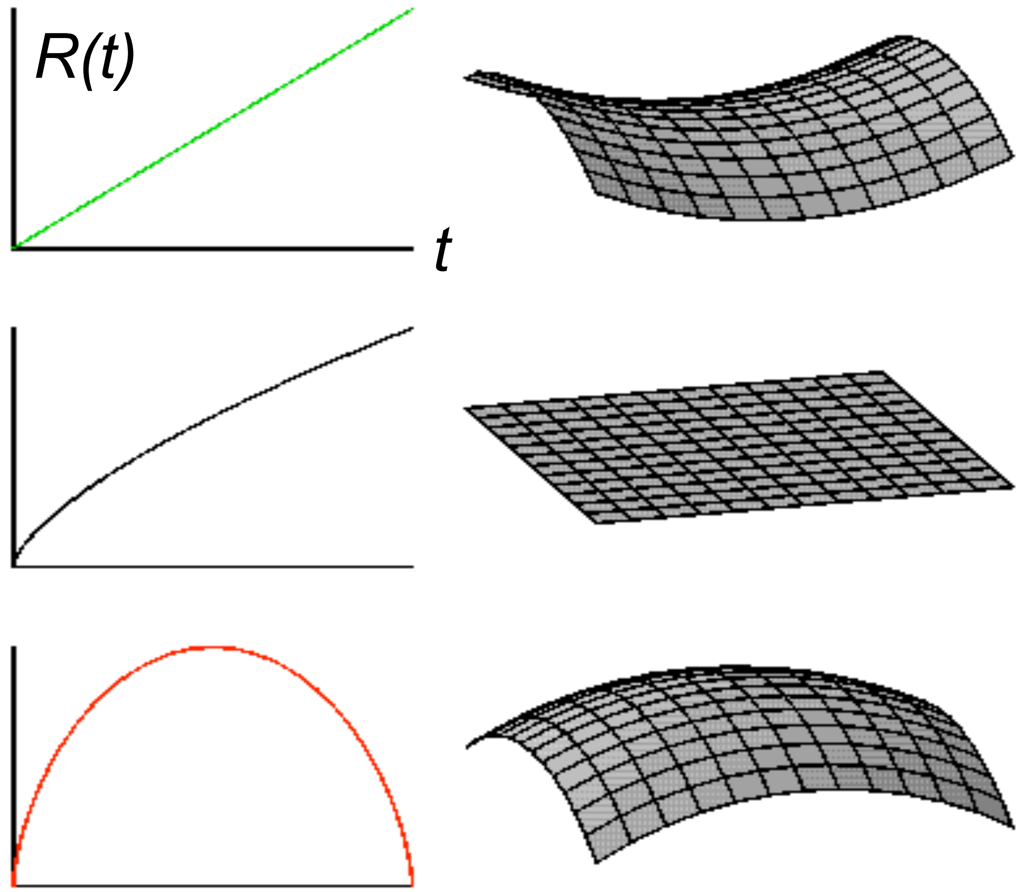
Static models unstable.

Fine tuning.



Density - Evolution - Geometry

$H^2 - (8\pi G/3)\,\rho = -kc^2 R^{-2}$, where $H=(dR/dt/R)$



$\rho < \rho_c$ — **Open** $k = -1$

$\rho = \rho_c$ — **Flat** $k = 0$

$\rho > \rho_c$ — **Closed** $k = +1$

$$\rho_c \equiv \frac{3\,H_0^{\,2}}{8\pi\,G} \approx 10^{-26}\,\mathrm{kg\ m^{-3}} \approx \frac{1.4\times10^{11}\,\mathrm{Msun}}{(\mathrm{Mpc})^3}$$

E.g.,: Empty Universe without vacuum

$$\dot{R}^2 = \left(\frac{8\pi\, G\, \rho + \Lambda}{3} \right) R^2 - k\, c^2$$

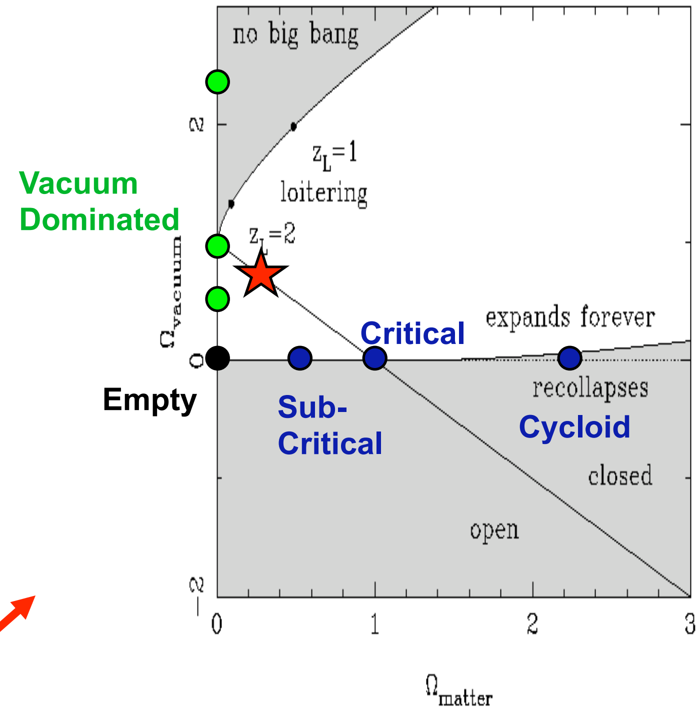
Set $\rho = 0, \quad \Lambda = 0.$ Then $\dot{R}^2 = -k\, c^2$

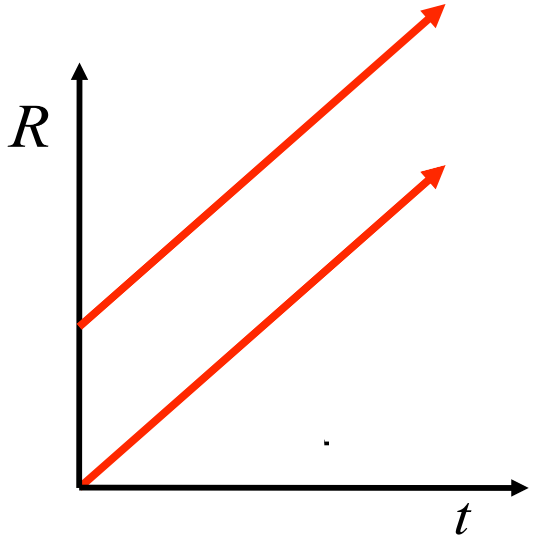
$\rightarrow \quad k = -1$ (negative curvature)

$$\dot{R} = c, \quad R = c\ t$$

$$H \equiv \frac{\dot{R}}{R} = \frac{1}{t}$$

$$\text{age}: \quad t_0 = \frac{R_0}{c} = \frac{1}{H_0}$$





Negative curvature drives rapid expansion/flattening

Four Pillars of Hot Big Bang

Galaxies moving apart from each other

- Redshift or receding from each other
- Universe was smaller.

Helium production outside stars

- Universe was hot, at least 3×10^9K to fuse 4H → He, to overcome a potential barrier of 1MeV.

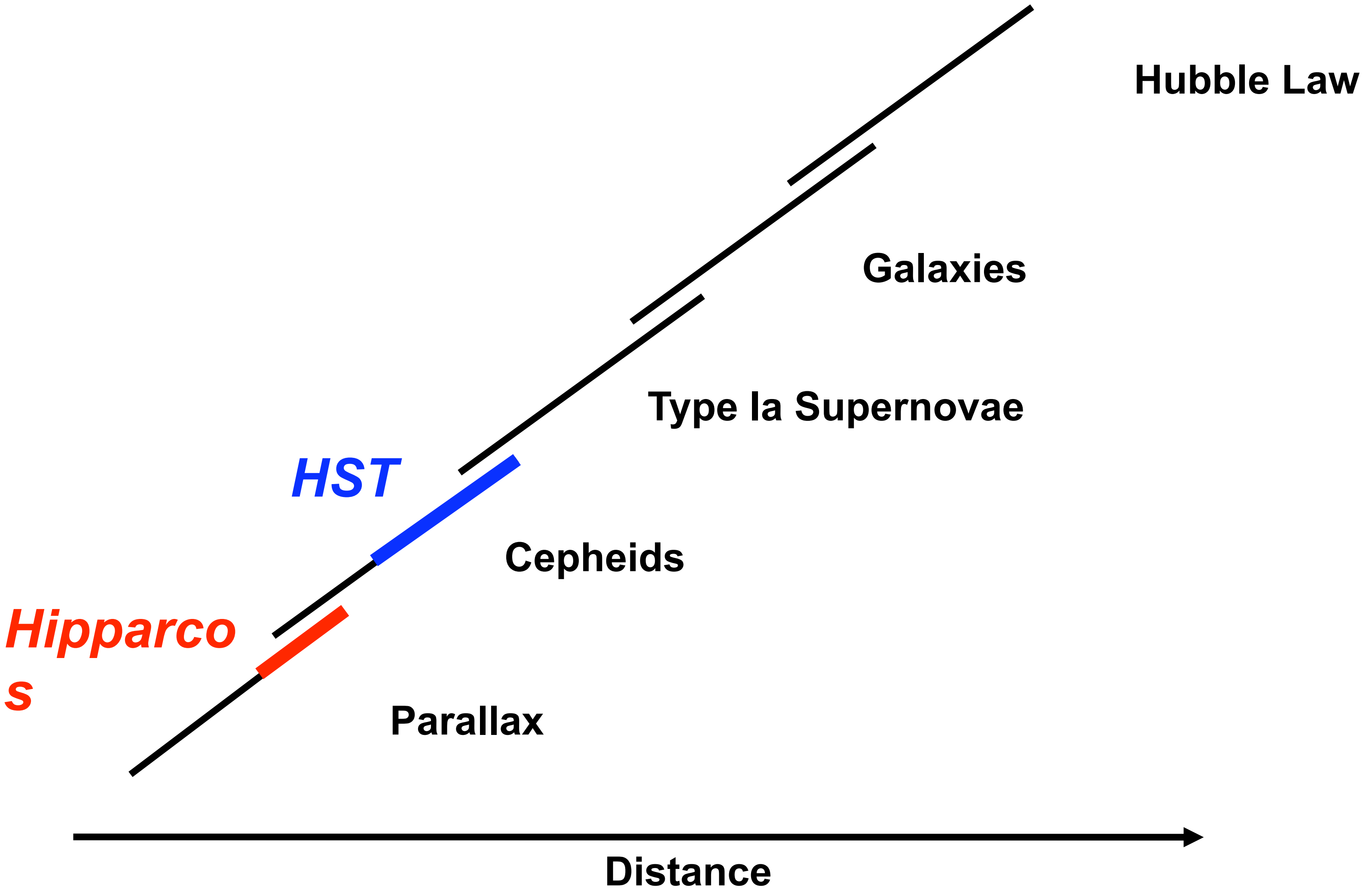
Nearly Uniform Radiation 3K Background (CMB)

- Universe has cooled, hence expanded by at least a factor 10^9. Photons (3K~10^{-5}eV) are only 10^{-3} of baryon energy density, so photon-to-proton number ratio ~ 10^{-3}(GeV/10^{-5}eV) ~ 10^9

Missing mass in galaxies and clusters (Cold DM)

- Cluster potential well is deeper than the potential due to baryons.
- CMB fluctuations: photons climb out of random potentials of DM.
- If 1/10 of the matter density in 1GeV protons, 9/10 in dark particles of e.g. 9GeV, then dark-to-proton number density ratio ~ 1

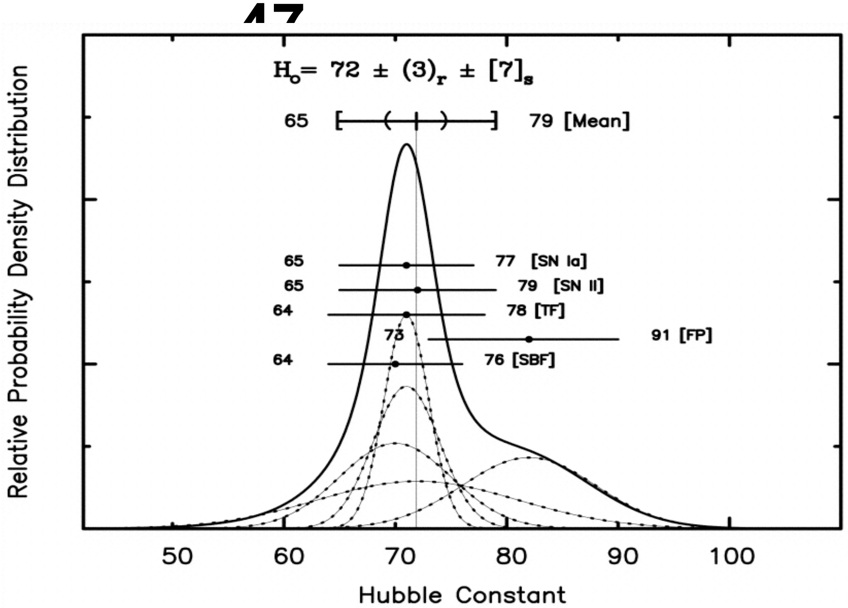
Cosmic Distance Ladder

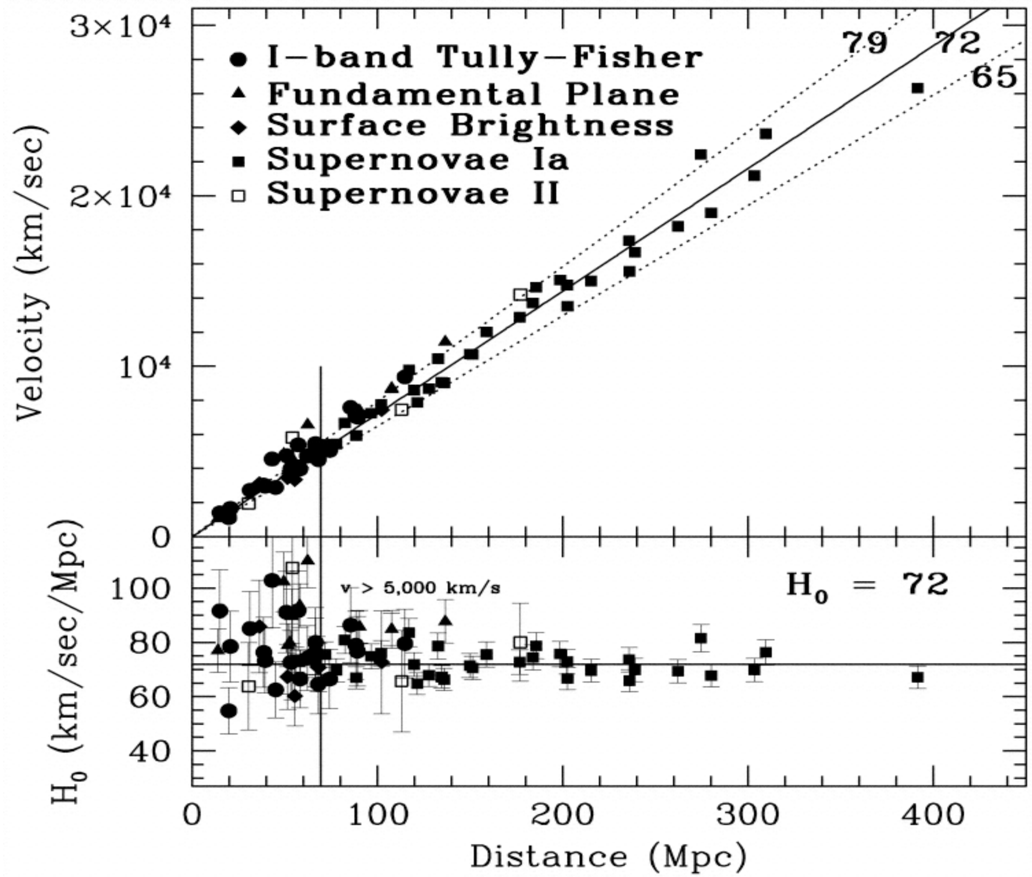


H_0 from the HST Key Project

$$H_0 \approx 72 \pm 3 \pm 7 \quad \text{km s}^{-1}\ \text{Mpc}^{-1}$$

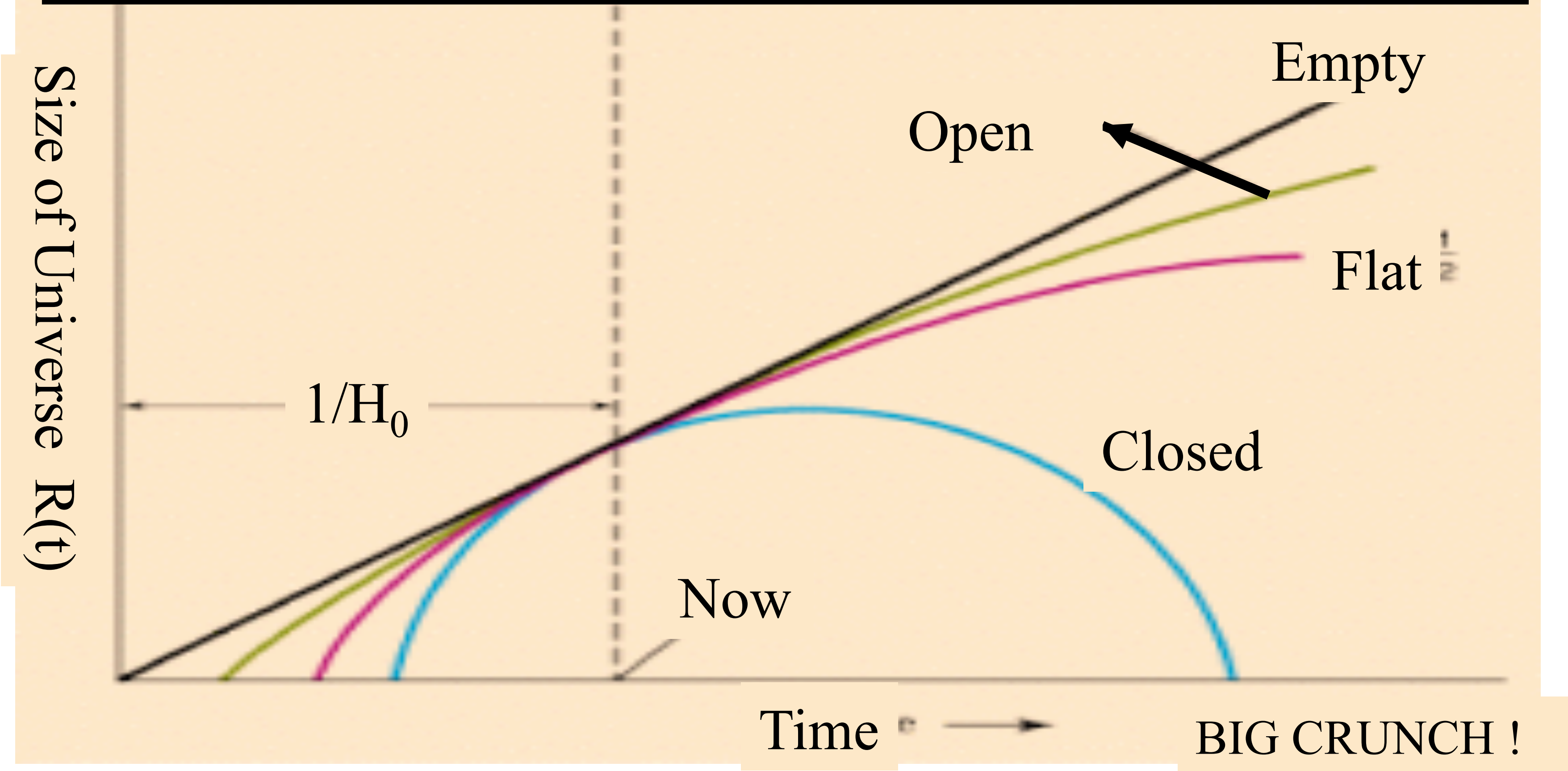
Freedman, et al. 2001 ApJ 553, 47





Re-collapse or Eternal Expansion ?

Inflation => expect FLAT GEOMETRY
CRITICAL DENSITY



Hubble Parameter Evolution -- H(z)

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k c^2}{R^2}$$

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda - \frac{k c^2}{H_0^{\,2} R_0^{\,2}} x^2$$

$$\text{evaluate at } x = 1 \quad \rightarrow \quad 1 = \Omega_0 - \frac{k c^2}{H_0^{\,2} R_0^{\,2}}$$

Dimensionless Friedmann Equation:

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1 - \Omega_0)\, x^2$$

$$x = 1 + z = R_0 / R$$

$$\rho_c = \frac{3 H_0^{\,2}}{8\pi G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}, \quad \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3 H_0^{\,2}}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$

Curvature Radius today:

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}} \quad \rightarrow \begin{cases} k = +1 & \Omega_0 > 1 \\ k = 0 & \Omega_0 = 1 \\ k = -1 & \Omega_0 < 1 \end{cases}$$

Density determines Geometry

Possible Universes

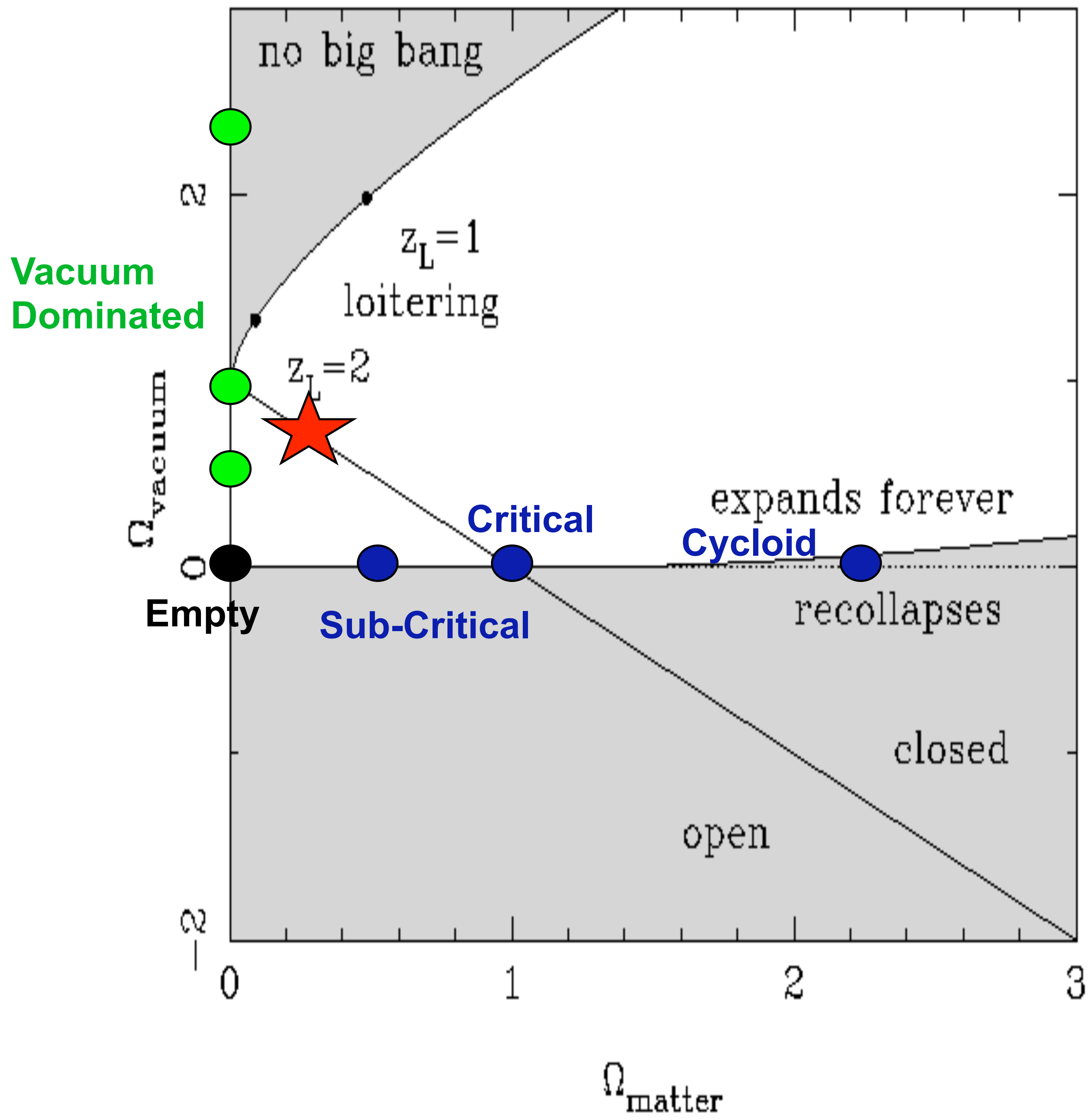
$$H_0 \approx 70 \frac{\text{km/s}}{\text{Mpc}}$$

$$\Omega_M \sim 0.3$$

$$\Omega_\Lambda \sim 0.7$$

$$\Omega_R \sim 8\times10^{-5}$$

$$\Omega = 1.0$$



Precision Cosmology

$h = 71 \pm 3$ expanding

$\Omega = 1.02 \pm 0.02$ flat

$\Omega_b = 0.044 \pm 0.004$ baryons

$\Omega_M = 0.27 \pm 0.04$ Dark Matter

$\Omega_\Lambda = 0.73 \pm 0.04$ Dark Energy

$t_0 = 13.7 \pm 0.2 \times 10^9$ yr now

$t_* = 180^{+220}_{-80} \times 10^6$ yr $z_* = 20^{+10}_{-5}$ reionisation

$t_R = 379 \pm 1 \times 10^3$ yr $z_R = 1090 \pm 1$ recombination

(From the WMAP 1-year data analysis)

Cosmology Milestones

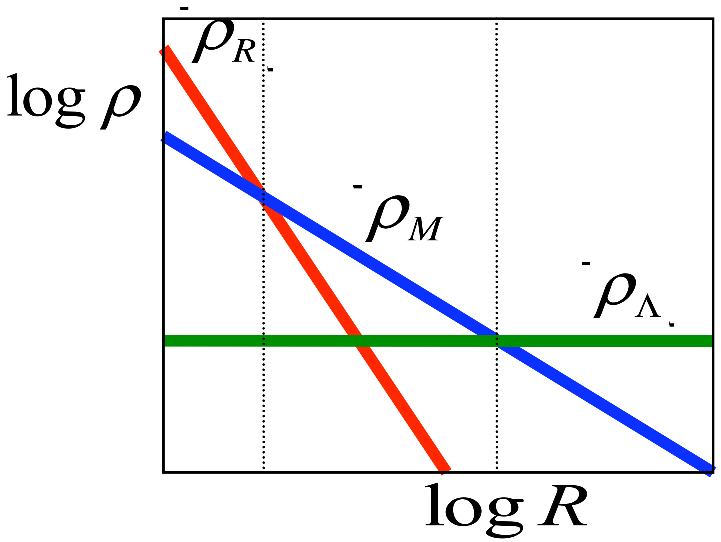
- 1925 Galaxy redshifts $\lambda = \lambda_0 \; (1 + z)$ $V = c\,z$
  - Isotropic expansion. (Hubble law $V = H_0 d$)
  - Finite age. ($t_0 = 13 \times 10^9$ yr)
- 1965 Cosmic Microwave Background (CMB)
  - Isotropic blackbody. $T_0 = 2.7$ K
  - Hot Big Bang $T = T_0 (1 + z)$
- 1925 General Relativity Cosmology Models :
  - Radiation era: $R \sim t^{1/2}$ $T \sim t^{-1/2}$
  - Matter era: $R \sim t^{2/3}$ $T \sim t^{-2/3}$
- 1975 Big Bang Nucleosynthesis (BBN)
  - light elements (^{1}H … ^{7}Li) $t \sim 3$ min $T \sim 10^9$ K
  - primordial abundances (75% H, 25% He) as observed!

Tutorial: 3 Eras: radiation-matter-vacuum

$$\text{radiation}: \quad \rho_R \propto R^{-4}$$

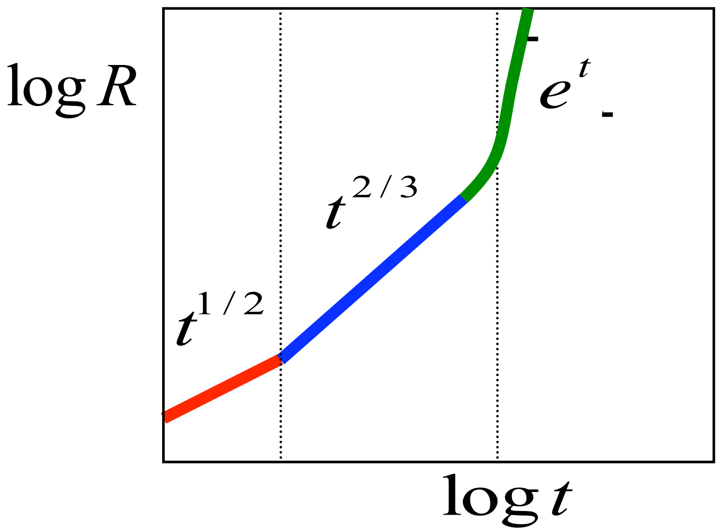
$$\text{matter}: \quad \rho_M \propto R^{-3}$$

$$\text{vacuum}: \quad \rho_\Lambda = \text{const}$$



$$a \equiv \frac{R}{R_0} = \frac{1}{1+z}$$

$$\rho = \frac{\rho_{R,0}}{a^4} + \frac{\rho_{M,0}}{a^3} + \rho_\Lambda$$



$$\rho_R = \rho_M \quad \text{at} \quad a \sim 10^{-4} \quad t \sim 10^4 \ \text{yr}$$

$$\rho_M = \rho_\Lambda \quad \text{at} \quad a \sim 0.7 \quad t \sim 10^{10} \ \text{yr}$$

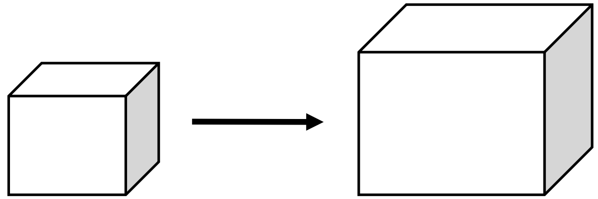
Presently vacuum is twice the density of matter.

5th concept: Equation of State w

Equation of state :

$$\rho \propto R^{-n} \qquad n = 3(1+w)$$

$$w \equiv \frac{\text{pressure}}{\text{energy density}} = \frac{p}{\rho c^2} = \frac{n}{3} - 1$$



Radiation : $(n = 4, w = 1/3)$

$$p_R = \frac{1}{3} \rho_R \ c^2$$

Matter : $(n = 3, w = 0)$

$$p_M \sim \rho_M \ c_S^{\ 2} << \rho_M \ c^2$$

Vacuum : $(n = 0, w = -1)$

$$p_\Lambda = -\rho_\Lambda \ c^2$$

Negative Pressure ! ?

$$d[energy] = work$$

$$d[\rho c^2 R^3] = -p \ d[R^3]$$

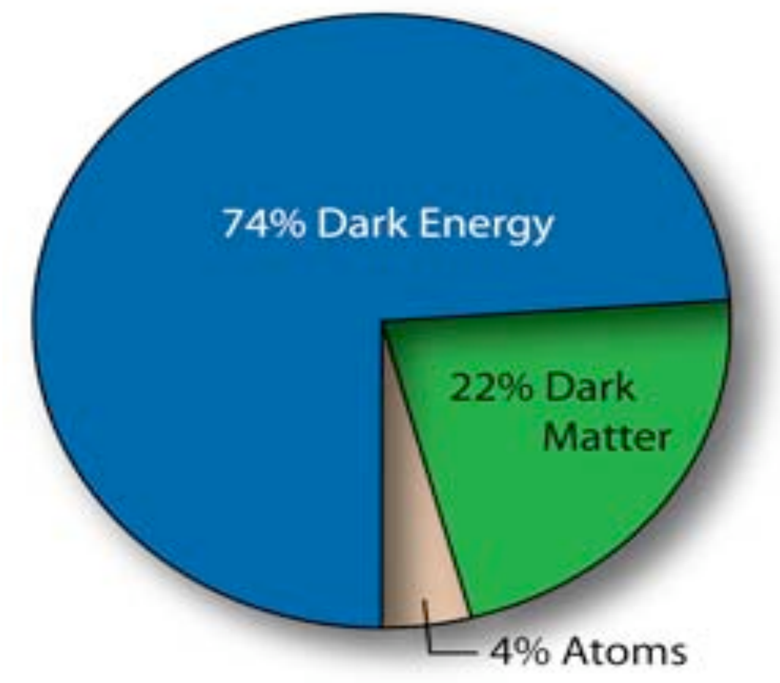
$$\rho c^2 \left(3 R^2 dR \right) + R^3 \ c^2 d\rho = -p \left(3 R^2 dR \right)$$

$$1 + \frac{R \, d\rho}{3 \rho \, dR} = -\frac{p}{\rho c^2} \equiv -w$$

$$w = -\frac{1}{3} \frac{d[\ln \rho]}{d[\ln R]} - 1$$

$$w = \frac{n}{3} - 1$$

Current Mysteries from Observations



Dark Matter ?

Holds Galaxies together
Triggers Galaxy formation

Dark Energy ?

Drives Cosmic Acceleration
and negative w.

Modified Gravity ?

General Relativity wrong ?

Density Parameters

critical density : density parameters (today) :

$$\rho_c \equiv \frac{3\,H_0^{\,2}}{8\pi\,G} \qquad \Omega_R \equiv \frac{\rho_R}{\rho_c} \quad \Omega_M \equiv \frac{\rho_M}{\rho_c} \quad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\,H_0^{\,2}}$$

total density parameter today :

$$\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda$$

density at a past/future epoch in units of today's critical density :

$$\Omega \equiv \frac{\rho}{\rho_c} = \sum_w \Omega_w x^{3(1+w)} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda \qquad x \equiv 1 + z = R_0 / R$$

in units of critical density at the past/future epoch :

$$\Omega(x) \equiv \frac{8\pi G\rho}{3H^2} = \frac{H_0^{\,2}}{H^2}\sum_w \Omega_w x^{3(1+w)} = \frac{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda}{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)\,x^2}$$

Note: radiation dominates at high z, can be neglected at lower z.

Key Points

- **Scaling Relation among**
  - **Redshift: z,**
  - **expansion factor: R**
    - **Distance between galaxies**
  - **Temperature of CMB: T**
    - **Wavelength of CMB photons: lambda**
- **Metric of an expanding 2D+time universe**
  - **Fundamental observers**
    - **Galaxies on grid points with fixed angular coordinates**
- **Energy density in**
  - **vacuum, matter, photon**
  - **How they evolve with R or z**
- **<u>If confused, recall the analogies of</u>**
  - **balloon, bread, a network on red giant star, microwave oven**

Sample a wide range of topics Theoretical and Observational

Universe of uniform density

- Metrics ds, Scale R(t) and Redshift
- EoS for mix of vacuum, photon, matter, geometry, distances

Thermal history

- Freeze-out of particles,
- Neutrinos, CDM wimps
- Nucleo-synthesis He/D/H

Structure formation

- Inflation and origin of perturbations
- Growth of linear perturbation
- Relation to CMB peaks, sound horizon

Quest of H0 (obs.)

- Applications of expansion models
- Distances Ladders

Cosmic Background

- COBE/MAP/PLANCK etc.
- Parameters of cosmos

Quest for Omega (obs.)

- Galaxy and SNe surveys
- Luminosity Functions

(thanks to slides from K. Horne)

6th concept: Distances in Non-Euclidean Curved Space

How Does Curvature affect Distance Measurements ?

Is the universe very curved?

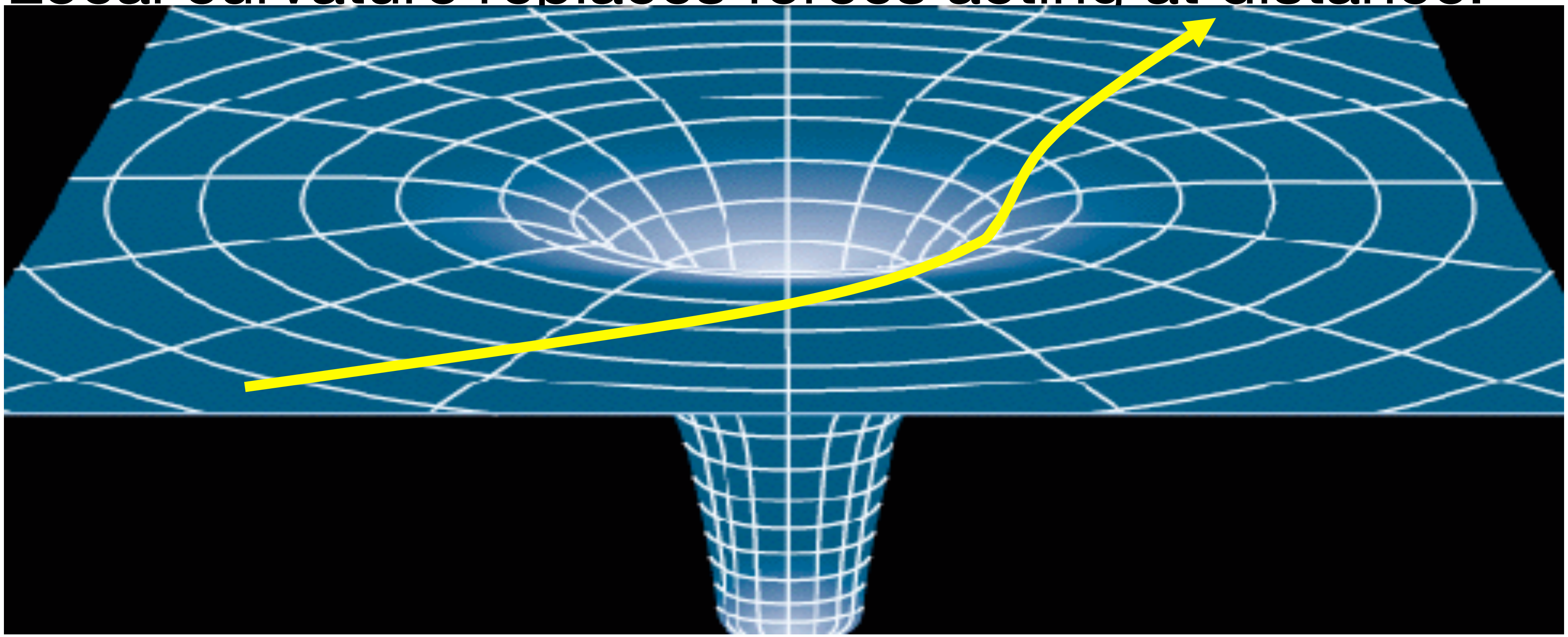
Geodesics

Gravity = curvature of space-time by matter/energy.

Freely-falling bodies follow **geodesic trajectories**.

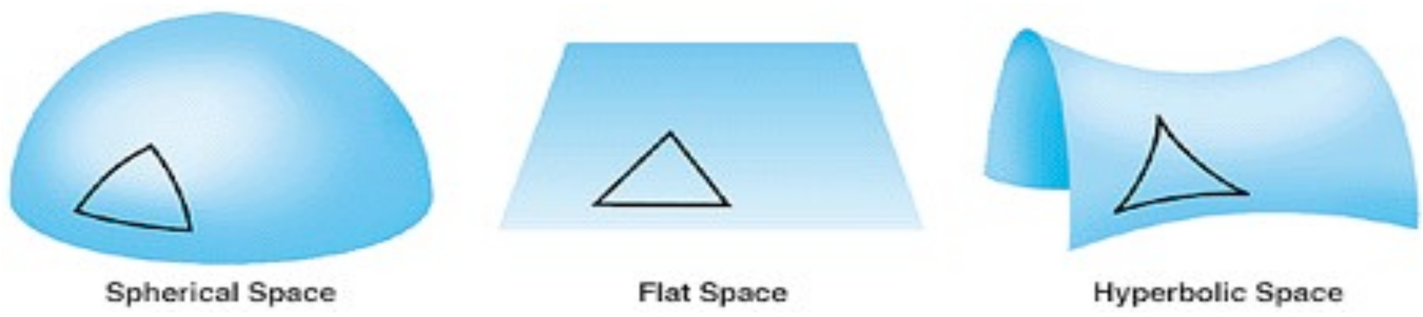
Shortest possible path in curved space-time.

Local curvature replaces forces acting at distance.



Is our Universe Curved?

	Closed	Flat	Open

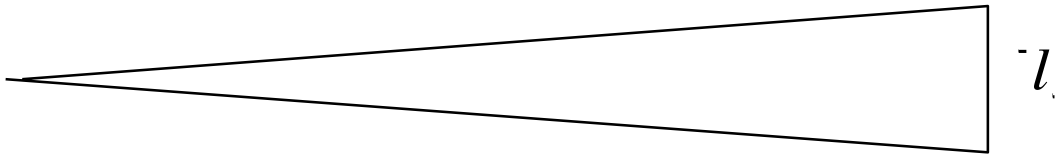


Curvature:	**+**	**0**	**--**
Sum of angles of triangle:	**> 180^{o}**	**= 180^{o}**	**< 180^{o}**
Circumference of circle:	**< 2 p *r***	**= 2 p *r***	**> 2 p *r***
Parallel lines:	**converge**	**remain parallel**	**diverge**
Size:	**finite**	**infinite**	**infinite**
Edge:	**no**	**no**	**no**

Distance Methods

- **Standard Rulers ==> Angular Size Distances**

$$\theta = \frac{l}{D}$$

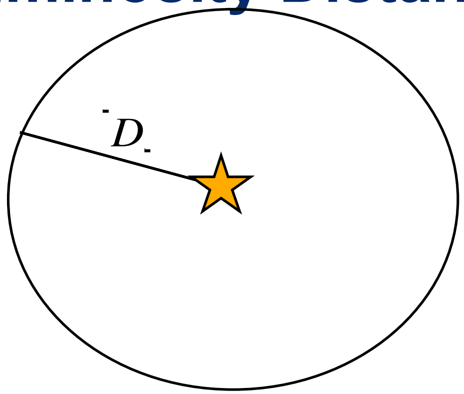


$$D_A = \frac{l}{\theta}$$

(for small angles << 1 radian)

- **Standard Candles ==> Luminosity Distances**

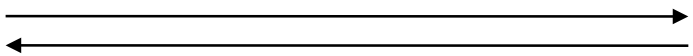
$$F = \frac{\text{energy/time}}{\text{area}} = \frac{L}{4\pi D^2}$$



$$D_L = \left(\frac{L}{4\pi F} \right)^{1/2}$$

- **Light Travel Time**

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{2D}{c}$$



$$D_t = \frac{c}{2t}$$

(e.g. within solar system)

Olber's Paradox

Why is the sky dark at night ?

Flux from all stars in the sky :

$$F = \int n_* \, F_* \, \mathrm{d}(\mathrm{Vol}) = \int_0^{\chi_{max}} n_* \left(\frac{L_*}{A(\chi)} \right) (A(\chi) \, R \, d\chi)$$

$$= n_* \, L_* \, R \, \chi_{max}$$

$$\Rightarrow \infty \quad \text{for flat space, } R \to \infty.$$

A dark sky may imply :

(1) an edge (we don't observe one)

(2) a curved space (finite size)

(3) expansion ($R(t)$ => finite age, redshift)

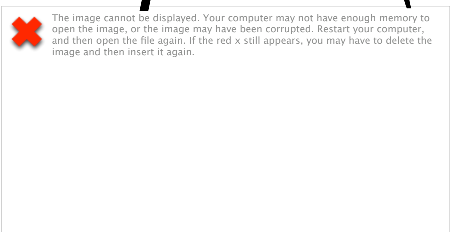
Minkowski Spacetime Metric

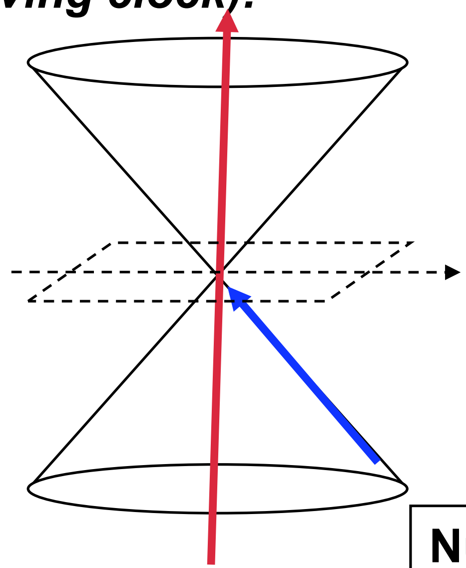
$$ds^2 = -c^2 dt^2 + dl^2$$

$$d\tau^2 = dt^2 - \frac{dl^2}{c^2} = dt^2\left(1 - \frac{1}{c^2}\left(\frac{dl}{dt}\right)^2\right)$$

Time-like intervals:
ds² < 0, dτ² > 0
Inside light cone.
Causally connected.

Proper time ***(moving clock):***





Space-like intervals:
ds² > 0 , dτ² < 0

Outside light cone.
Causally disconnected.

World line of massive particle at rest.

Null intervals light cone:
v = c, *ds² = 0*

Photons arrive from our past light cone.

Flat Space: Euclidean Geometry

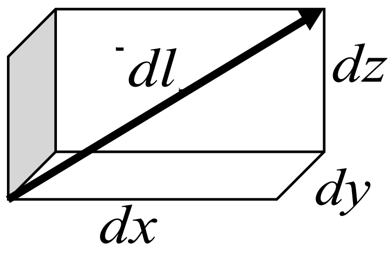
Cartesian coordinates :

$$1\,\mathrm{D}: \quad dl^2 = dx^2$$

$$2\,\mathrm{D}: \quad dl^2 = dx^2 + dy^2$$

$$3\,\mathrm{D}: \quad dl^2 = dx^2 + dy^2 + dz^2$$

$$4\,\mathrm{D}: \quad dl^2 = dw^2 + dx^2 + dy^2 + dz^2$$



Metric tensor : coordinates -> distance

$$dl^2 = (\,dx \quad dy \quad dz\,)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}dx\\ dy\\ dz\end{pmatrix}$$

Orthogonal coordinates
<--> diagonal metric

$$g_{xx} = g_{yy} = g_{zz} = 1$$

$$g_{xy} = g_{xz} = g_{yz} = 0$$

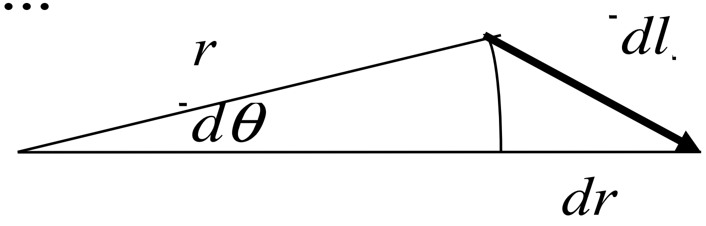
$$\text{symmetric}: \; g_{ij} = g_{ji}$$

Summation convention :

$$dl^2 = g_{ij}\, dx^i\, dx^j \equiv \sum_i \sum_j g_{ij}\, dx^i\, dx^j$$

Polar Coordinates

Radial coordinate r, angles $\phi, \theta, \alpha, \ldots$



1 D: $dl^2 = dr^2$

2 D: $dl^2 = dr^2 + r^2\ d\theta^2$

3 D: $dl^2 = dr^2 + r^2\ (d\theta^2 + \sin^2\theta\ d\phi^2)$

4 D: $dl^2 = dr^2 + r^2\ [d\theta^2 + \sin^2\theta\ (d\phi^2 + \sin^2\phi\ d\alpha^2)]$

$$dl^2 = dr^2 + r^2\ d\psi^2 \qquad \text{generic angle: } d\psi^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \ldots$$

$$dl^2 = (\, dr \quad d\theta \quad d\phi\,)\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

$g_{rr} = ?\quad g_{r\theta} = ?$

$g_{\theta\theta} = ?$

$g_{\phi\phi} = ?$

$g_{\alpha\alpha} = ?$

metric of space embedded in Sphere of radius R

R = radius of curvature

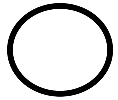
$1-\mathrm{D}: \quad R^2 = x^2$

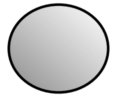
$2-\mathrm{D}: \quad R^2 = x^2 + y^2$

$3-\mathrm{D}: \quad R^2 = x^2 + y^2 + z^2$

$4-\mathrm{D}: \quad R^2 = x^2 + y^2 + z^2 + w^2$

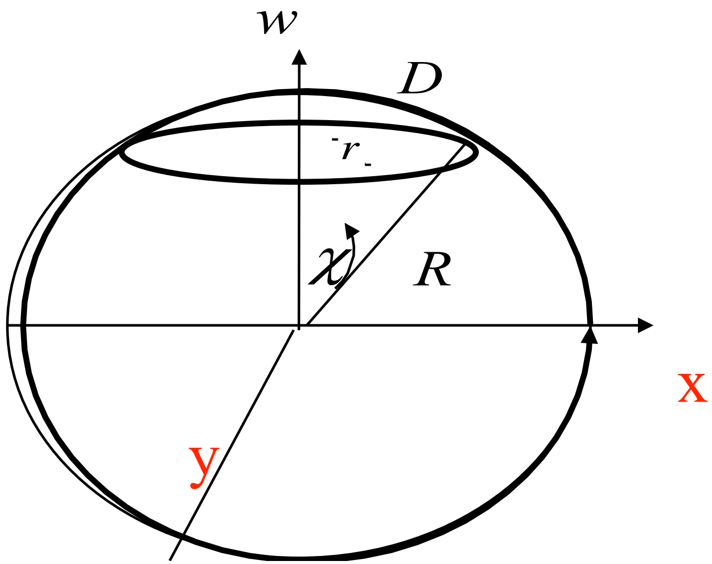
0 - D 2 points 

1 - D circle 

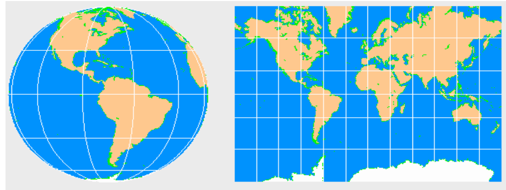
2 - D surface of 3 - sphere 

3 - D surface of 4 - sphere

?



coordinate systems



Distance varies in time:

$$D(t)$$

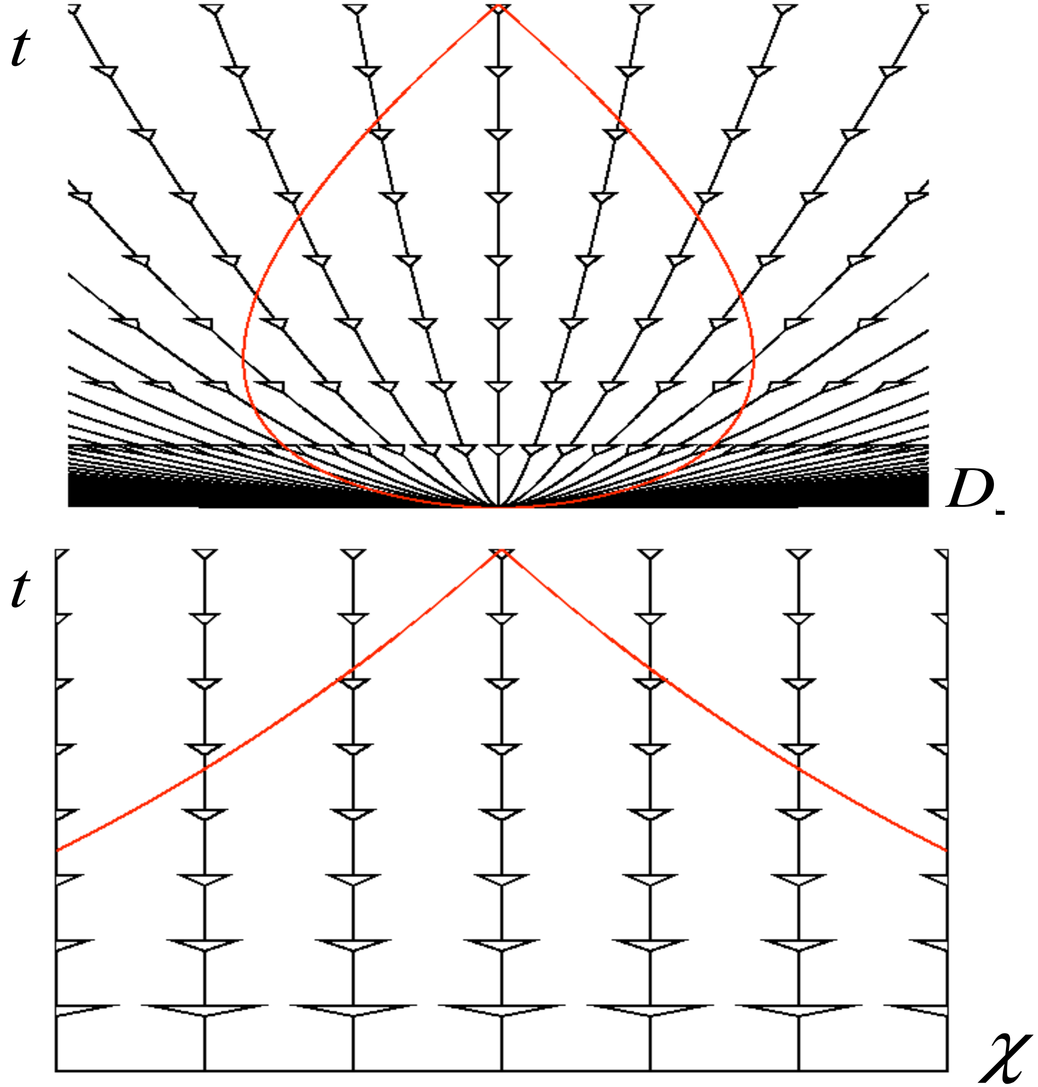
"Fiducial observers" (Fidos)

$$D(t) = R(t)\,\chi$$

"Co-moving" coordinates

$$\chi \quad \text{or} \quad D_0 \equiv R_0\,\chi$$

Labels the Fidos



Reading: Non-Euclidean Metrics

$$k = -1, 0, +1 \quad (\text{ open, flat, closed })$$

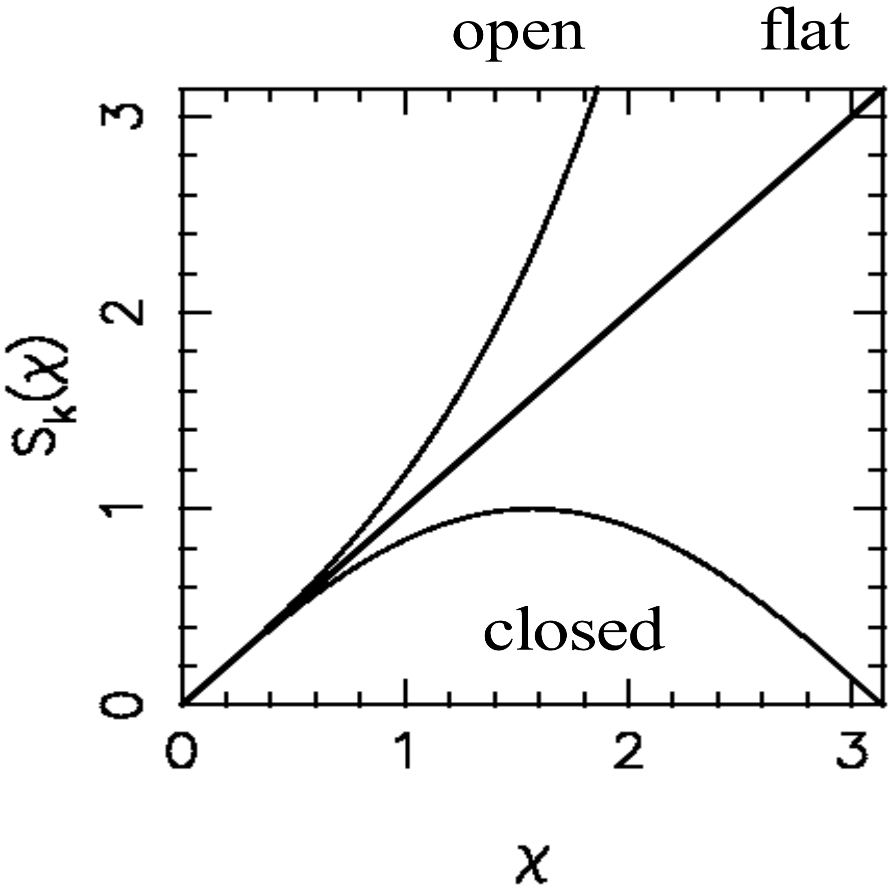
$$dl^2 = \frac{dr^2}{1 - k\,(r/R)^2} + r^2 d\psi^2$$

dimensionless radial coordinates :

$$u = r/R = S_k(\chi)$$

$$dl^2 = R^2 \left(\frac{du^2}{1 - k\,u^2} + u^2 d\psi^2 \right)$$

$$= R^2 \left(d\chi^2 + S_k^2(\chi)\, d\psi^2 \right)$$



$$S_{-1}(\chi) \equiv \sinh(\chi) \ , \quad S_0(\chi) \equiv \chi \ , \quad S_{+1}(\chi) \equiv \sin(\chi)$$

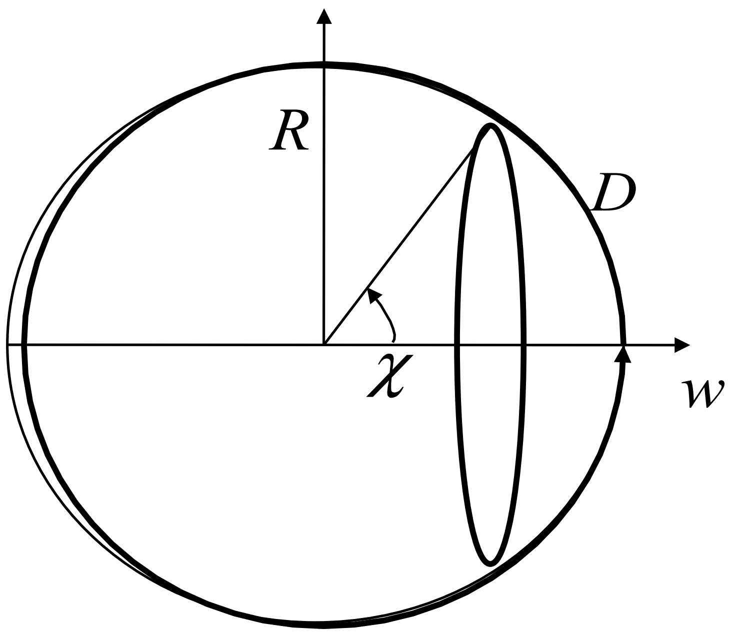
Reading: Circumference

metric :

$$dl^2 = \frac{dr^2}{1 - k\,(r/R)^2} + r^2\, d\theta^2$$

radial distance (for $k = +1$):

$$D = \int_0^r \frac{dr}{\sqrt{1 - k\,(r/R)^2}} = R \sin^{-1}(r/R)$$



circumference :

$$C = \int_0^{2\pi} r\, d\theta = 2\pi\, r$$

"circumferencial" distance : $r \equiv \dfrac{C}{2\pi} = R\, S_k(D/R) = R\, S_k(\chi)$

If $k = +1$, coordinate r breaks down for $r > R$

Reading: Circumference

metric :

$$dl^2 = R^2 \left(d\chi^2 + S_k^2(\chi)\, d\theta^2 \right)$$

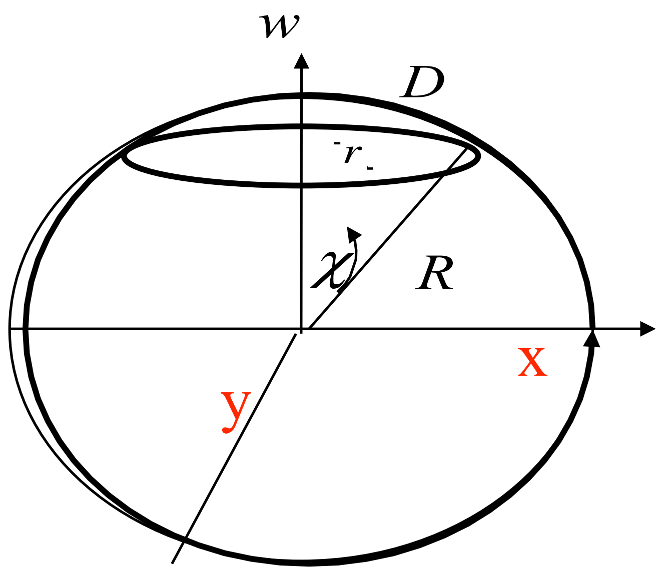
radial distance :

$$D = \int \sqrt{g_{\chi\chi}}\, d\chi = \int_0^{\chi} R\, d\chi = R\,\chi$$

circumference :

$$C = \oint \sqrt{g_{\theta\theta}}\, d\theta = \int_0^{2\pi} R\, S_k(\chi)\, d\theta = 2\pi\, R\, S_k(\chi)$$

$$= 2\pi\, D\, \frac{S_k(\chi)}{\chi}$$



Same result for any choice of coordinates.

Reading: Angular Diameter

metric :

$$dl^2 = R^2 \, (\, d\chi^2 + S_k^2(\chi) \, d\theta^2 \,)$$

radial distance :

$$D = \int \sqrt{g_{\chi\chi}} \, d\chi = \int_0^{\chi} R \, d\chi = R \, \chi$$

linear size : ($l << D$)

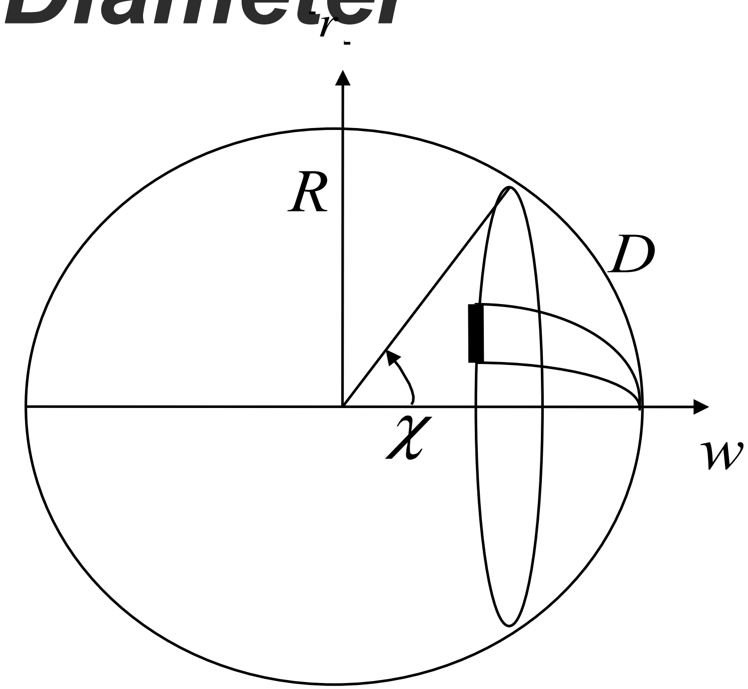
$$l = \int \sqrt{g_{\theta\theta}} \, d\theta = R \, S_k(\chi) \, \theta$$

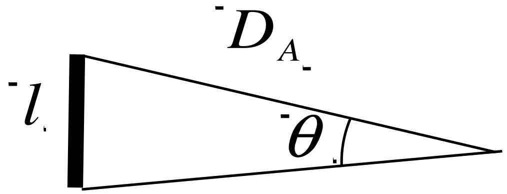
angular size :

$$\theta = \frac{l}{D_A}$$

$D = R \, \chi =$ Radial Distance

$D_A = R \, S_k(\chi) =$ Angular Diameter Distance





Reading: Area of Spherical Shell

radial coordinate χ, angles θ, ϕ :

$$dl^2 = R^2 \left[d\chi^2 + S_k^2(\chi) \, (d\theta^2 + \sin^2\theta \, d\phi^2) \right]$$

area of shell :

$$A = \int \sqrt{g_{\theta\theta}} \, d\theta \sqrt{g_{\phi\phi}} \, d\phi$$

$$= R^2 \, S_k^2(\chi) \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi$$

$$= 4\pi \, R^2 \, S_k^2(\chi)$$

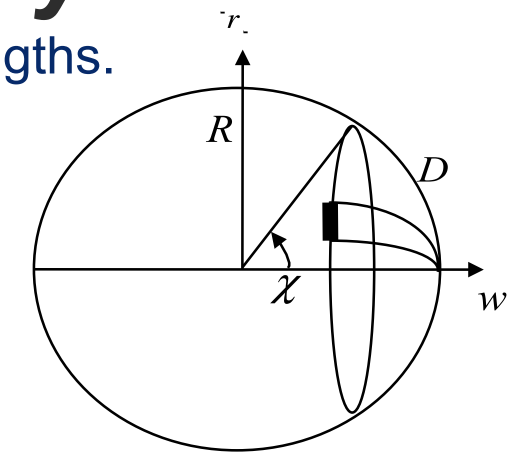
flux :

$$F = \frac{L}{A} = \frac{L}{4\pi \, D_L^2}$$

$$D_L = R \, S_k(\chi) = \text{ Luminosity Distance}$$

[we will work with flats only] Curved Space Summary

- The **metric** converts coordinate steps (grids) to physical lengths.
- Use the metric to compute lengths, areas, volumes, …

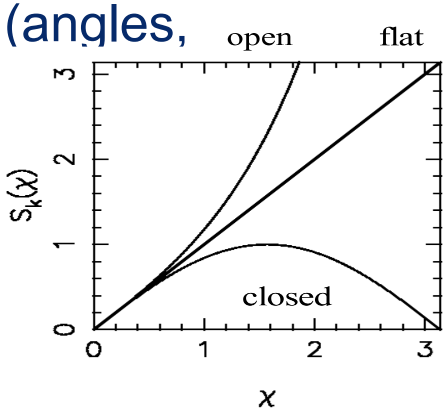


- Radial distance: $$D \equiv \int \sqrt{g_{rr}}\, dr = R\,\chi$$
- "Circumferencial" distance

$$r \equiv \frac{C}{2\pi} = \left(\frac{A}{4\pi}\right)^{1/2} = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi = R\,S_k(\chi) = R\,S_k(D/R)$$

- "Observable" distances, defined in terms of local observables (angles, fluxes), give r, not D.

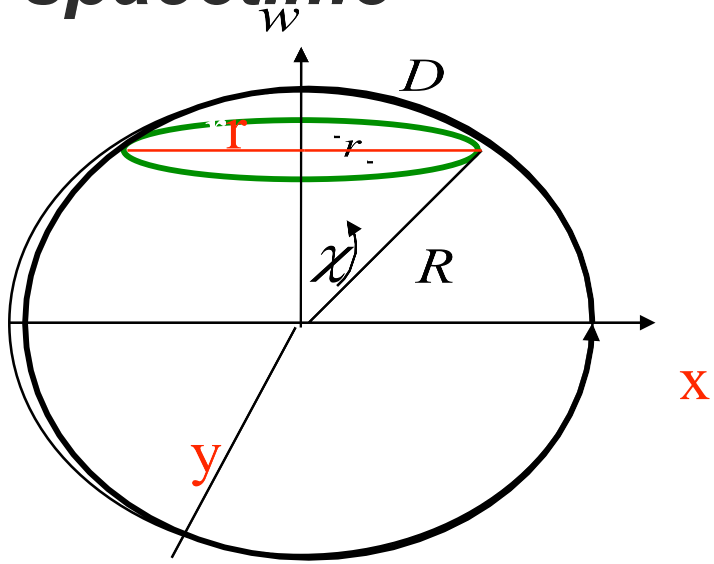
$$D_A \equiv \frac{l}{\theta} = r \qquad D_L \equiv \left(\frac{L}{4\pi F}\right)^{1/2} = r$$



- r < D (positive curvature, $S_{+1}(x)=\sin x$) (negative, $S_{-1}(x)=\sinh x$) or r=D (flat, $S_0(x)=x$)

7th Concept: Robertson-Walker metric
uniformly curved, evolving spacetime

$$ds^2 = -c^2 dt^2 + R^2(t)\,\left(d\chi^2 + S_k^2(\chi)\,d\psi^2\right)$$



$$S_k(\chi) = \begin{cases} \sin\chi & (k=+1) \quad \text{closed} \\ \chi & (k=0) \quad \text{flat} \\ \sinh\chi & (k=-1) \quad \text{open} \end{cases}$$

$$d\psi^2 \equiv d\theta^2 + \sin^2\theta\,d\phi^2$$

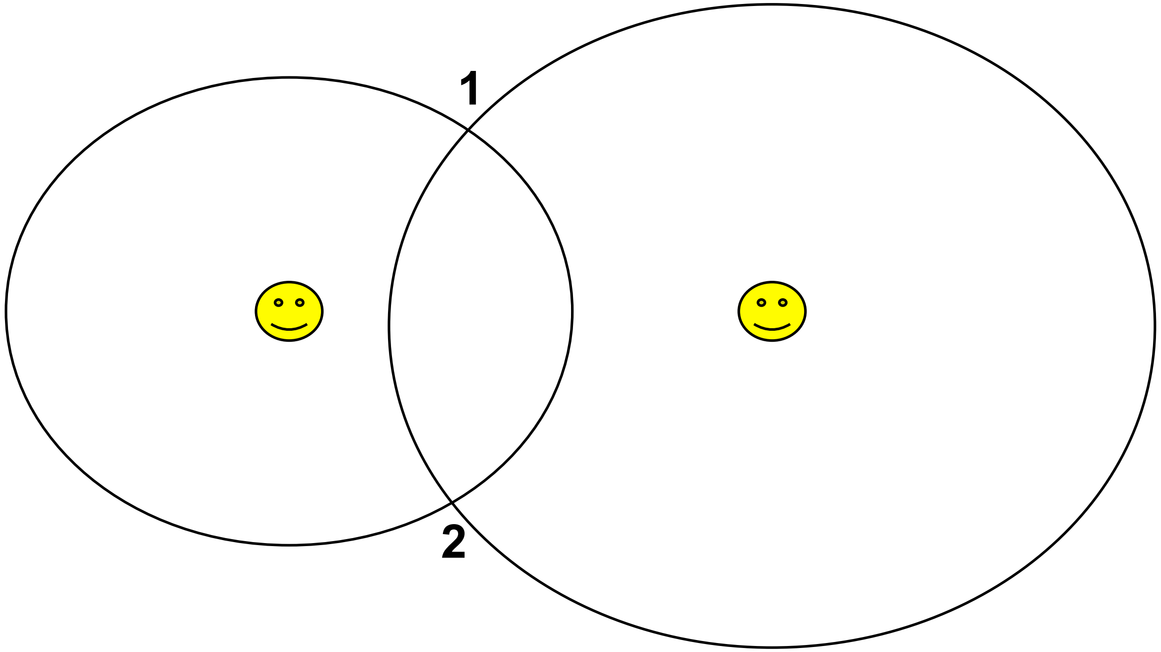
$$a(t) \equiv R(t)/R_0$$

$$R_0 \equiv R(t_0)$$

$$\text{radial distance} = D(t) = R(t)\,\chi$$

$$\text{circumference} = 2\pi\,r(t) \qquad r(t) = \ldots = R(t)\,S_k(\chi)$$

Cosmological Principle (assumed) + Isotropy (observed) => Homogeneity



$\rho_1 = \rho_2$ otherise not isotropic for equidistant fidos

Distances-Redshift relation

- We observe the **redshift :**

$$z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 \qquad \lambda = \text{observed},$$

$$\lambda_0 = \text{emitted (rest)}$$

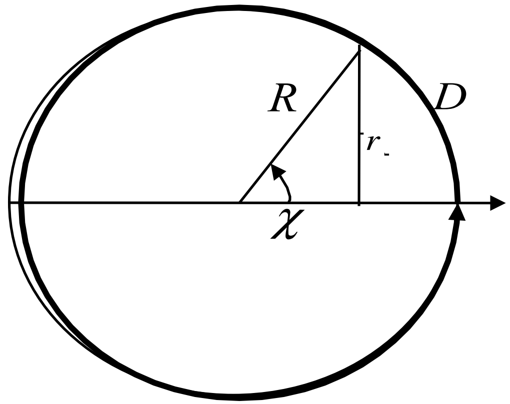
- Hence we know the **expansion factor:**

$$x \equiv 1 + z = \frac{\lambda}{\lambda_0} = \frac{\lambda(t_0)}{\lambda(t)} = \frac{R(t_0)}{R(t)} = \frac{R_0}{R(t)}$$

- Need the time of light emitted $\quad t(z) = ?$
- Need coordinate of the source $\quad \chi(z) = ?$
- Need them as functions of $\quad H_0 \quad \Omega_M \quad \Omega_\Lambda$
- Distances

$$D(t,\chi) = R(t)\,\chi \qquad D_A = r_0(\chi)/(1+z)$$

$$r(t,\chi) = R(t)\,S_k(\chi) \qquad D_L = r_0(\chi)(1+z)$$



- E.g. **D_L is 4 x D_A for an object at z=1.**

Tutorial: Time -- Redshift relation

$$x = 1 + z = \frac{R_0}{R}$$

$$\frac{dx}{dt} = -\frac{R_0}{R^2}\frac{dR}{dt}$$

$$= -\frac{R_0}{R}\frac{\dot{R}}{R}$$

$$= -x\ H(x)$$

$$\therefore\ dt = \frac{-dx}{x\ H(x)} = \frac{-dz}{(1+z)\ H(z)}$$

Memorise this derivation!

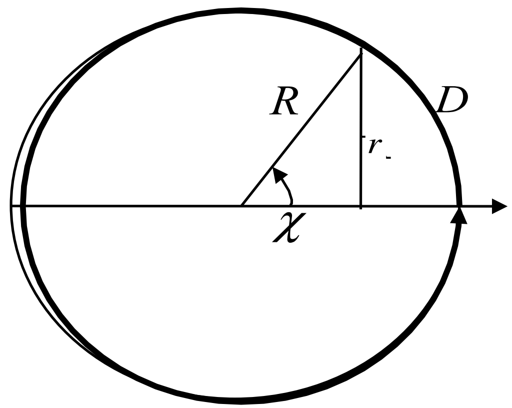
Hubble parameter : $H \equiv \frac{\dot{R}}{R}$

Tutorial: Time and Distance vs Redshift

$$\frac{d}{dt}\left(x = 1 + z = \frac{R_0}{R} \right) \rightarrow \quad dt = \frac{-dx}{x\,H(x)}$$

Look - back time :

$$t(z) = \int_t^{t_0} dt = \int_{1+z}^{1} \frac{-dx}{x\,H(x)} = \int_1^{1+z} \frac{dx}{x\,H(x)}$$



Age: $t_0 = t(z \rightarrow \infty)$

Distance : $D = R\,\chi \qquad r = R\,S_k(\chi)$

$$\chi(z) = \int d\chi = \int_t^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{R_0} \int_1^{1+z} \frac{R_0}{R(t)} \frac{dx}{x\,H(x)} = \frac{c}{R_0} \int_1^{1+z} \frac{dx}{H(x)}$$

Horizon : $\chi_H = \chi(z \rightarrow \infty)$

Need to know *R(t)*, or R_0 and *H(x)*.