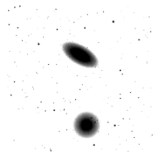
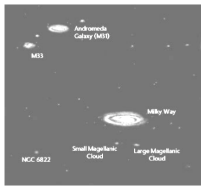
Gravitational Dynamics: An Introduction HongSheng Zhao

C1.1.1 Our Galaxy and Neighbours



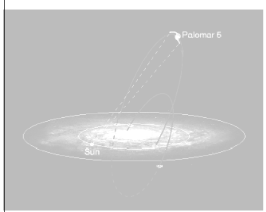


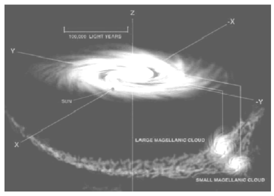
- How structure in universe form/evolve?
- Galaxy Dynamics Link together early universe & future.

Our Neighbours

- M31 (now at 500 kpc) separated from MW a Hubble time ago
- Large Magellanic Cloud has circulated our Galaxy for about 5 times at 50 kpc
  - **argue** both neighbours move with a typical 100-200km/s velocity relative to us.

Outer Satellites on weak-g orbits around Milky Way





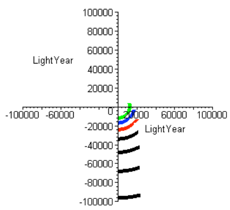
R>10kpc: Magellanic/Sgr/Canis streams
R>50kpc: Draco/Ursa/Sextans/Fornax...

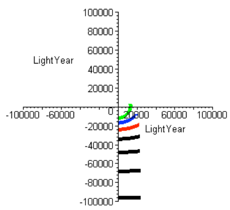
~ 50 globulars on weak-g (R<150 kpc)
~100 globulars on strong-g (R< 10 kpc)

C1.1.2 Milky Way as Gravity Lab

- Sun has circulated the galaxy for 30 times
  - velocity vector changes direction +/- 200km/s twice each circle (R = 8 kpc)
  - **Argue** that the MW is a nano-earth-gravity Lab
  - **Argue** that the gravity due to 10^{10} stars only within 8 kpc is barely enough. Might need to add Dark Matter.

Sun escapes unless our Galaxy has Dark Matter





C1.1.3 Dynamics as a tool

- Infer additional/dark matter
  - E.g., Weakly Interacting Massive Particles
    - proton mass, but much less interactive
    - Suggested by Super-Symmetry, but undetected
  - A $billion$ industry to find them.
    - What if they don't exist?

…

- Test the law of gravity:
  - valid in nano-gravity regime?
  - Uncertain outside solar system:
    - GM/r^2 or cst/r ?

Outer solar system

- The Pioneer experiences an anomalous non-Keplerian acceleration of 10^{-8} cm s^{-2}
    - What is the expected acceleration at 10 AU?
    - What could cause the anomaly?

Gravitational Dynamics can be applied to:

- Two body systems:binary stars
- Planetary Systems, Solar system
- Stellar Clusters:open & globular
- Galactic Structure:nuclei/bulge/disk/halo
- Clusters of Galaxies
- The universe:large scale structure

Topics

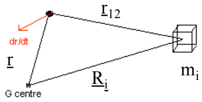
- Phase Space Fluid f(x,v)
  - Eq^n of motion
  - Poisson's equation
- Stellar Orbits
  - Integrals of motion (E,J)
  - Jeans Theorem
- Spherical Equilibrium
  - Virial Theorem
  - Jeans Equation
- Interacting Systems
  - Tides→Satellites→Streams
  - Relaxation→collisions
- MOND

C2.1 How to model motions of 10^{10}stars in a galaxy?

- Direct N-body approach (as in simulations)
  - At time t particles have $(m_i,x_i,y_i,z_i,vx_i,vy_i,vz_i)$, i=1,2,...,N (feasible for $N<<10^6$).
- Statistical or fluid approach (N very large)
  - At time t particles have a spatial density distribution n(x,y,z)*m, e.g., uniform,
  - at each point have a velocity distribution G(vx,vy,vz), e.g., a 3D Gaussian.

C2.2 N-body Potential and Force

- In N-body system with mass $m_1 \ldots m_N$, the gravitational acceleration $\underline{g}(r)$ and potential $\varphi(r)$ at position $\underline{r}$ is given by:



$$\vec{F} = m\vec{g}(r) = -\sum_{i=1}^{N} \frac{G \cdot m \cdot m_i \cdot \hat{r}_{12}}{\left|\vec{r} - \vec{R}_i\right|^2} = -m\vec{\nabla}_r \phi$$

$$\Phi = m\phi(r) = -\sum_{i=1}^{N} \frac{G \cdot m \cdot m_i}{\left|\vec{r} - \vec{R}_i\right|}$$

Example: Force field of two-body system in Cartesian coordinates

$$\phi(\vec{r}) = -\sum_{i=1}^{2} \frac{G \cdot m_i}{\left|\vec{r} - \vec{R}_i\right|}, \text{ where } \vec{R}_i = (0,0,-i) * a, m_i = m_\circ$$

Sketch the configuration, sketch equal potential contours

$\phi(x, y, z) = ?$

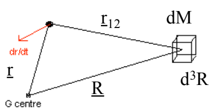
$$\vec{g}(\vec{r}) = (g_x, g_y, g_z) = -\nabla\phi(\vec{r}) = (-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial z})$$

$$\left\|\vec{g}(\vec{r})\right\| = \sqrt{(g_x^2 + g_y^2 + g_z^2)} = ?$$

sketch field lines. at what positions is force = 0?

C2.3 A fluid element: Potential & Gravity

- For large N or a continuous fluid, the gravity dg and potential $d\varphi$ due to a small mass element dM is calculated by replacing m_i with dM:



$$d\vec{g} = -\frac{G \cdot dM \cdot \hat{r}_{12}}{\left|\vec{r} - \vec{R}_i\right|^2}$$

$$d\phi = -\frac{G \cdot dM}{\left|\vec{r} - \vec{R}\right|}$$

Lec 2 (Friday, 10 Feb): Why Potential $\varphi(r)$?

- Potential per unit mass $\varphi(r)$ is scalar,
  - function of $\underline{r}$ only,
  - Related to but easier to work with than force (vector, 3 components)
  - Simply relates to orbital energy $E = \varphi(r) + \frac{1}{2} v^2$

C2.4 Poisson's Equation

- PE relates the potential to the density of matter generating the potential by:

$$\nabla \cdot \nabla\phi = -\vec{\nabla} \cdot \vec{g} = 4\pi G\rho(r)$$

- [BT2.1]

C2.5 Eq. of Motion in N-body

- Newton's law: a point mass m at position $\underline{r}$ moving with a velocity $d\underline{r}/dt$ with Potential energy $\Phi(\underline{r}) = m\varphi(r)$ experiences a Force F=mg , accelerates with following Eq. of Motion:

$$\frac{d}{dt}\left[\frac{d\vec{r}(t)}{dt}\right] = \frac{\vec{F}}{m} = \frac{-\vec{\nabla}_{\vec{r}}\Phi(r)}{m}$$

Example 1: trajectories when G=0

- Solve Poisson's Eq. with G=0 ➔
  - **F**=0, ➔ $\Phi(\underline{r})$=cst, ➔
- Solve EoM for particle i initially at ($\mathbf{X}_{i,0}$, $\mathbf{V}_{0,i}$)
  - $d\mathbf{V}_i/dt = \mathbf{F}_i/m_i = 0$ ➔ $\mathbf{V}_i$ = cst = $\mathbf{V}_{0,i}$
  - $d\mathbf{X}_i/dt = \mathbf{V}_i = \mathbf{V}_{i,0}$ ➔ $\mathbf{X}_i(t) = \mathbf{V}_{i,0}\, t + \mathbf{X}_{i,0}$,
  - where **X**, **V** are vectors,
  - ➔straight line trajectories
- E.g., photons in universe go straight
  - occasionally deflected by electrons,
  - Or bent by gravitational lenses

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What have we learned?

- Implications on gravity law and DM.
- Poisson's eq. and how to calculate gravity
- Equation of motion

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How N-body system evolves

- Start with initial positions and velocities of all N particles.
- Calculate the mutual gravity on each particle
  - Update velocity of each particle for a small time step dt with EoM
  - Update position of each particle for a small time step dt
- Repeat previous for next time step.
- ➔ N-body system fully described

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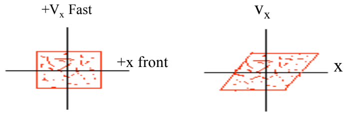
C2.6 Phase Space of Galactic Skiers

- N_{skiers} identical particles moving in a small bundle in phase space (Vol $=\Delta_x \Delta_v$),
- phase space deforms but maintains its area.



- Gap widens between faster & slower skiers
  - but the phase volume & No. of skiers are constants.

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"Liouvilles Theorem on the piste"

- Phase space density of a group of skiers is const.

  $f = m\ N_{skiers} / \Delta x\, \Delta v_x$ = const

  Where m is mass of each skier,

  [BT4.1]

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C2.7 density of phase space fluid: Analogy with air molecules

- air with uniform density n=10^{23} cm^{-3}

  Gaussian velocity rms velocity σ =0.3km/s in x,y,z directions:

$$f(x,v) = \frac{m \times n_o \exp\left(-\frac{v_x^2+v_y^2+v_x^2}{2\sigma^2}\right)}{(\sqrt{2\pi}\sigma)^3}$$

- Estimate f(0,0,0,0,0,0) in pc^{-3} $(km/s)^{-3}$

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Lec 3 (Valentine Tuesday)

C2.8 Phase Space Distribution Function (DF)

PHASE SPACE DENSITY: No. of sun-like stars per unit volume per velocity volume f(**x**,**v**)

$$f(x,v) = \frac{dN \times m_{sun}}{dx^3 dv^3} = \frac{\text{number of suns} \times m_{sun}}{\text{space volume} \times \text{velocity volume}}$$

$$\approx \frac{1 \times m_{sun}}{pc^3 \times (100\,kms^{-1})^3}$$

C2.9 add up stars: integrate over phase space

- star mass density: integrate velocity volume

$$\rho(\vec{x}) = m_{sun} \times n(\vec{x}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\vec{x},\vec{v})\,d\vec{v}_x d\vec{v}_y d\vec{v}_z$$

- The total mass : integrate over phase space

$$M_{total} = \int \rho(x) d^3x = \int f(\vec{x},\vec{v}) d^3\vec{v} d^3\vec{x}$$

- define spatial density of stars n(x)

$$n \equiv \int f d^3v$$

- and the mean stellar velocity v(x)

$$n\bar{v}_i \equiv \text{flux in i-direction} = \int f v_i d^3v$$

- E.g., Conservation of flux (without proof)

$$\frac{\partial n}{\partial t} + \frac{\partial (n\bar{v}_1)}{\partial x_1} + \frac{\partial (n\bar{v}_2)}{\partial x_2} + \frac{\partial (n\bar{v}_3)}{\partial x_3} = 0$$

C3.0 Star clusters differ from air:

- Stars collide far less frequently
  - size of stars<<distance between them
  - Velocity distribution not isotropic
- Inhomogeneous density ρ(r) in a Grav. Potential φ(**r**)

Example 2: A 4-body problem

- Four point masses with G m = 1 at rest (x,y,z)=(0,1,0),(0,-1,0),(-1,0,0),(1,0,0). Show the initial total energy
  Einit = 4 * (½ + $2^{-1/2}$ + $2^{-1/2}$) /2 = 3.8
- Integrate EoM by brutal force for one time step =1 to find the positions/velocities at time t=1.
  - Use $V=V_0 + g\,t = g$ = (u, u, 0) ; u = $2^{1/2}/4 + 2^{1/2}/4 + ¼$ = 0.95
  - Use $x= x_0 + V_0\,t = x_0$ = (0, 1, 0).
- How much does the new total energy differ from initial?
  E - Einit = ½ (u^2 +u^2) * 4 = 2 u^2 = 1.8

Often-made Mistakes

- Specific energy or specific force confused with the usual energy or force
- Double-counting potential energy between any pair of mass elements, kinetic energy with v^2
- Velocity vector V confused with speed,
- 1/|r| confused with 1/|x|+1/|y|+1/|z|

What have we learned?

Potential to Gravity $g = -\nabla\phi$

Potential to density $\rho = \frac{1}{4\pi G}\nabla^2\phi$

Density to potential $\phi(\bar{r}) = -\int \frac{G\rho d^3\bar{r}'}{|\bar{r}-\bar{r}'|}$

Motion to gravity $g = dv/dt$

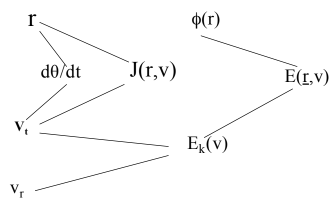
Concepts

- Phase space density
  - incompressible
  - Dimension Mass/[Length^3 Velocity^3]
  - **Show** a pair of non-relativistic Fermionic particle occupy minimal phase space $(x*v)^3 > (h/m)^3$, hence has a maximum phase density $=2m\ (h/m)^{-3}$

Where are we heading to? Lec 4, Friday 17 Feb

- potential and eqs. of motion
  - in general geometry
  - Axisymmetric
  - spherical

Link phase space quantities



C 3.1: Laplacian in various coordinates

Cartesians:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical:

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Example 3: Energy is conserved in STATIC potential

- The orbital energy of a star is given by:

$$E = \frac{1}{2}v^2 + \phi(\bar{r},t)$$

$$\frac{dE}{dt} = \bar{v}\frac{d\bar{v}}{dt} + \frac{d\bar{r}}{dt}\nabla\phi + \frac{\partial\phi}{\partial t} = 0 + \frac{\partial\phi}{\partial t}$$

0 since $\frac{d\bar{v}}{dt} = -\nabla\phi$ and $\frac{d\bar{r}}{dt} = \bar{v}$

0 for static potential.

So orbital Energy is Conserved dE/dt=0 only in "time-independent" potential.

Example 4: Static Axisymmetric density → Static Axisymmetric potential

- We employ a cylindrical coordinate system (R, ϕ,z) e.g., centred on the galaxy and align the z axis with the galaxy axis of symmetry.
- Here the potential is of the form $\phi(R,z)$.
- Density and Potential are Static and Axisymmetric
  - independent of time and azimuthal angle

$$\phi(R,z) \Rightarrow \rho(R,z) = \frac{1}{4\pi G}\left[R\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right) + \frac{\partial^2\phi}{\partial z^2}\right]$$

$$g_r = -\frac{\partial\phi}{\partial R} \qquad g_z = -\frac{\partial\phi}{\partial z}$$

C3.2: Orbits in an axisymmetric potential

- Let the potential which we assume to be symmetric about the plane z=0, be $\phi(R,z)$.
- The general equation of motion of the star is

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\phi(R,z) \qquad \text{Eq. of Motion}$$

- Eqs. of motion in cylindrical coordinates

$$\ddot{z} = -\frac{\partial\phi}{\partial z}, \quad \ddot{R} - R\dot{\theta}^2 = -\frac{\partial\phi}{\partial R}, \quad 2\dot{R}\dot{\theta} + R\ddot{\theta} = \frac{d}{Rdt}(R^2\dot{\theta}) = -\frac{\partial\phi}{R\partial\theta} = 0$$

Conservation of angular momentum z-component Jz if axisymmetric

$$J_Z = R^2\dot{\theta} \Rightarrow \frac{d}{dt}Jz = \frac{d}{dt}(R^2\dot{\theta}) = 0$$

- The component of angular momentum about the z-axis is conserved.
- If $\phi(R,z)$ has no dependence on θ then the azimuthal angular momentum is conserved
  - or because z-component of the torque $\underline{r}\times\underline{F}=0$. (Show it)

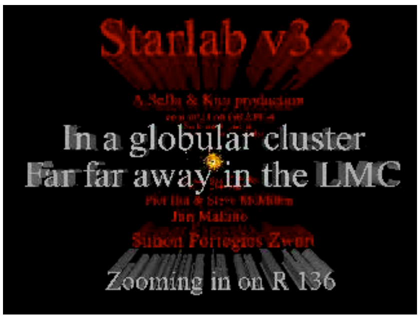
C4.1: Spherical Static System

- Density, potential function of radius |r| only
- Conservation of
  - energy E,
  - angular momentum J (all 3-components)
  - Argue that a star moves orbit which confined to a plane perpendicular to J vector.

C 4.1.0: Spherical Cow Theorem

- Most astronomical objects can be approximated as spherical.
- Anyway non-spherical systems are too difficult to model, almost all models are spherical.

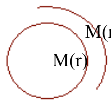
Globular: A nearly spherical static system



C4.2: From Spherical Density to Mass

$$M(R+dr) = M(R) + dM$$

$$dM = \rho(r) d\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \rho(r) dr$$



$$\rho(r) = \frac{dM}{d\left(\frac{4}{3}\pi r^3\right)} = \frac{dM}{4\pi r^2 dr}$$

$$M(R) = \int \rho d\left(\frac{4}{3}\pi r^3\right)$$

C4.3: Theorems on Spherical Systems

- NEWTONS 1st THEOREM: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell
- NEWTONS 2nd THEOREM: The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the matter were concentrated at its centre. [BT 2.1]

C4.4: Poisson's eq. in Spherical systems

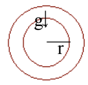
- Poisson's eq. in a spherical potential with no θ or Φ dependences is:

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = 4\pi G\rho(r)$$

- BT2.1.2

Example 5: Interpretation of Poissons Equation

- Consider a spherical distribution of mass of density ρ(r).



$$g = -\frac{GM(r)}{r^2}$$

$$\phi = \int_r^\infty g(r)dr \qquad \text{since } \phi = 0 \text{ at } \infty \text{ and is} < 0 \text{ at r}$$

$$= -\int_r^\infty \frac{GM(r)}{r^2} dr$$

$$\text{Mass Enclosed} = \int_r^\infty 4\pi r^2 \rho(r) dr$$

- Take d/dr and multiply r^2 →

$$r^2\frac{d\phi}{dr} = -gr^2 = GM(r) = \left(G\int 4\pi r^2\rho(r)dr\right)$$

- Take d/dr and divide r^2 →

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2 g\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(GM) = 4\pi G\rho(r)$$

$$\rightarrow \nabla^2\phi = -\vec{\nabla}.g = 4\pi G\rho$$

C4.5: Escape Velocity

- ESCAPE VELOCITY= velocity required in order for an object to escape from a gravitational potential well and arrive at ∞ with zero KE.

=0 often

$$\phi(r) = \phi(\infty) - \frac{1}{2}v_{esc}^2$$

$$\rightarrow v_{esc}(r) = \sqrt{2\phi(\infty) - 2\phi(r)}$$

Example 6: Plummer Model for star cluster

- A spherically symmetric potential of the form:

$$\phi = -\frac{GM}{\sqrt{r^2 + a^2}}$$

e.g., for a globular cluster a=1pc, M=10^5 Sun Mass show Vesc(0)=30km/s

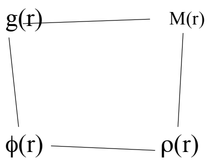
- Show corresponding to a density (use Poisson's eq):

$$\rho = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}}$$

What have we learned?

- Conditions for conservation of orbital energy, angular momentum of a test particle
- Meaning of escape velocity
- How Poisson's equation simplifies in cylindrical and spherical symmetries

Lec 5, (Tue 21 Feb)
Links of quantities in spheres



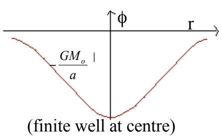
A worked-out example 7: Hernquist Potential for stars in a galaxy

$$\phi_*(r) = -\frac{GM_0}{a + r}$$, use Poisson eq. show

$$\rho_*(r) = \frac{M_0}{2\pi a^3}\left(\frac{r}{a}\right)^{-1}\left(1 + \frac{r}{a}\right)^{-2}$$

- E.g., a=1000pc, $M_0=10^{10}$ solar, show central escape velocity Vesc(0)=300km/s,
- Show M_0 has the meaning of total mass
  - Potential at large r is like that of a point mass M_0
  - Integrate the density from r=0 to inifnity also gives M_0

Potential of globular clusters and galaxies looks like this:



$\phi(0) = -const$ (finite well at centre)

$\phi(r) \propto r^{-1}$ (Kepler for large r)

→ *Centre* is the minimum of potential with escape velocity

$$v_{esc}(0) = \sqrt{\frac{2GM_o}{a}}$$

C4.6: Circular Velocity

- CIRCULAR VELOCITY= the speed of a test particle in a circular orbit at radius r.

$$|g| = \frac{v_{cir}^2}{r} = \nabla\phi = \frac{GM(r)}{r^2}$$

$$\Rightarrow M(r) = \frac{v_{cir}^2 r}{G}$$

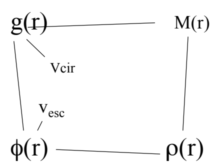
For a point mass M: $v_c(r) = \sqrt{\frac{GM}{r}}$

Show in a uniform density sphere $v_c(r) = \sqrt{\frac{4\pi G\rho}{3}}\,r$ since $M(r) = \frac{4}{3}\pi r^3 \rho$

What have we learned?

- How to apply Poisson's eq.
- How to relate
  - Vesc with potential and
  - Vcir with gravity
- The meanings of
  - the potential at very large radius,
  - The enclosed mass

Lec 6, (Fri, 24 Feb) Links of quantities in spheres



C4.7: Motions in spherical potential [BT3.1]

Equation of motion

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \vec{g} = -\nabla\phi$$

If spherical

$$g_r = -\frac{\partial\phi}{\partial r}$$

$$g_\theta = -\frac{\partial\phi}{\partial\theta} = 0$$

If no gravity

$$\mathbf{x}(t) = \mathbf{v}_0 t + \mathbf{x}_0$$

$$\mathbf{v}(t) = \mathbf{v}_0$$

Conserved if spherical static

$$E = \frac{1}{2}v^2 + \phi(r)$$

$$L = J = \mathbf{x} \otimes \mathbf{v} = r v_t \cdot \hat{n}$$

Another Proof: Angular Momentum is Conserved if spherical

- $\vec{L} = \vec{r} \times \vec{v}$

→ $$\frac{d\vec{L}}{dt} = \frac{d(\vec{r}\times\vec{v})}{dt} = \frac{d\vec{r}}{dt}\times\vec{v} + \vec{r}\times\frac{d\vec{v}}{dt} = 0 + \vec{r}\times\vec{g}$$

Since $\frac{\partial\phi}{\partial\theta} = 0$ then the spherical force g is in the r direction, no torque

→both cross products on the RHS = 0.

So Angular Momentum L is Conserved

$$\frac{d\vec{L}}{dt} = 0$$

C4.8: Star moves in a plane (r,θ) perpendicular to L

- Direction of angular momentum

$$\underline{r} \times \underline{\dot{r}} = \underline{L}$$

- equations of motion are
  - radial acceleration: $\ddot{r} - r\dot{\theta}^2 = g(r)$
  - tangential acceleration: $2\dot{r}\dot{\theta} + r\ddot{\theta}^2 = 0$

$$r^2\dot{\theta} = \text{constant} = L$$

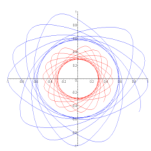
C4.8.1: Orbits in Spherical Potentials

- The motion of a star in a centrally directed field of force is greatly simplified by the familiar law of conservation (WHY?) of angular momentum.

$$\vec{L} = \vec{r} \times \frac{d\vec{r}}{dt} = \text{const}$$

$$= r^2\frac{d\theta}{dt} = 2\frac{\text{area swept}}{\text{unit time}}$$

Keplers 3rd law



C 4.9: Radial part of motion

- Energy Conservation (WHY?)

$$E = \phi(r) + \frac{1}{2}\left(\frac{d\bar{r}}{dt}\right)^2 + \frac{1}{2}\left(r\frac{d\theta}{dt}\right)^2$$

ϕ_{eff}

$$= \phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\left(\frac{d\bar{r}}{dt}\right)^2$$

$$\frac{dr}{dt} = \pm\sqrt{2E - 2\phi_{eff}(r)}$$

C5.0: Orbit in the z=0 plane of a disk potential ϕ(R,z).

- Energy/angular momentum of star (per unit mass)

$$E = \tfrac{1}{2}\left[\dot{R}^2 + \left(R\dot{\theta}\right)^2\right] + \Phi(R,0)$$

$$= \tfrac{1}{2}\left[\dot{R}^2\right] + \left[\frac{J_z^{\,2}}{2R^2} + \Phi(R,0)\right]$$

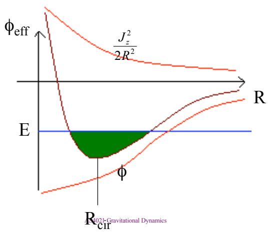
$$= \tfrac{1}{2}\left[\dot{R}^2\right] + \Phi_{\text{eff}}(R,0)$$

- orbit bound within

$$\Rightarrow E \geq \Phi_{\text{eff}}(R,0)$$

C5.1: Radial Oscillation

- An orbit is bound between two radii: a loop
- Lower energy E means thinner loop (nearly circular closed) orbit

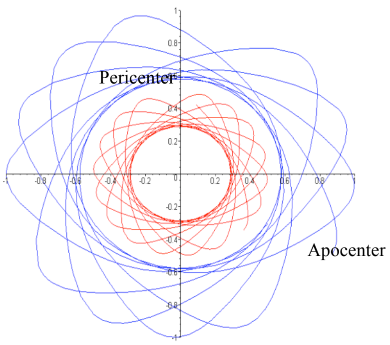


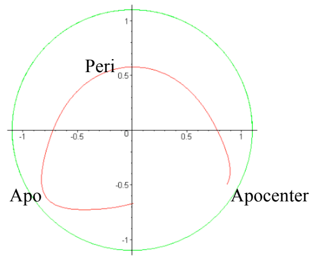
C5.2: Eq of Motion for planar orbits

- EoM:

$$\ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R}; \qquad z=0$$

If circular orbit R=cst, $z = 0$ => $\frac{\partial \Phi_{\text{eff}}}{\partial R} = 0$ at $R = R_c$





C5.3: Apocenter and pericenter

- No radial motion at these turn-around radii
  - dr/dt $=V_r$ =0 at apo and peri
- Hence
- $J_z = R\,V_t$
  $= R_a V_a = R_p V_p$
- $E = ½ (V_r^2 + V_t^2) + \Phi(R,0) = ½ V_r^2 + \Phi_{eff}(R,0)$
  $= ½ V_a^2 + \Phi(R_a,0)$
  $= ½ V_p^2 + \Phi(R_p,0)$

Lec7: Orbit in axisymmetric disk potential ϕ(R,z).

- Energy/angular momentum of star (per unit mass)

$$E = \tfrac{1}{2}\left[\dot{R}^2 + \left(R\dot{\theta}\right)^2 + \dot{z}^2\right] + \Phi$$
$$= \tfrac{1}{2}\left[\dot{R}^2 + \dot{z}^2\right] + \frac{J_z^{\,2}}{2R^2} + \Phi$$
$$= \tfrac{1}{2}\left[\dot{R}^2 + \dot{z}^2\right] + \Phi_{eff}(R,z)$$

- orbit bound within

$$\Longrightarrow E \geq \Phi_{eff}(R,z)$$

C5.4 EoM for nearly circular orbits

- EoM: $\ddot{R} = -\frac{\partial \Phi_{eff}}{\partial R}; \qquad \ddot{z} = -\frac{\partial \Phi_{eff}}{\partial z}$

$$\frac{\partial \Phi_{eff}}{\partial R} = \frac{\partial \Phi_{eff}}{\partial z} = 0 \quad \text{at} \quad R = R_g, z = 0$$

- Taylor expand

$$\Phi_{eff} = \tfrac{1}{2}\left(\frac{\partial^2 \Phi_{eff}}{\partial R^2}\right)_{(R_g,0)} x^2 + \tfrac{1}{2}\left(\frac{\partial^2 \Phi_{eff}}{\partial z^2}\right)_{(R_g,0)} z^2 + \ldots$$
$$= \kappa^2 x^2/2 + \quad \nu^2 z^2/2 + \ldots$$

– x=R-R_g

Sun's Vertical and radial epicycles

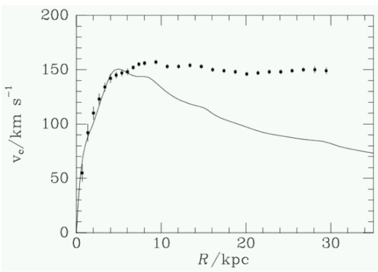
- harmonic oscillator +/-10pc every 10^8 yr

κ – epicyclic frequency :

ν – vertical frequency :

$$\ddot{R} = -\kappa^2 R, \qquad \text{and} \qquad \ddot{z} = -\nu^2 z$$

Lec 8, C7.0: Stars are not enough: add Dark Matter in galaxies [BT10.4]



NGC 3198 (Begeman 1987)

Bekenstein & Milgrom (1984) Bekenstein (2004), Zhao & Famaey (2006)

- Modify gravity g,
  - Analogy to E-field in medium of varying Dielectric

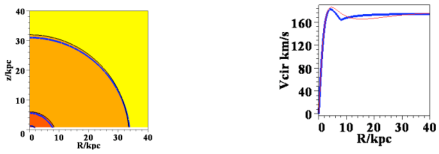
$$-\nabla \bullet \left(\frac{\mathbf{g}}{4\pi \overline{\mathrm{G}}}\right) = \rho_*(r)$$

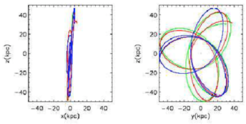
$\overline{\mathrm{G}}(g/a_0) = (1 + a_0\ /g)\ \mathrm{G}$
$\sim \mathrm{G}$ if g= $|\nabla\phi| > a_0$
$\sim \mathrm{G}\ a_0/g > \mathrm{G}$ if g $< a_0$

Not Standard Belief!

- Gradient of Conservative potential

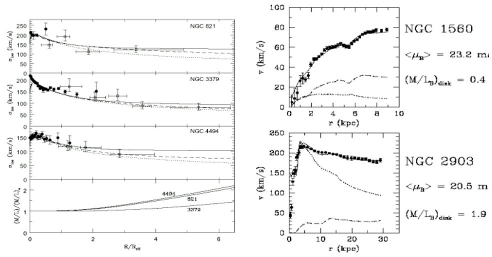
MOND and DM similar in potential, rotation curves and orbits... Personal view!





Researches of Read & Moore (2005)

Explained: Fall/Rise/wiggles in Ellip/Spiral/Dwarf galaxies



Views of Milgrom & Sanders, Sanders & McGaugh

What have we learned?

- Orbits in a spherical potential or in the mid-plane of a disk potential
- How to relate Pericentre, Apocentre through energy and angular momentum conservation.
- Rotation curves of galaxies
  - Need for Dark Matter or a boosted gravity

Tutorial: Singular Isothermal Sphere

- Has Potential Beyond r_o: $\phi(r) = -\frac{GM_0}{r}$
- And Inside $r<r_0$ $\phi(r) = v_0^2 \ln\frac{r}{r_o} + \phi_o$
- Prove that the potential AND gravity is continuous at $r=r_o$ if $\phi_0 = -GM_0/r_0 = -v_0^2$
- Prove density drops sharply to 0 beyond r0, and inside r0

$$\rho(r) = \frac{V_0^2}{4\pi G r^2}$$

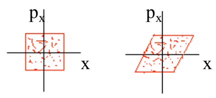
- Integrate density to prove total mass=M0
- What is circular and escape velocities at r=r0?
- Draw diagrams of M(r), Vesc(r), Vcir(r), |Phi(r)|, rho(r), |g(r)| vs. r (assume V0=200km/s, r0=100kpc).

Another Singular Isothermal Sphere

- Consider a potential $\Phi(r)=V_0^2\ln(r)$.
- Use Jeans eq. to show the velocity dispersion σ (assume isotropic) is constant V_0^2/n for a spherical tracer population of density $A*r^{-n}$; Show we required constants $A = V_0^2/(4*Pi*G)$. and n=2 in order for the tracer to become a self-gravitating population. Justify why this model is called Singular Isothermal Sphere.
- Show stars with a phase space density $f(E)= \exp(-E/\sigma^2)$ inside this potential well will have no net motion <V>=0, and a constant rms velocity σ in all directions.
- Consider a black hole of mass m on a rosette orbit bound between pericenter r_0 and apocenter $2r_0$. Suppose the black hole decays its orbit due to dynamical friction to a circular orbit $r_0/2$ after time t_0. How much orbital energy and angular momentum have been dissipated? By what percentage has the tidal radius of the BH reduced? How long would the orbital decay take for a smaller black hole of mass m/2 in a small galaxy of potential $\Phi(r)=0.25V_0^2\ln(r)$. ? Argue it would take less time to decay from r_0 to $r_0/2$ then from $r_0/2$ to 0.

Lec 9: C8.0: Incompressible df/dt=0

- N_{star} identical particles moving in a small bundle in phase space (Vol=$\Delta_x \Delta_p$),
- phase space deforms but maintains its area.
  - Likewise for y-p_y and z-p_z.



$\frac{dVol}{d\lambda} = 0, \frac{dNstar}{d\lambda} = 0,$ 'LIOUVILLES THEOREM'

Phase space density f=Nstars/$\Delta_x \Delta_p$ ~ const

C8.1: Stars flow in phase-space [BT4.1]

- Flow of points in phase space ~ stars moving along their orbits.
- phase space coords: $(\underline{x},\underline{v}) \equiv \underline{w} \equiv (w_1, w_2, ..., w_6)$

$$\dot{\underline{w}} = (\dot{\underline{x}}, \dot{\underline{v}}) = (\underline{v}, -\nabla\Phi)$$

C8.2 Collisionless Boltzmann Equation

- Collisionless df/dt=0:

$$\frac{\mathrm{d}}{\mathrm{dt}} f(x,v,t) = \left(\frac{\partial}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial}{\partial w_\alpha} \right) f(w,t) = 0$$

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] = 0$$

- Vector form

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f - \nabla\Phi \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

C8.3 DF & its 0th ,1st , 2nd moments

$$\overline{A}(\mathbf{x}) = \frac{d^3\mathbf{x} \iiint A f(x,v) d^3\mathbf{v}}{d^3\mathbf{x}\rho}$$

$$d^3\mathbf{x}\rho = dM = d^3\mathbf{x} \iiint f(x,v) d^3\mathbf{v}$$

where $A(\mathbf{x},\mathbf{v}) = 1, V_x, \mathbf{V_x V_y}, \ldots$

e.g., verify $\overline{V^2} = \overline{V_x^2} + \overline{V_y^2} + \overline{V_z^2}$

- E.g: rms speed of air particles in a box dx^3 :

$$f(x,v) = \frac{m \times n_o \exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2\sigma^2} \right)}{(\sqrt{2\pi}\sigma)^3}$$

verify $\overline{v_x} = 0, \ \overline{v_x v_x} - \overline{v_x}\ \overline{v_x} = \sigma^2, \ \ \overline{v_x v_y} - \overline{v_x}\ \overline{v_y} = 0$

$$\overline{v^2} = \frac{\int v^2 dN}{\int dN} = \frac{d^3x \int v^2 f d^3v}{d^3x \int f d^3v} = \frac{\int_0^\infty e^{-\frac{v^2}{2\sigma^2}} 4\pi v^4 dv}{\int_0^\infty e^{-\frac{v^2}{2\sigma^2}} 4\pi v^2 dv} = 3\sigma^2,$$

C8.4: CBE ➔ Moment/Jeans Equations [BT4.2]

- Phase space incompressible

df(w,t)/dt=0, where w=[x,v]: CBE

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] = 0$$

- taking moments U=1, v_j, $v_j v_k$ by integrating over all possible velocities

$$\int \frac{\partial f}{\partial t} U d^3\underline{v} + \int v_i \frac{\partial f}{\partial x_i} U d^3\underline{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} U d^3\underline{v} = 0$$

C8.5: 0th moment (continuity) eq.

- define spatial density of stars n($\underline{x}$)

$$n \equiv \int f d^3v$$

- and the mean stellar velocity v($\underline{x}$)

$$\bar{v}_i \equiv \frac{1}{n} \int f v_i d^3v$$

- then the zeroth moment equation becomes

$$\frac{\partial n}{\partial t} + \sum_{i=1,2,3} \frac{\partial \left(n\bar{v}_i \right)}{\partial x_i} = 0$$

C8.6: 2^{nd} moment Equation

$$\rightarrow \frac{\partial \overline{v_j}}{\partial t} + \sum_{i=1,2,3} \overline{v_i} \frac{\partial \left(\overline{v}_j\right)}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} - \sum_{i=1,2,3} \frac{\partial \left(n\sigma_{ij}^2\right)}{n\partial x_i}$$

$$\sigma_{ij}^2 = \overline{(v_i - \overline{v_i})(v_j - \overline{v_j})} = \overline{V_i V_j} - \overline{V}_i \overline{V}_j$$

$$\frac{\partial \underline{v}}{\partial t} + \left(\underline{v} \cdot \nabla\right)\underline{v} = -\nabla\Phi - \frac{1}{\rho}\nabla P$$

similar to the Euler equation for a fluid flow:

– last term of RHS represents pressure force

$$-\frac{\partial}{\partial x_i}\left(n\sigma_{ij}^2\right) \Leftrightarrow -\nabla P$$

Lec 10, C8.7: Meaning of pressure in star system

- What prevents a non-rotating star cluster collapse into a BH?
  - No systematic motion as in Milky Way disk.
  - But random orbital angular momentum of stars!

C8.8: Anisotropic Stress Tensor

- describes a pressure which is $P_{ij} = n\sigma_{ij}^2$
  - perhaps stronger in some directions than other
- the tensor is symmetric, can be diagonalized
  - velocity ellipsoid with semi-major axes given by

$$\sigma_{ij}^2 = \sigma_{ii}^2 \delta_{ij}$$

$$\sigma_{11}, \sigma_{22}, \sigma_{33}$$

C8.9: Prove Tensor Virial Theorem (BT4.3)

$$\Rightarrow <\vec{v}\vec{v}> = <\vec{r}\nabla\phi>$$

$$2K_{ij} = -W_{ij}$$

$$<v_x v_x> = <x\partial_x \phi>$$

$$<v_x v_y> = <x\partial_y \phi> \text{ etc}$$

$$<v^2> = <v_x^2> + <v_y^2> + <v_z^2>$$

$$= <\vec{r}.\nabla\phi>$$

$$= <r\frac{d\phi}{dr}>$$

- Many forms of Viral theorem, E.g.

Scaler Virial Theorem

$$\Rightarrow 2K + W = 0$$

C9.0: Jeans theorem [BT4.4]

- For most stellar systems the DF depends on (x,v) through generally 1,2 or 3 integrals of motion (conserved quantities),
- $I_i(x,v)$, i=1..3 ➔ $f(x,v) = f(I_1(x,v), I_2(x,v), I_3(x,v))$
- E.g., in Spherical Equilibrium, f is a function of energy E(x,v) and ang. mom. vector L(x,v).'s amplitude and z-component

$$f(x,v) = f(E, \|\bar{L}\|, \bar{L}\cdot\hat{z})$$

C9.1: An anisotropic incompressible spherical fluid, e.g. $f(E,L) = \exp(-E/\sigma_0^2)L^{2\beta}$ [BT4.4.4]

- Verify $<V_r^2> = \sigma_0^2$, $<V_t^2> = 2(1-\beta)\,\sigma_0^2$
- Verify $<V_r> = 0$
- Verify CBE is satisfied along the orbit or flow:

$$\frac{df(E,L)}{dt} = \frac{\partial f(E,L)}{\partial E}\frac{dE}{dt} + \frac{\partial f(E,L)}{\partial L}\frac{dL}{dt} = 0 + 0$$

0 for static potential, 0 for spherical potential

So f(E,L) indeed constant along orbit or flow

C9.2: Apply JE & PE to measure Dark Matter [BT4.2.1d]

- A bright sub-component of observed density n*(r) and anisotropic velocity dispersions $\langle V_t^2 \rangle = 2(1-\beta)\langle V_r^2 \rangle$
- in spherical potential φ(r) from total (+dark) matter density ρ(r)

JE: $$-\frac{1}{n^*}\frac{d}{dr}\left(n^*\overline{v_r^2}\right)+\frac{\overline{v_t^2}-2\overline{v_r^2}}{r}=\frac{d\Phi}{dr}$$

PE: $$\frac{G\int_0^r \rho(r)4\pi r^2 dr}{r^2}=\frac{d\Phi}{dr}$$

C9.3: Measure total Matter density ρ(r)

Assume anistropic parameter beta, substitute JE for stars of density n* into PE, get

$$-\frac{d}{r^2 dr}\left[\frac{r^2}{4\pi G}\left(\frac{2\beta\overline{v_r^2}}{r}+\frac{d\left(n^*\overline{v_r^2}\right)}{n^* dr}\right)\right]=\rho(r)$$

- all quantities on the LHS are, in principle, determinable from observations.

C9.4: Spherical Isotropic f(E) Equilibriums [BT4.4.3]

- ISOTROPIC β=0:The distribution function f(E) only depends on |V| the modulus of the velocity, same in all velocity directions.

$$f(E), E=|\vec{v}|^2/2+\phi(r)$$

$$\text{show } \sigma^2=\sigma_x^2=\sigma_y^2=\sigma_z^2 \quad =\sigma_r^2=\frac{1}{2}\sigma_{\text{tangential}}^2$$

$$<\bar{v}_x\bar{v}_y>=0$$

Note:the tangential direction has θ and ϕ components

C9.5: subcomponents

- Non-SELF-GRAVITATING: There are additional gravitating matter
- The matter density that creates the potential is NOT equal to the density of stars.
  - e.g., stars orbiting a black hole is non-self-gravitating.

C9.6: subcomponents add up to the total gravitational mass

$$\langle A+B\rangle=\langle A\rangle+\langle B\rangle$$

Phase density of stars plus dark matter

$f=f_1+f_2$

density of stars plus dark matter

$\rho=\rho_1+\rho_2$

Total density ρ ==> Φ shared total potential

What have we learned?

- Meaning of anistrpic pressure and dispertion.
- Usage of Jeans theorem [phase space]
- Usage of Jeans eq. (dark matter)
- Link among quantities in sphere.

C10.0: Galactic disk mass density from vertical equilibrium

- Use JE and PE in cylindrical coordinates.
- drop terms 1/R or d/dR (d/dz terms dominates).
  - |z|< 1kpc < R~8kpc near Sun

$$\frac{\partial}{n_* \partial z}\left(n_* \overline{v_z^2}\right) \approx -\frac{\partial \Phi}{\partial z} \qquad 4\pi G\rho \approx \frac{\partial^2 \Phi}{\partial^2 z}$$

- Combine vertical hydrostatic eq. and gravity eq. :

$\Rightarrow$ total density in a column 1kpc high

$$= \int_{z=-1\text{kpc}}^{z=1\text{kpc}} dz\rho = \left[\frac{2}{4\pi G n_*}\frac{\partial}{\partial z}\left(n_* \overline{v_z^2}\right)\right]_{z=1kpc}$$

- RHS are observables
- => JE/PE useful in weighing the galaxy disk.

A toy galaxy

Learn to analyse

If you like challenge…

$$\phi(R,z) = 0.5 v_0^2 \ln(R^2 + 2z^2) - v_0^2 (1 + (R^2 + z^2)/1kpc^2)^{-1/2},$$

$v0 = 100 km/s$. Argue 1st & 2nd terms of above galaxy potential resemble dark halo and stars respectively.
Calculate the circular velocity and dark halo density on equator (R,z) = (1kpc,0)
Estimate the total mass of stars and dark matter inside 10kpc.
Estimate the star column density inside |z|<0.1kpc, R=1kpc.

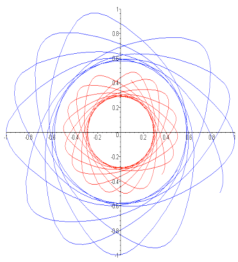
Helpful Math/Approximations (To be shown at AS4021 exam)

- Convenient Units: $1\text{km/s} \approx \frac{1\text{pc}}{1\text{Myr}} \approx \frac{1\text{kpc}}{1\text{Gyr}}$
- Gravitational Constant: $G \approx 4\times10^{-3}\text{pc} \ \ (\text{km/s})^2 M_{sun}^{-1}$, $G \approx 4\times10^{-6}\text{kpc} \ \ (\text{km/s})^2 M_{sun}^{-1}$
- Laplacian operator in various coordinates:

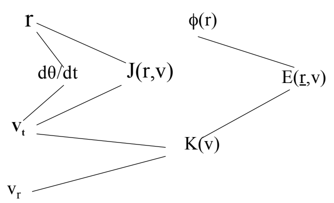
$$\nabla\bullet\nabla = \partial_x^2 + \partial_y^2 + \partial_z^2 \text{ (rectangular)}$$
$$= R^{-1}\partial_R(R\partial_R) + \partial_z^2 + R^{-2}\partial_\phi^2 \text{ (cylindrical)}$$
$$= \frac{\partial_r(r^2\partial_r)}{r^2} + \frac{\partial_\theta(\sin\theta\partial_\theta)}{r^2\sin\theta} + \frac{\partial_\phi^2}{r^2\sin\theta^2} \text{ (spherical)}$$

- Phase Space Density f(x,v) relation with the mass in a small position cube and velocity cube: $dM = f(x,v)dx^3dv^3$

Thinking of a globular cluster …



C10.1: Link phase space quantities



C10.2: Link quantities in spheres