Tutorial Questions AS4021

- You can print these into a sheet of tutorial questions

- M31 (now at 500 kpc) separated from MW a Hubble time ago
- Large Magellanic Cloud has circulated our Galaxy for about 5 times at 50 kpc
  - **argue** both neighbours move with a typical 100-200km/s velocity relative to us.

- Sun has circulated the galaxy for 30 times
  - velocity vector changes direction +/- 200km/s twice each circle (R = 8 kpc)
  - **Argue** that the MW is a nano-earth-gravity Lab
  - **Argue** that the gravity due to 10^{10} stars only within 8 kpc is barely enough. Might need to add Dark Matter.

Outer solar system

- The Pioneer experiences an anomalous non-Keplerian acceleration of 10^{-8} cm s^{-2}
  - What is the expected acceleration at 10 AU?
  - Explain a few possible causes for the anomaly.

Example: Force field of two-body system in Cartesian coordinates

$$\phi(\vec{r}) = -\sum_{i=1}^{2} \frac{G \cdot m_i}{\left|\vec{r} - \vec{R}_i\right|}, \text{ where } \vec{R}_i = (0,0,-i) * a, m_i = m_\circ$$

Sketch the configuration, sketch equal potential contours

$$\phi(x, y, z) = ?$$

$$\vec{g}(\vec{r}) = (g_x, g_y, g_z) = -\nabla\phi(\vec{r}) = (-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial z})$$

$$\left\|\vec{g}(\vec{r})\right\| = \sqrt{(g_x^2 + g_y^2 + g_z^2)} = ?$$

sketch field lines. at what positions is force = 0?

C2.7 density of phase space fluid: Analogy with air molecules

- air with uniform density n=10^{23} cm^{-3}
  Gaussian velocity rms velocity σ =0.3km/s in x,y,z directions:

$$f(x, v) = \frac{m \times n_o \exp\left(-\frac{v_x^2 + v_y^2 + v_x^2}{2\sigma^2}\right)}{(\sqrt{2\pi}\sigma)^3}$$

- Estimate f(0,0,0,0,0,0) in pc^{-3} $(km/s)^{-3}$

Example 2: A 4-body problem

- Four point masses with G m = 1 at rest (x,y,z)=(0,1,0),(0,-1,0),(-1,0,0),(1,0,0). Show the initial total energy
  $E_{init} = 4 * (½ + 2^{-1/2} + 2^{-1/2}) /2 = 3.8$
- Integrate EoM by brutal force for one time step =1 to find the positions/velocities at time t=1.
  - Use $V=V_0 + g\, t = g = (u, u, 0)$; $u = 2^{1/2}/4 + 2^{1/2}/4 + ¼ = 0.95$
  - Use $x= x_0 + V_0\, t = x_0 = (0, 1, 0)$.
- How much does the new total energy differ from initial?
  $E - E_{init} = ½ (u^2 +u^2) * 4 = 2 u^2 = 1.8$

Concepts

- Phase space density
  - incompressible
  - Dimension Mass/[$Length^3$ $Velocity^3$]
  - a pair of non-relativistic Fermionic particle occupy minimal phase space $(x*v)^3 > (h/m)^3$, Show it has a maximum phase density =2m $(h/m)^{-3}$

Example 6: Plummer Model for star cluster

- A spherically symmetric potential of the form:

$$\phi = -\frac{GM}{\sqrt{r^2 + a^2}}$$

  e.g., for a globular cluster a=1pc, $M=10^5$ Sun Mass show Vesc(0)=30km/s
- Show corresponding to a density (use Poisson's eq):

$$\rho = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}}$$

A worked-out example 7: Hernquist Potential for stars in a galaxy

$$\phi_*(r) = -\frac{GM_0}{a+r}$$, use Poisson eq. show

$$\rho_*(r) = \frac{M_0}{2\pi a^3}\left(\frac{r}{a}\right)^{-1}\left(1+\frac{r}{a}\right)^{-2}$$

- E.g., a=1000pc, $M_0=10^{10}$ solar, show central escape velocity Vesc(0)=300km/s,
- Show M_0 has the meaning of total mass
  - Potential at large r is like that of a point mass M_0
  - Integrate the density from r=0 to inifnity also gives M_0

- For a uniform sphere of density ρ_0 and radius r_0.

Compute the total mass.

Compute the potential as function of radius. Plot the potential and gravity as functions of radius.

Compute the pressure at the center of the sphere, assuming isotropic dispersion.

Compute the total potential energy.

Fist session lec5

Tutorial: Singular Isothermal Sphere

- Has Potential Beyond r_o: $\phi(r) = -\frac{GM_0}{r}$
- And Inside $r<r_0$ $\phi(r) = v_0^2 \ln\frac{r}{r_o} + \phi_o$
- Prove that the potential AND gravity is continuous at $r=r_o$ if $\phi_0 = -GM_0/r_0 = -v_0^2$
- Prove density drops sharply to 0 beyond r0, and inside r0

$$\rho(r) = \frac{V_0^2}{4\pi G r^2}$$

- Integrate density to prove total mass=M0
- What is circular and escape velocities at r=r0?
- Draw diagrams of M(r), Vesc(r), Vcir(r), |Phi(r)|, ρ (r), |g(r)| vs. r (assume V0=200km/s, r0=100kpc).

Another Singular Isothermal Sphere

- Consider a potential $\Phi(r)=V_0^2\ln(r)$.
- Use Jeans eq. to show the velocity dispersion σ (assume isotropic) is constant V_0^2/n for a spherical tracer population of density $A*r^{-n}$; Show we required constants $A = V_0^2/(4*Pi*G)$. and n=2 in order for the tracer to become a self-gravitating population. Justify why this model is called Singular Isothermal Sphere.
- Show stars with a phase space density $f(E)= \exp(-E/\sigma^2)$ inside this potential well will have no net motion $<V>=0$, and a constant rms velocity σ in all directions.
- Consider a black hole of mass m on a rosette orbit bound between pericenter r_0 and apocenter $2r_0$. Suppose the black hole decays its orbit due to dynamical friction to a circular orbit $r_0/2$ after time t_0. How much orbital energy and angular momentum have been dissipated? By what percentage has the tidal radius of the BH reduced? How long would the orbital decay take for a smaller black hole of mass m/2 in a small galaxy of potential $\Phi(r)=0.25V_0^2\ln(r)$. ? Argue it would take less time to decay from r_0 to r_0 /2 then from $r_0/2$ to 0.

- <u>For An anisotropic incompressible spherical fluid, e.g, f(E,L) =exp(-E/σ_0^2)$L^{2\beta}$ [BT4.4.4]</u>
- Verify $<V_r^2> = \sigma_0^2$, $<V_t^2>=2(1-\beta)\,\sigma_0^2$
- Verify $<V_r> = 0$

- For a spherical potential, Prove angular momentum x-component is conserved in a spherical potential; Is the angular momentum conserved if the potential varies with time.

C9.4: Spherical Isotropic f(E) Equilibriums [BT4.4.3]

- ISOTROPIC β=0:The distribution function f(E) only depends on |V| the modulus of the velocity, same in all velocity directions.

$$f(E), E = |\vec{v}|^2/2 + \phi(r)$$

$$\text{show } \sigma^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 \quad = \sigma_r^2 = \frac{1}{2}\sigma_{\text{tangential}}^2$$

$$< \vec{v}_x \vec{v}_y >= 0$$

A toy galaxy

$$\phi(R,z) = 0.5v_0^2 \ln(R^2 + 2z^2) - v_0^2(1 + (R^2 + z^2)/1kpc^2)^{-1/2},$$

$v0 = 100 km/s$. Argue 1st & 2nd terms of above galaxy potential resemble dark halo and stars respectively.
Calculate the circular velocity and dark halo density on equator (R,z) = (1kpc,0)
Estimate the total mass of stars and dark matter inside 10kpc.
Estimate the star column density inside |z|<0.1kpc, R=1kpc.

Size and Density of a BH

- A black hole has a finite (schwarzschild) radius $R_{bh}=2\,G\,M_{bh}/c^2 \sim 2au\,(M_{bh}/10^8M_{sun})$
  - verify this! What is the mass of 1cm BH?
- A BH has a density $(3/4Pi)\,M_{bh}/R_{bh}^3$, hence smallest holes are densest.
  - Compare density of 10^8Msun BH with Sun (or water) and a giant star (10Rsun).

Short question

- Recalculate the instantaneous Roche Lobe for satellite on radial orbit, but assume
  - Host galaxy potential $\Phi(R) = V_0^2 \ln(R)$
  - Satellite self-gravity potential $\varphi(r) = v_0^2 \ln(r)$, where v_0, V_0 are constants.
  - Show $M = V_0^2 R/G$, $m = v_0^2 r/G$,
  - Hence Show $r_t/R = \text{cst}\ v_0/V_0$, $\text{cst} = k^{1/2}$

Short questions

- Turn the Sun's velocity direction (keep amplitude) such that the Sun can fall into the BH at Galactic Centre. How accurate must the aiming be in term of angles in arcsec? Find input values from speed of the Sun, BH mass and distances from literature.
- Consider a giant star (of 100solar radii, 1 solar mass) on circular orbit of 0.1pc around the BH, how big is its tidal radius in terms of solar radius? The star will be drawn closer to the BH as it grows. Say BH becomes 1000 as massive as now, what is the new tidal radius in solar radius?

Motions in spherical potential

Equation of motion

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \vec{g} = -\nabla\phi$$

If spherical

$$g_r = -\frac{\partial\phi}{\partial r}$$

$$g_\theta = -\frac{\partial\phi}{\partial\theta} = 0$$

If no gravity

$$\mathbf{x}(t) = \mathbf{v}_0 t + \mathbf{x}_0$$

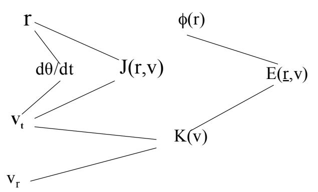
$$\mathbf{v}(t) = \mathbf{v}_0$$

Conserved if spherical static

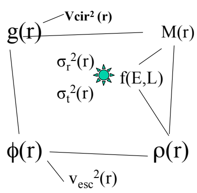
$$E = \frac{1}{2}v^2 + \phi(r)$$

$$L = J = \mathbf{x} \otimes \mathbf{v} = r v_t \cdot \hat{n}$$

Link phase space quantities



Link quantities in spheres



Helpful Math/Approximations (To be shown at AS4021 exam)

- Convenient Units
  $$1\text{km/s} \approx \frac{1\text{pc}}{1\text{Myr}} \approx \frac{1\text{kpc}}{1\text{Gyr}}$$
- Gravitational Constant
  $$G \approx 4\times10^{-3}\text{pc}\ \ (\text{km/s})^2 M_{\text{sun}}^{-1}$$
  $$G \approx 4\times10^{-6}\text{kpc}\ \ (\text{km/s})^2 M_{\text{sun}}^{-1}$$
- Laplacian operator in various coordinates
  $$\nabla\bullet\nabla = \partial_x^2 + \partial_y^2 + \partial_z^2 \text{ (rectangular)}$$
  $$= R^{-1}\partial_R(R\partial_R) + \partial_z^2 + R^{-2}\partial_\phi^2 \text{ (cylindrical)}$$
  $$= \frac{\partial_r(r^2\partial_r)}{r^2} + \frac{\partial_\theta(\sin\theta\partial_\theta)}{r^2\sin\theta} + \frac{\partial_\phi^2}{r^2\sin\theta^2} \text{ (spherical)}$$
- Phase Space Density f(x,v) relation with the mass in a small position cube and velocity cube
  $$dM = f(x,v)dx^3dv^3$$