MATH 332: Vector Analysis
Formulas

Vector Algebra

$$x_1 = x, \quad x_2 = y, \quad x_3 = z$$

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k$$

$$\mathbf{A} = A_i \mathbf{e}_i$$

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

$$|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A} = A_i A_i$$

$$\mathbf{A} \times \mathbf{B} = \varepsilon_{ijk} A_j B_k \mathbf{e}_i = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k$$

$$\mathbf{A} \times \mathbf{A} = 0$$

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \varepsilon_{ijk} A_i B_j C_k$$

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = [\mathbf{B}, \mathbf{C}, \mathbf{A}] = [\mathbf{C}, \mathbf{A}, \mathbf{B}]$$

Line parallel to $\mathbf{A}$:

$$x_i = x_{(0)\,i} + A_i t$$

$$t = \frac{x_i - x_{(0)\,i}}{A_i}, \qquad A_i \neq 0$$

Plane orthogonal to $\mathbf{N}$

$$N_i(x_i - x_{(0)\,i}) = 0$$

Identities:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Tensors:

$$\delta_{ij} = \delta_{ji}$$

$$\delta_{ii} = 3$$

$$\delta_{ij} A_j = A_i$$

$$\delta_{ij} A_i B_j = A_i B_i = \mathbf{A} \cdot \mathbf{B}$$

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$$

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$

$$\varepsilon_{ijj} = \varepsilon_{jij} = \varepsilon_{jji} = 0$$

$$\varepsilon_{ijk}\delta_{ij} = \varepsilon_{ijk}\delta_{ik} = \varepsilon_{ijk}\delta_{jk} = 0$$

$$\varepsilon_{ijk}A_jA_k = \varepsilon_{ijk}A_iA_k = \varepsilon_{ijk}A_iA_j = 0$$

$$\delta^i_j = \delta^j_i = \delta^{ij} = \delta_{ij}$$

$$\varepsilon^{ijk} = \varepsilon_{ijk}$$

$$\varepsilon_{ijk}\varepsilon^{mnl} = \delta^m_i\delta^n_j\delta^l_k + \delta^m_j\delta^n_k\delta^l_i + \delta^m_k\delta^n_i\delta^l_j - \delta^m_i\delta^n_k\delta^l_j - \delta^m_j\delta^n_i\delta^l_k - \delta^m_k\delta^n_j\delta^l_i$$

$$\varepsilon_{ijk}\varepsilon^{mnk} = \delta^m_i\delta^n_j - \delta^m_j\delta^n_i$$

$$\varepsilon_{ijk}\varepsilon^{mjk} = 2\delta^m_i$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6$$

Vector Functions

Position

$$\mathbf{R} = x_i\mathbf{e}_i = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Velocity

$$\mathbf{v} = \frac{d\mathbf{R}}{dt}$$

Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{R}}{dt^2}$$

Arc Length

$$ds = |\mathbf{v}|dt$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau$$

Speed

$$|\mathbf{v}| = \frac{ds}{dt}$$

Tangent

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Curvature

$$k = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

Radius of curvature

$$\rho = \frac{1}{k}$$

Principal Normal

$$\mathbf{N} = \frac{1}{k|\mathbf{v}|}\frac{d\mathbf{T}}{dt}$$

Binormal

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

Torsion

$$\tau = \frac{1}{|\mathbf{v}|}\mathbf{B}\cdot\frac{d\mathbf{N}}{dt}$$

Scalar and Vector Fields

Partial derivatives

$$\partial_i = \frac{\partial}{\partial x_i}$$

$$\partial_1 = \partial_x, \quad \partial_2 = \partial_y, \quad \partial_3 = \partial_z$$

Nabla (Del) Operator

$$\nabla = \mathbf{e}_i\partial_i = \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z$$

Gradient

$$\mathbf{grad}\ f = \nabla f = \mathbf{e_i}\partial_i f$$

$$\nabla_i f = \partial_i f$$

Directional derivative

$$\frac{df}{ds} = \frac{d\mathbf{R}}{ds}\cdot\mathbf{grad} f = \frac{dx_i}{ds}\partial_i f$$

Flow curves

$$\frac{dx_i}{ds} = \beta F_i$$

$$\beta ds = \frac{dx_i}{F_i}$$

Divergence

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \partial_i F_i$$

Curl

$$\mathbf{curl}\ \mathbf{F} = \nabla \times \mathbf{F} = \varepsilon_{ijk}\partial_j F_k \mathbf{e}_i$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$(\mathbf{curl}\ \mathbf{F})_i = \varepsilon_{ijk}\partial_j F_k$$

Laplacian

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \partial_i \partial_i = \partial_x^2 + \partial_y^2 + \partial_z^2$$

Vector identities:

$$\nabla \times \nabla\varphi = 0, \qquad \boxed{\mathbf{curl}\ \mathbf{grad} = 0}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0, \qquad \boxed{\text{div } \mathbf{curl}} = 0$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$

$$\nabla(f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f\nabla\mathbf{F}$$

$$\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F})$$

$$\nabla f(\varphi) = \frac{df}{d\varphi}\nabla\varphi$$

$$\nabla\mathbf{R} = 3$$

$$\nabla \times \mathbf{R} = 0$$

$$(\mathbf{F} \cdot \nabla)\mathbf{R} = \mathbf{F}$$

Cylindrical Coordinates:

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$$

$$dV = \rho\, d\rho\, d\theta\, dz$$

$$\nabla f = \left(\mathbf{e}_\rho \partial_\rho + \mathbf{e}_\theta \frac{1}{\rho}\partial_\theta + \mathbf{e}_z \partial_z\right) f$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\partial_\rho(\rho F_\rho) + \frac{1}{\rho}\partial_\theta F_\theta + \partial_z F_z$$

$$\nabla \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \partial_\rho & \partial_\theta & \partial_z \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

$$\Delta f = \frac{1}{\rho}\,\partial_\rho\left(\rho\,\partial_\rho\, f\right) + \frac{1}{\rho^2}\partial_\theta^2\, f + \partial_z^2\, f$$

Spherical Coordinates

$$ds^2 = dr^2 + r^2 d\varphi^2 + r^2 \sin^2\varphi d\theta^2$$

$$dV = r^2 \sin\varphi\, dr\, d\varphi\, d\theta$$

$$\nabla f = \left(\mathbf{e}_r \partial_r + \mathbf{e}_\varphi \frac{1}{r}\partial_\varphi + \mathbf{e}_\theta \frac{1}{r\sin\varphi}\partial_\theta\right) f$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\partial_r(r^2 F_r) + \frac{1}{r\sin\varphi}\partial_\varphi(\sin\varphi\, F_\varphi)$$

$$+\frac{1}{r\sin\varphi}\partial_\theta F_\theta$$

$$\nabla \times \mathbf{F} = \frac{1}{r^2\sin\varphi}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\varphi & r\sin\varphi, \mathbf{e}_\theta \\ \partial_r & \partial_\varphi & \partial_\theta \\ F_r & rF_\varphi & r\sin\varphi F_\theta \end{vmatrix}$$

$$\Delta f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\varphi}\partial_\varphi(\sin\varphi\partial_\varphi f)$$

$$+\frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f$$

Line, Surface and Volume Integrals

Line Integrals

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \frac{d\mathbf{R}}{dt}\ dt$$

$$= \int_a^b F_i\ \frac{dx_i}{dt}\ dt$$

Potentials

$$\mathbf{F} = \nabla\varphi \ \Leftrightarrow\ \varphi(x, y, z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{R}$$

$$\int_P^Q \nabla\varphi \cdot d\mathbf{R} = \varphi(Q) - \varphi(P)$$

$$\mathbf{F} = \nabla\times\mathbf{G} \;\Leftrightarrow\; \mathbf{G}(x,y,z) = \int_0^1 \mathbf{F}(tx,ty,tz)\times\mathbf{R}\,t\,dt$$

Unit Normal:
to a surface $\mathbf{R} = \mathbf{R}(u,v)$

$$\mathbf{n} = \frac{\partial_u\mathbf{R}\times\partial_v\mathbf{R}}{|\partial_u\mathbf{R}\times\partial_v\mathbf{R}|}$$

to a surface $f(x,y,z) = C$

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}$$

Surface Element

$$d\mathbf{S} = \partial_u\mathbf{R}\times\partial_v\mathbf{R}\,du\,dv$$

$$d\,S = |\partial_u\mathbf{R}\times\partial_v\mathbf{R}|\;du\,dv$$

For a surface given by

$$z = f(x,y),\;\; a \le x \le b,\;\; y_1(x) \le y \le y_2(x)$$

$$dS = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\;dy\,dx = \frac{dx\,dy}{|\cos\gamma|}$$

Flux through S

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_S \mathbf{F}\cdot\mathbf{n}\,d\,S$$

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} \mathbf{F}\cdot\mathbf{n}\,\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\;dy\,dx$$

Divergence Theorem

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

Green's Theorem

$$\int_C (F_1 dx + F_2 dy) = \iint_D (\partial_x F_2 - \partial_y F_1)dxdy$$

Stokes' Theorem

$$\iint_D (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{R}$$