AS5001 (= SUPAAAA)
ADA= "Advanced" (Astronomical) Data Analysis

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ADA web page: http://star-www.st-and.ac.uk/~kdh1/ada/ada.html

All lecture pdfs, homework, projects, videos on Moodle.

Supplementary Texts:

Press et al. (CUP) ***Numerical Recipes : The Art of Scientific Computing***
(on the web at Numerical.Recipes)
Wall & Jenkins (CUP) ***Practical Statistics for Astronomers***
Gregory (CUP) ***Bayesian Logical Data Analysis for the Physical Sciences***

Opinionated Lessons in Statistics, *by Bill Press.* *OpinionatedLessons.org*

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ADA= "Advanced" (Astronomical) Data Analysis

Goal: Build concepts and skills for analysing quantitative data.

~15 Lectures: develop basic principles, illustrate with examples, extend step-by-step to build expertise for advanced analysis of datasets.

50% 2 Homework sets: test understanding, build skills

50% 2 Projects: analyse real datasets (Keck, HST)

NO EXAM :)

Work steadily, ask questions, get help when you don't understand, and you will succeed.

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ADA 01 - 10am Mon 12 Sep 2022

Astronomical Data + Noise
Statistical vs Systematic errors
Probability distributions (pdf, cdf)
Mode, Mean, Median
Variance, standard deviation, MAD
Skewness, Kurtosis
Parameterised distributions
(Uniform, Gaussian, Lorentzian, Poisson, Exponential, Chi^2)

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ADA Lecture 1 Outline

- ***Astronomical Data Sets***
- ***Noise :***
  - ***statistical vs systematic errors***
- ***Probability distributions :***
  - ***Mean vs Median***
  - ***Variance (standard deviation) vs MAD***
  - ***Central moments (skewness, kurtosis)***
- ***Survey of parameterised distributions***
  - ***Uniform, Gaussian, Lorentzian, Poisson, Exponential, Chi-squared***

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Astronomical Datasets

- (Almost) all our information about the Universe arrives as photons. (neutrinos, gravitational waves)
- ***Photon properties:***
  position: $\vec{x}$
  time: t
  direction: α, δ
  energy: $E = h\nu = hc/\lambda$
  polarisation: (Stokes parameters, $\vec{p} = I,Q,U,V$)
- Astronomical datasets are (usually) photon distributions confined by a detector to (some subset of) these properties:

$$D_i = \int P_i(\vec{x},t,\alpha,\delta,\lambda,\vec{p})\; f(\vec{x},t,\alpha,\delta,\lambda,\vec{p})\; d(\vec{x},t,\alpha,\delta,\lambda,\vec{p}) \;+\; Noise_i$$

Photon detection probability for data point *i*   **Photon distribution**

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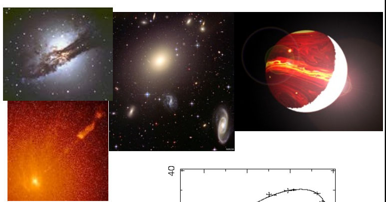
Astronomical Datasets

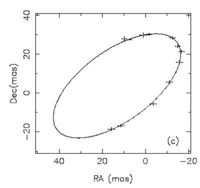
- ***Direct imaging:***
  - ***size***
  - ***structure***

$$D(\alpha,\delta)$$



- ***Astrometry:***
  - ***distance***
  - ***parallax***
  - ***motion***
  - ***proper motion***
  - ***visual binary orbits***

$$D(\alpha,\delta,t)$$



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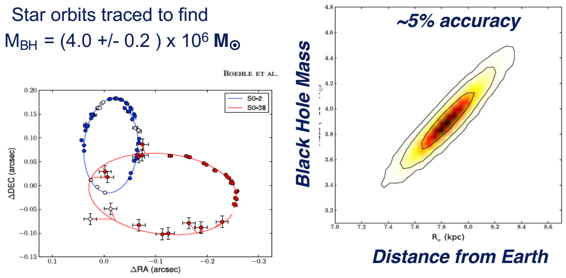
Black Hole Mass from Stellar Orbits

- Black Hole **in the Galactic Centre**
- Star orbits traced to find
  M_{BH} = (4.0 +/- 0.2) x 10^6 **$M_\odot$**



Boehle, Ghez et al (2016) ApJ

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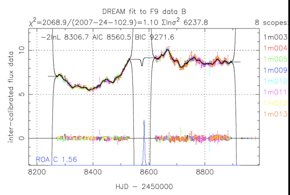
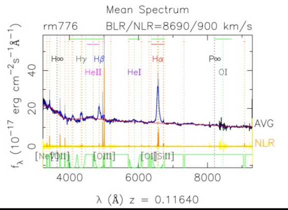
Astronomical Datasets

- ***Light curves:*** $D(t)$
  - *Time variations*
  - *Orbital periods*



- ***Spectra:*** $D(\lambda)$
  - *Physical conditions*
  - *Temperature, density*
  - *Velocities => masses*



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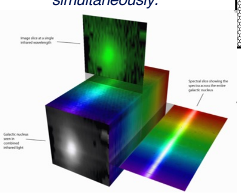
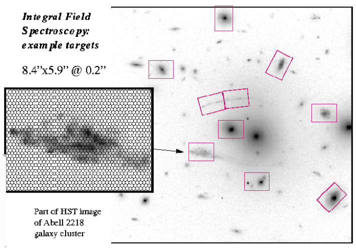
Integral-field Spectroscopy: $D(\alpha,\delta,\lambda)$

- *Close-packed array of fibres (or lenslets) giving spectra over a grid of positions on the sky.*
- *Probes spatial and spectral structure simultaneously.*





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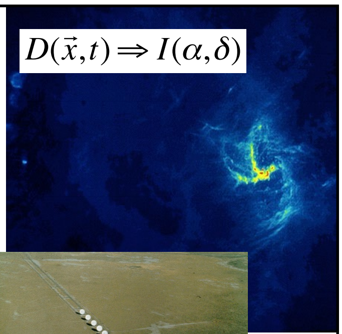
Interferometry: $D(\vec{x},t) \Rightarrow I(\alpha,\delta)$

- An example of ***Indirect Imaging***
- Use information about ***arrival time at different locations*** to infer angular structure of source.
- Picture: 6 cm radio map of "mini-spiral" of gas around Sgr A* (=black hole at the centre of our Milky Way galaxy).



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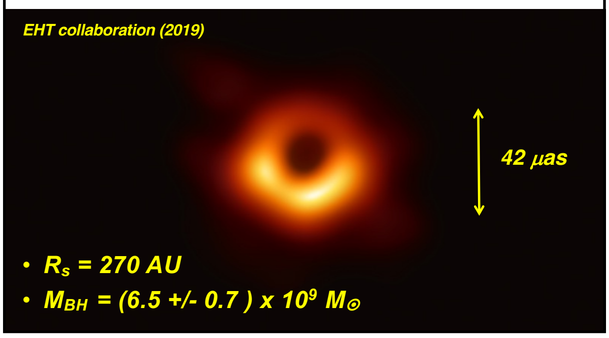
Black Hole in M87
imaged by the Event Horizon Telescope



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Data are Data

There are many different types of data.

Photon properties define the dimensions of (most) astronomical datasets.

But: The same analysis techniques apply to all quantitative datasets.

(Astronomical or otherwise.)

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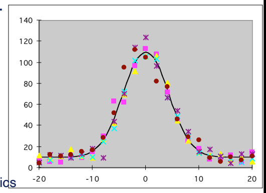
Data are affected by Noise

- Repetitions of the same experiment or observation give different results.
- e.g. spectral-line profile:



- Sources of noise:
- ***Quantum (Poisson) noise***
  - finite number of photons
- ***Thermal noise***
  - thermal fluctuations in the detector/electronics
- ***Rare events***
  - cosmic ray hits, instrument failures

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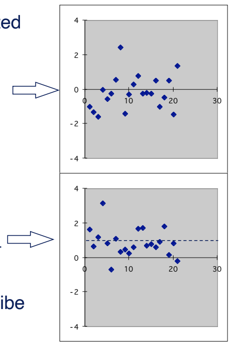
Data Values as "Random Variables"

- Consider an ensemble of repeated measurements.
- Data values *"dance"* around.
- ***Statistical errors:***
  - From random nature of measurement process.
  - Can be reduced by averaging repeat measurements.
- ***Systematic errors (bias):***
  - Due to flawed measurement technique.
  - Bias remains after averaging repeat measurements.
- ***Probability distributions*** describe this "dance" of the data values.



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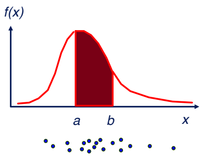
Probability Distributions (PDFs)

- ***Probability distribution** f(x)*
- aka: *probability density function* (pdf)
- defines the probability that *x* lies in some range:

$$P(a < x \le b) \equiv \int_a^b f(x)\,dx$$



- ***Probabilities add up to 1.***
- If *x* can take any value between -∞ and +∞ then

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

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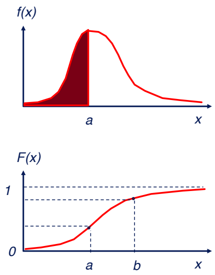
Cumulative Probability Functions (CDFs)

- Integrating *f(x)* gives the ***cumulative probability*** *F(a)* that $x \le a$:

$$F(a) \equiv P(x \le a) \equiv \int_{-\infty}^{a} f(x)\,dx$$

$$F(-\infty) = 0 \qquad F(+\infty) = 1$$

$$P(a < x \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$$



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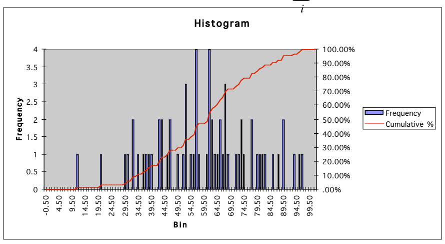
Discrete Probability Distributions

- Example:
  - Exam marks
  - Photons per pixel

$$f(x) \equiv \sum_i p_i\,\delta(x - x_i)$$

$$F(x) \equiv \sum_i p_i \text{ for all } x_i \le x$$



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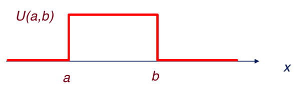
Uniform Distribution *U(a,b)*

- Also known as a "**boxcar**" or "**tophat**" distribution:

$$f(x) = \frac{1}{|b-a|} \quad \text{for } a < x < b$$

$$f(x) = 0 \quad \text{otherwise.}$$



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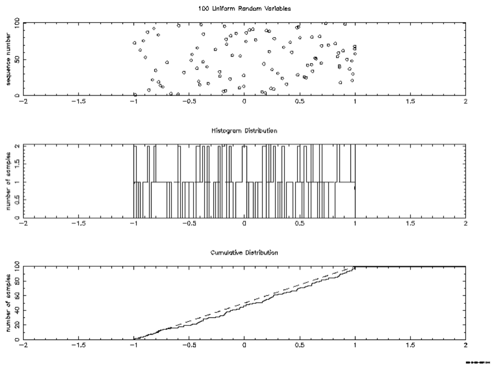

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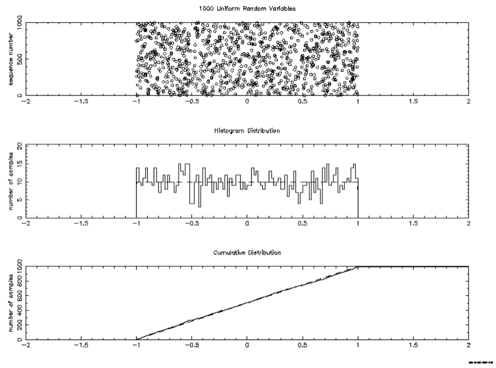

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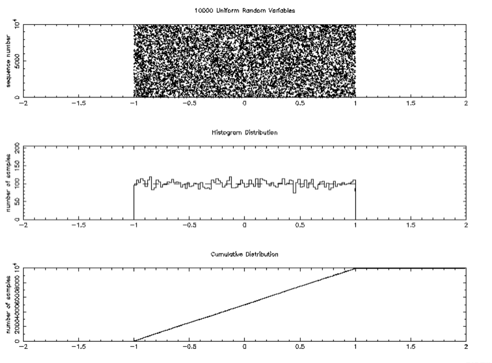

Note that the histograms converge to *f(x)* and *F(x).*

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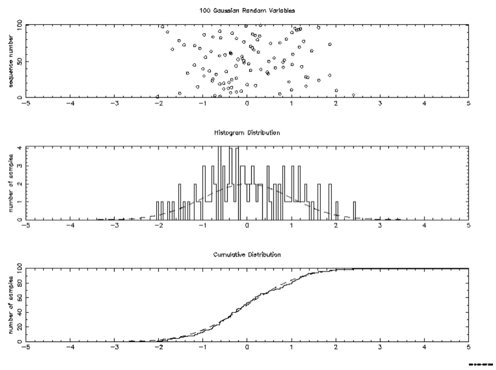

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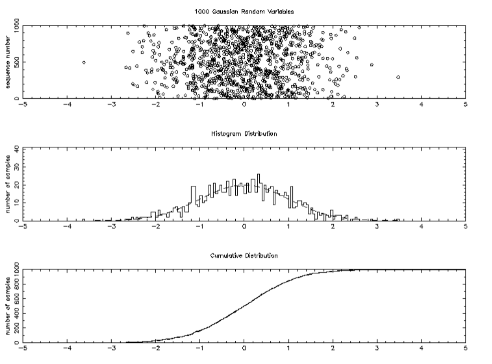

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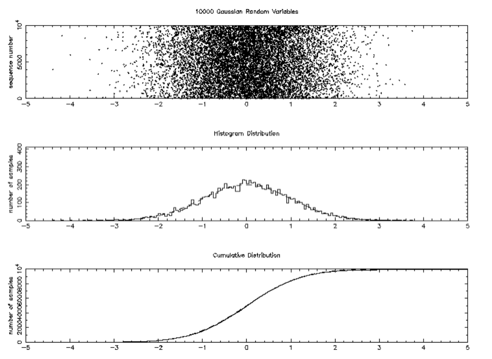

Note that the histograms converge to *f(x)* and *F(x).*

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Moments of Distributions

- The **moments** of a distribution characterise its **location**, **width** and **shape**.
- Strong physical analogy with moments in mechanics of rigid bodies:
  - Centre of mass = first moment
  - Moment of inertia = second (central) moment
  - Higher moments => info on the shape of the distribution

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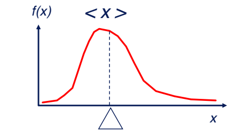
Location measures: Mode, Mean and Median

- ***Mode*** (highest probability density)
- ***Mean*** (centre of mass)
  = probability-weighted average of x

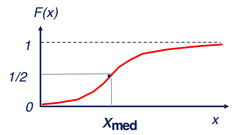
$$\langle x \rangle \equiv \int f(x)\, x\, dx$$



- ***Median*** (50th percentile)

$$F(x_{\mathrm{med}}) \equiv \frac{1}{2}$$

$$P(x < x_{\mathrm{med}}) = P(x > x_{\mathrm{med}})$$



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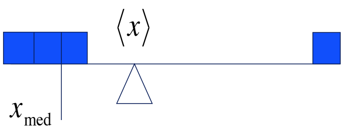
Mean vs Median

- **Median** is less sensitive to the long wings of a distribution -- the outliers.



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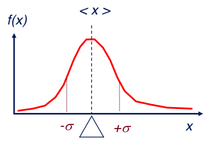
Width Measures: Standard Deviation, MAD

- **Standard deviation** σ measures **width** of distribution.
- **Variance** σ² (moment of inertia)

$$\sigma^2(x) = \sigma_x^2 = \mathrm{Var}(x) \equiv \left\langle [x - \langle x \rangle]^2 \right\rangle$$

$$= \int f(x) [x - \langle x \rangle]^2\, dx$$



Mean Absolute Deviation (MAD):

$$\mathrm{MAD} \equiv \left\langle \left| x - x_{\mathrm{med}} \right| \right\rangle$$

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Shape : Higher-order (Central) Moments

- General form: $m_n \equiv \left\langle \left[\frac{x - \langle x \rangle}{\sigma} \right]^n \right\rangle$

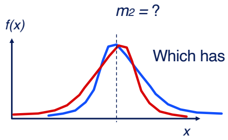
(n^{th} central moment in units of σ^n)

$m_1 = ?$

$m_2 = ?$

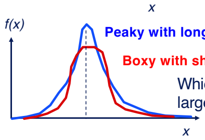
Higher central moments $n = 3, 4, \ldots$ define the **shape** of the distribution.

- **Skewness (m_3) :** (asymmetric tails)



Which has $m_3 > 0$?

- **Kurtosis (m_4) :**



Peaky with long tails

Boxy with short tails

Which has larger m_4 ?

If you know ***all*** the moments, you know the full shape.

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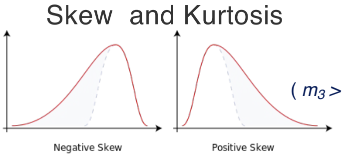
Skew and Kurtosis

($m_3 < 0$)

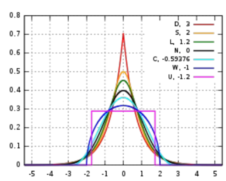


($m_3 > 0$)

"Leptokurtic" ($m_4 > 3$) with longer tails, like a kangaroo (leaps)

"Mesokurtic" ($m_4 = 3$) like a Gaussian.

"Platykurtic" ($m_4 < 3$) with shorter tails, like a platypus.



$m_4 > 3$ increases peak and wings relative to a Gaussian

Excess Kurtosis ($m_4 - 3$) defined relative to the kurtosis of a Gaussian.

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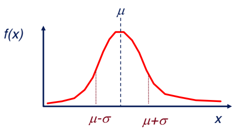
Gaussian Distribution $G(\mu, \sigma^2)$

- Also known as a **Normal** distribution. $N(\mu, \sigma^2)$
- Physical example: thermal Doppler broadening

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



- 2 parameters:
- Mean (expected) value:
- $E(x) = \langle x \rangle = \mu$
- Variance: $\mathrm{Var}(x) = \sigma^2(x) = \sigma^2$
- Standard deviation (dispersion) σ
- Full width at half maximum (FWHM)

$$\mathrm{FWHM} = \sqrt{8\ln 2}\,\sigma \approx 2.3\sigma$$

- 32% probability that x is outside $\mu \pm \sigma$
- 4.5% for x outside $\mu \pm 2\sigma$
- 0.3% for x outside $\mu \pm 3\sigma$

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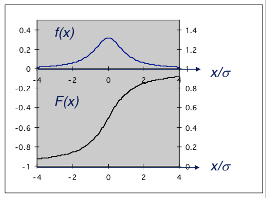
Lorentzian (Cauchy) Distribution $L(\mu,\sigma)$

- Peak at $x = \mu$, HWHM $= \sigma$.
- Physical example: damping wings of spectral lines.



$$f(x) = \frac{\sigma}{\pi}\frac{1}{\sigma^2 + (x-\mu)^2}$$

$$F(x) = \frac{1}{\pi}\tan^{-1}\left(\frac{x-\mu}{\sigma}\right) + \frac{1}{2}$$

- Pathological: wings so broad that all moments diverge! ☹

$$\langle x \rangle \equiv \frac{\sigma}{\pi}\int_{-\infty}^{\infty}\frac{x\,dx}{\sigma^2 + (x-\mu)^2} \propto \ln\left(\left|1+x^2\right|\right)\Big|_{-\infty}^{\infty} = \infty - \infty$$

$$\left\langle (x-\mu)^2 \right\rangle = \infty$$

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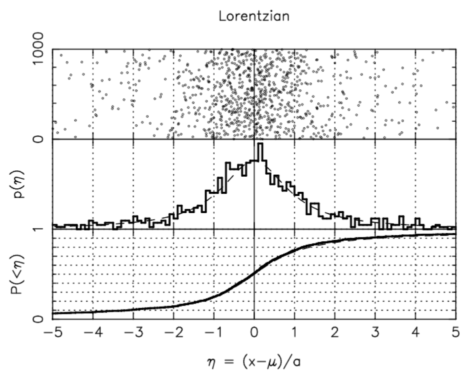

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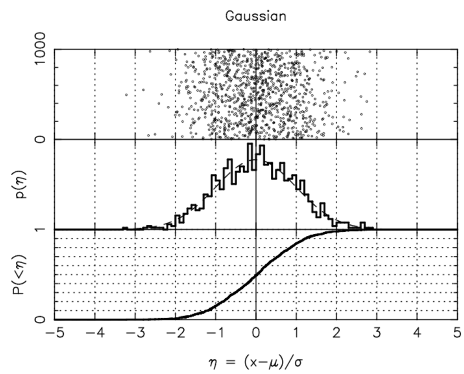

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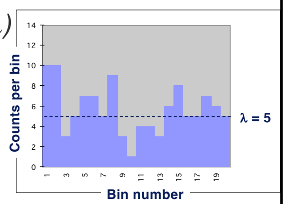
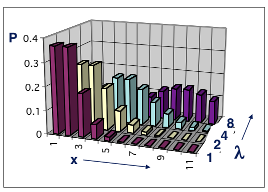
Poisson Distribution $P(\lambda)$



- A discrete distribution
- Describes counting statistics:
  - Raindrops in bucket per time interval
  - Photons per pixel during exposure
- **λ = mean count rate**
  - **Not necessarily an integer !**



$$f(x) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\delta(x-n)$$

$$P(x=n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad n = 0,1,2,\ldots$$

$$\langle x \rangle = \lambda$$

$$\sigma^2(x) = \lambda \Rightarrow \boxed{\sigma(x) = \sqrt{\langle x \rangle}}$$

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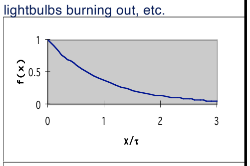
Exponential Distribution $E(\tau)$

- Distribution of time intervals between random events
  - Raindrops, photons, radioactive decays, lightbulbs burning out, etc.

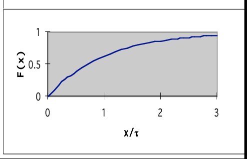
$$f(x) = \frac{1}{\tau}e^{-x/\tau}$$

$$F(x) = 1 - e^{-x/\tau}$$



$$\langle x \rangle = \tau = \text{ mean time between events}$$

$$\mathrm{Var}[x] = \left\langle (x-\tau)^2 \right\rangle = \tau^2$$



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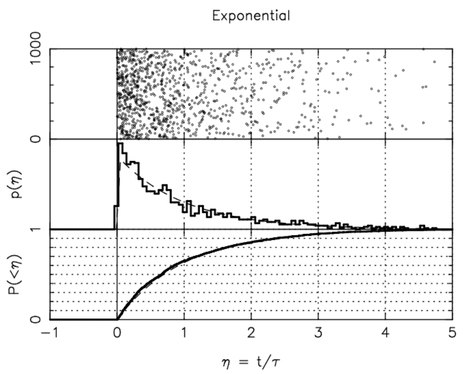

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Chi-Squared Distribution χ^2_N

- Sum of squares of N independent Gaussian random variables

$\chi^2_N \equiv$ Chi-Squared with N degrees of freedom

X and Y are independent Gaussian random variables.

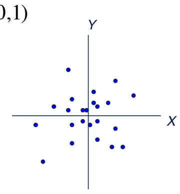
i.e. $X \sim G(0,1)$ $\quad Y \sim G(0,1)$

then $X^2 \sim \chi^2_1$ $\quad Y^2 \sim \chi^2_1$

$X^2 + Y^2 \sim \chi^2_2$

and so on for each new degree of freedom:

$\chi^2_N + \chi^2_M \sim \chi^2_{N+M}$



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Chi-Squared = "Badness of Fit"

$$\chi^2 \equiv \sum_{i=1}^{N} \left(\frac{D_i - \mu_i(\alpha)}{\sigma_i} \right)^2 \sim \chi^2_{N-P}$$

D_i = data value

σ_i = 1 - σ error bar

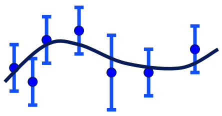
$\mu_i(\alpha)$ = model predicted data value

α = parameters of the model

N = number of data points

P = number of fitted parameters

$N - P$ = degrees of freedom



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χ^2 distribution with N degrees of freedom

$$f(x) = \frac{1}{\Gamma(N/2)\, 2^{N/2}}\, x^{(N/2-1)}\, e^{-x/2}$$

$\Gamma(1) = 1 \quad \Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n) = (n-1)! \quad \Gamma(x+1) = x\,\Gamma(x)$

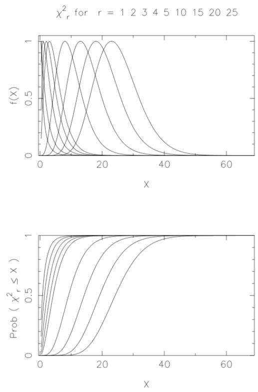
e.g. $\Gamma(3/2) = (1/2)\,\Gamma(1/2) = \sqrt{\pi}/2$

$\chi^2_1: \quad f(x) = \left(\frac{e^{-x}}{2\,\pi\, x} \right)^{1/2}$

$\chi^2_2: \quad f(x) = \frac{1}{2} e^{-x/2}$

$\langle \chi^2_N \rangle = N$

$\sigma^2(\chi^2_N) = 2N$



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χ^2_N and reduced χ^2_N distribution

- Sum of squares of N independent Gaussian random variables

χ^2_N = chi – squared with N degrees of freedom

$\langle \chi^2_N \rangle = N$

$\sigma^2\left(\chi^2_N\right) = 2N$

$\sigma\left(\chi^2_N\right) = \sqrt{2N}$

Reduced χ^2_N

$\left\langle \frac{\chi^2_N}{N} \right\rangle = 1$

$\sigma^2\left(\frac{\chi^2_N}{N}\right) = \frac{2}{N}$

$\sigma\left(\frac{\chi^2_N}{N}\right) = \sqrt{\frac{2}{N}}$

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