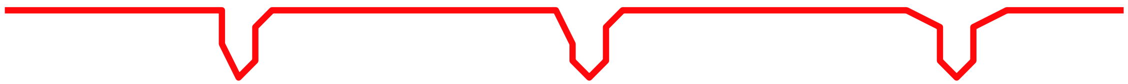
ADA 13 -- 9am Tue 11 Oct 2022

Time Series Analysis, Ephemerides

Fourier Analysis:
Fourier frequencies and basis functions,
Nyquist sampling.

Periodogram analysis (part 1):
sidelobes, aliasing, harmonics

Timing Analysis - Defining an Ephemeris



Timings: Observed times of a fiducial point in a periodic lightcurve, e.g. mid-eclipse.

$$t_i \pm \sigma_i$$

The Ephemeris:

$$t = t_0 + P\,E = \text{predicted time}$$

$$t_0 = \text{epoch of phase 0}$$

$$P = \text{period}$$

$$E = n + \phi = \text{cycle number + phase}$$

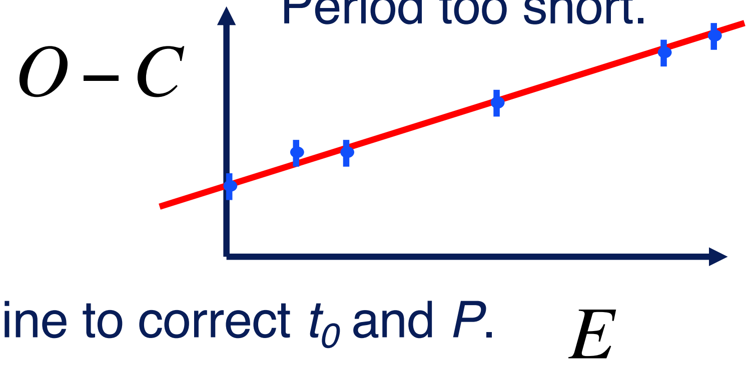
$$O\text{-}C = \text{observed - calculated}$$

$$= t_i - (t_0 + P\,E_i)$$

$$n_i = \text{NINT}\left[\frac{t_i - t_0}{P}\right]$$

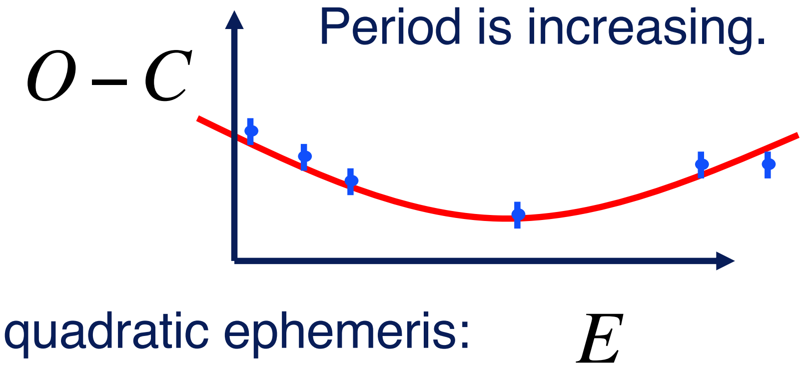
$$\text{phase:} \quad \phi_i = \frac{t_i - t_0}{P} - n_i\,, \quad 0 < \phi < 1$$

Epoch too early.
Period too short.



Fit a line to correct t_0 and P.

Period is increasing.



Fit quadratic ephemeris:

$$t = t_0 + P_0\,E + B\,E^2$$

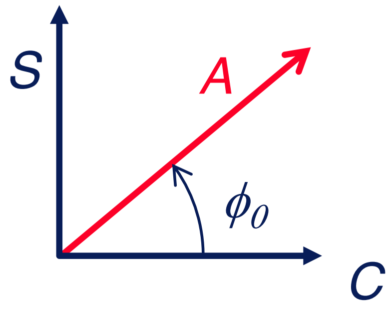
$$P = dt/dE = P_0 + 2\,B\,E$$

$$\dot{P} = dP/dt = 2\,B/P$$

Hunting for Sinusoidal Signals

(e.g. Planet hunting -- circular orbit radial velocity curve)

Search a time series for a sinusoidal oscillation of unknown frequency ω :



- Fit a sinusiod (scale 3 patterns):

$$X(t) = X_0 + A\cos(\omega t + \phi_0)$$

$$= X_0 + C\cos\omega t + S\sin\omega t$$

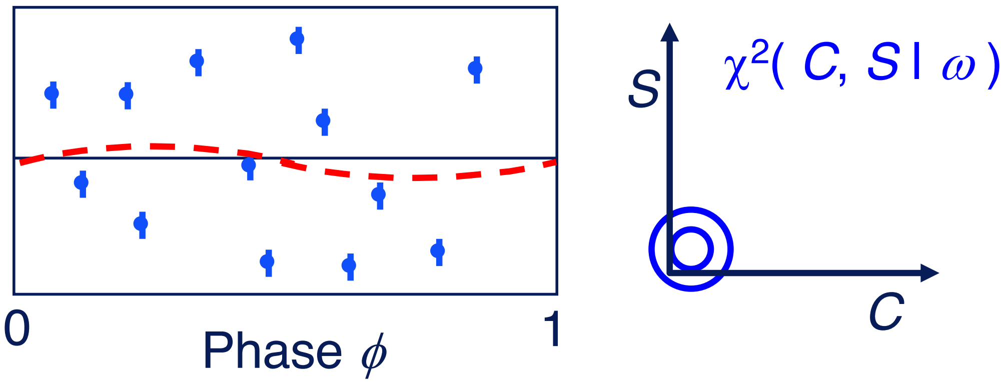
$$\text{Amplitude}: \quad A^2 = C^2 + S^2$$

$$\text{Phase at } t = 0: \quad \phi_0 = \tan^{-1}(-S/C)$$

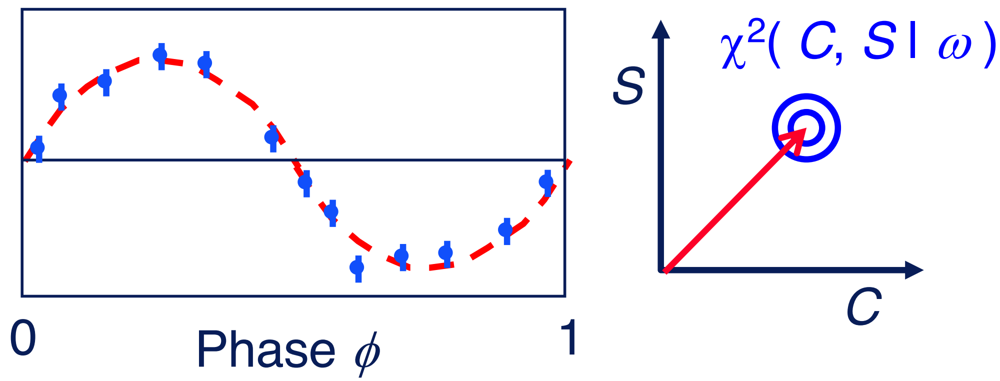
Programming hint:
Use `phi=atan2(–S,C)` if you care about which quadrant ϕ ends up in!

- "Fold" data on a trial period $P = 2\pi / \omega$

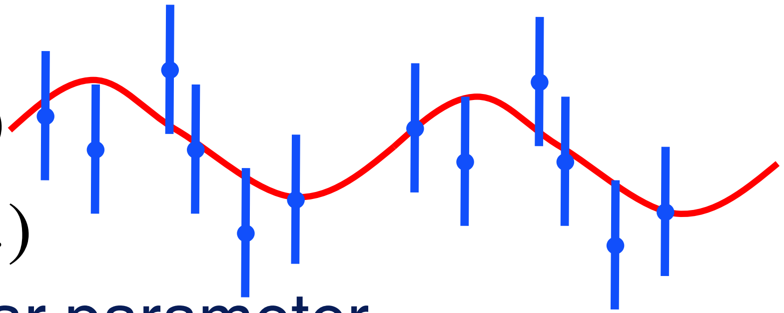
Wrong ω : bad χ^2, small A



Correct ω : good χ^2, large A



Sinusoid + Background



$$\text{Data}: X_i \pm \sigma_i \text{ at } t = t_{\text{i}}$$

$$\text{Model}: X(t) = X_0 + S\sin(\omega t) + C\cos(\omega t)$$

$$\text{3 Patterns}: 1, \quad s_i = \sin(\omega t_i), \quad c_i = \cos(\omega t_i)$$

3-parameter linear regression + 1 non-linear parameter

Iterated Optimal Scaling:

$$\hat{X}_0 = \frac{\sum \left(X_i - \hat{S}\, s_i - \hat{C}\, c_i \right) / \sigma_i^2}{\sum 1/\sigma_i^2}, \quad \text{Var}\left[\hat{X}_0\right] = \frac{1}{\sum 1/\sigma_i^2}$$

$$\hat{S} = \frac{\sum \left(X_i - \hat{X}_0 - \hat{C}\, c_i \right) s_i / \sigma_i^2}{\sum s_i^2/\sigma_i^2}, \quad \text{Var}\left[\hat{S}\right] = \frac{1}{\sum s_i^2/\sigma_i^2}$$

$$\hat{C} = \frac{\sum \left(X_i - \hat{X}_0 - \hat{S}\, s_i \right) c_i / \sigma_i^2}{\sum c_i^2/\sigma_i^2}, \quad \text{Var}\left[\hat{C}\right] = \frac{1}{\sum c_i^2/\sigma_i^2}$$

Iterate (if patterns not orthogonal).

Variance formulas assume orthogonal parameters, otherwise give too small error bars.

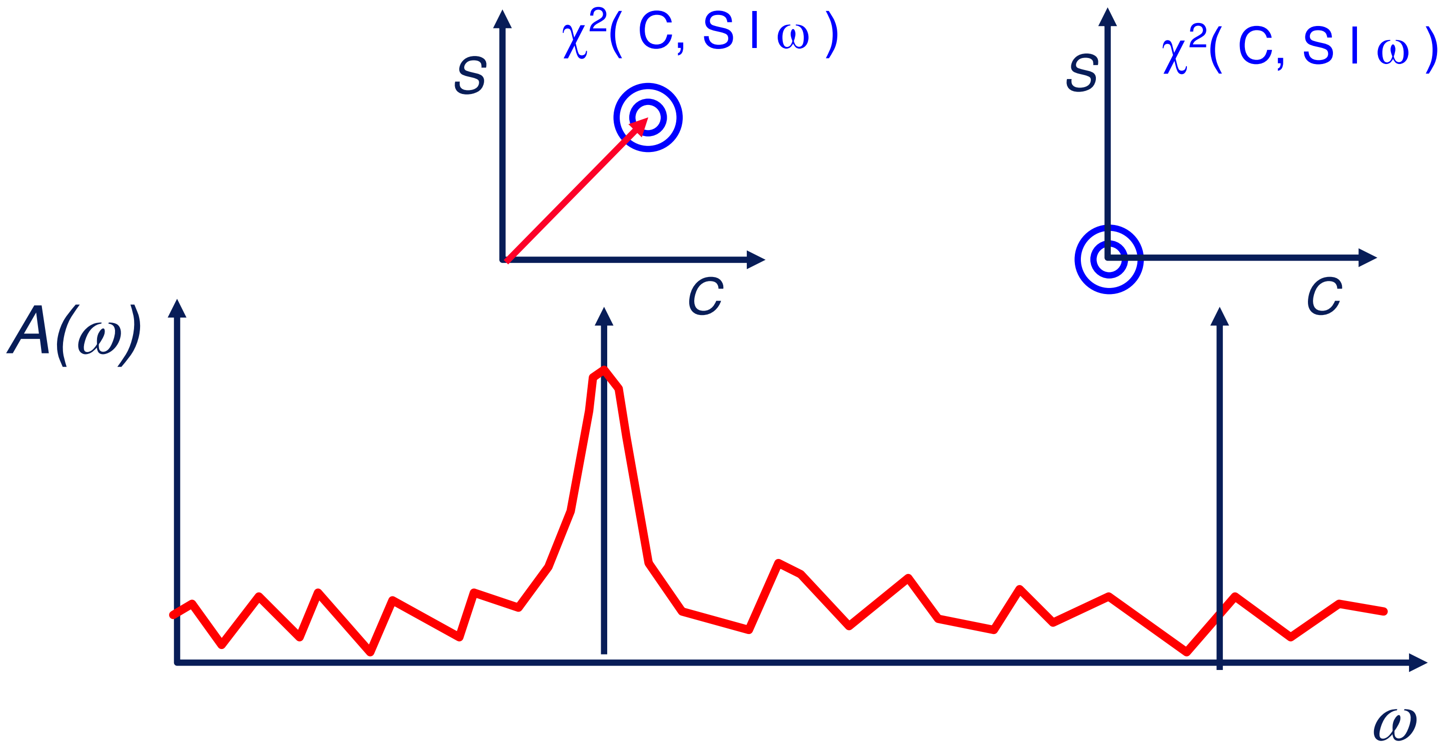
Use the inverse-Hessian matrix, e.g. when phase coverage is not close to uniform.

Periodogram : grid scan in frequency

Model is non-linear in ω (or $P = 2\pi / \omega$, or $f = 1/P$).

Use **grid-search**: fit sine curve for a grid of ω values.

Periodogram: plot $A(\omega)$ and/or $\chi^2(\omega)$.



Periodogram of a finite data train

Purely sinusoidal time variation.

Sampled at *N* regularly spaced time intervals Δt

The **periodogram**:

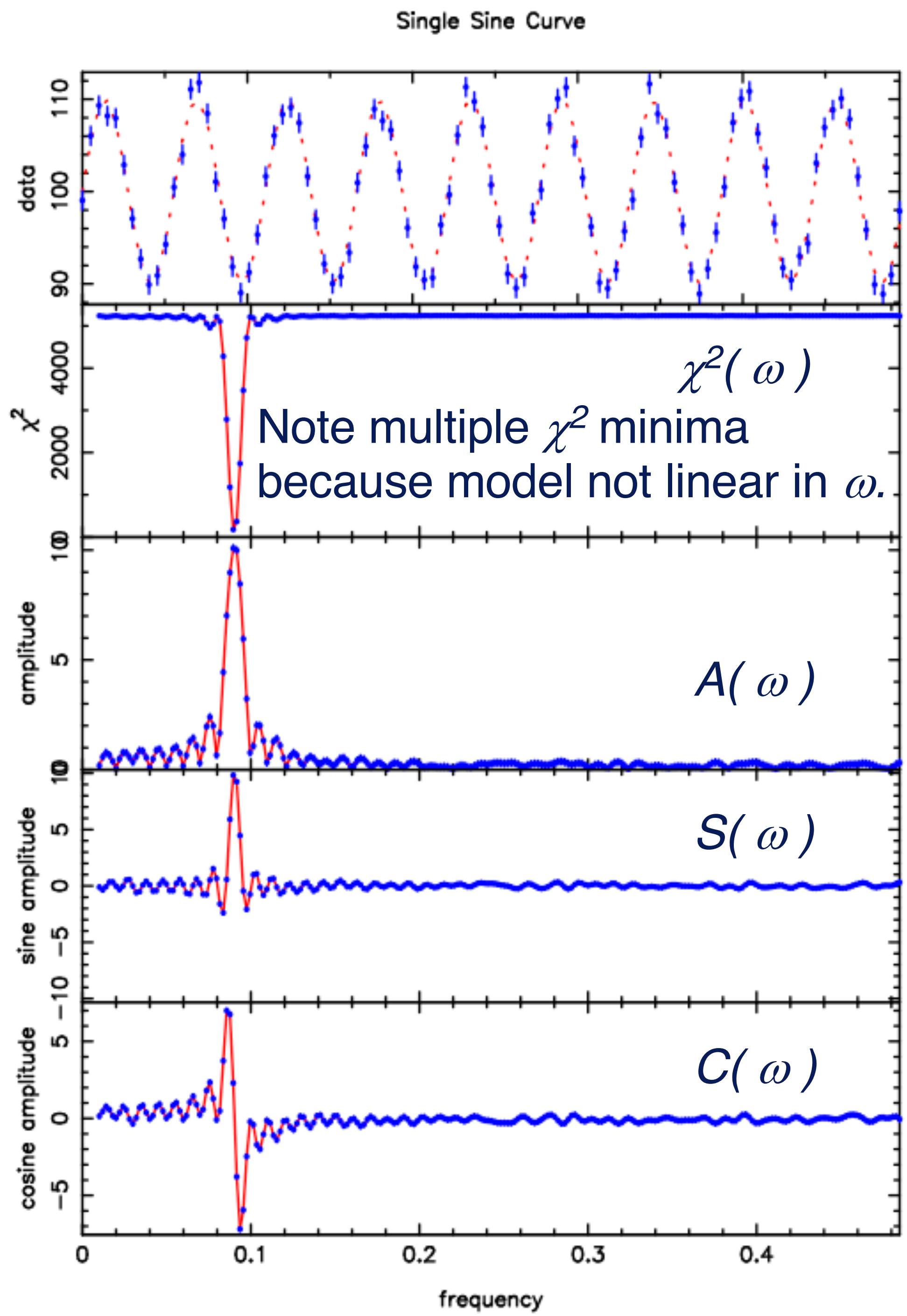
Note χ^2 minimum and peak in *A* at correct ω.

Use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

Note sidelobes and finite width of peak.

Why not a delta function?

(Spectral leakage)



Spectral Leakage due to finite timespan T

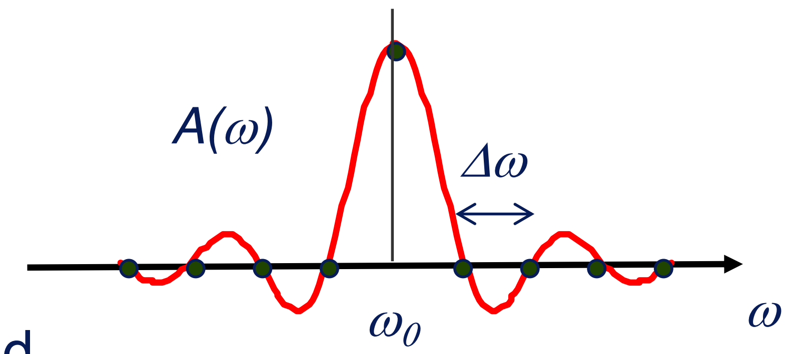
A pure sinusoid at frequency ω_0 "leaks" into adjacent frequencies ω due to the finite timespan T of the data.

$$\hat{A}(\omega) \approx A_0 \frac{\sum (\sin \omega_0 t_i)(\sin \omega t_i) / \sigma_i^2}{\sum (\sin \omega t_i)^2 / \sigma_i^2}$$

= Optimal Scaling of the pattern sin(ωt) to fit data varying as $A_0 \sin(\omega_0 t)$.

Special case : **Evenly spaced data ,**
at times $t_i = t_0 + i\, \Delta t$ for $i = 1,..N$,
and **Equal error bars,** $\sigma_i = \sigma$:

$$\hat{A}(\omega) = A_0 \frac{\sin \pi x}{\pi x} \quad \text{where} \quad x = \frac{\omega - \omega_0}{\Delta\omega}$$



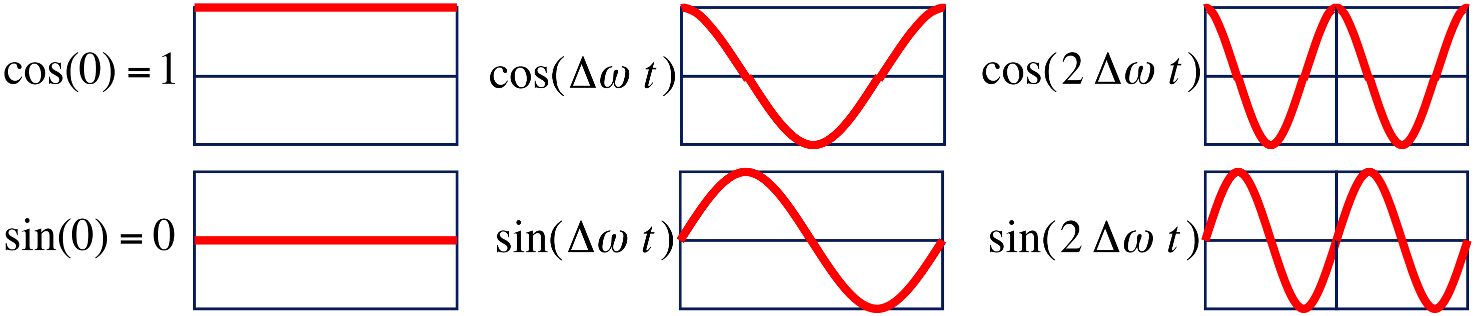
This "Sinc" function has a ***1/x* envelope** and **evenly spaced zeroes** at frequency step

$$\Delta\omega = 2\pi / N \Delta t = 2\pi / T .$$

"De-tuning" by $\Delta\omega$ gives an **orthogonal function** with 1 extra cycle per time $T = N \Delta t$.

Fourier Frequencies and Basis Functions

$$f(t) = \sum_{k=0}^{K_{\max}} \left[S_k \sin(\omega_k t) + C_k \cos(\omega_k t) \right]$$



De-tuning by Δω gives an orthogonal function with 1 extra cycle during time *T*.

Orthogonal for **evenly-spaced data** with **equal error bars**.

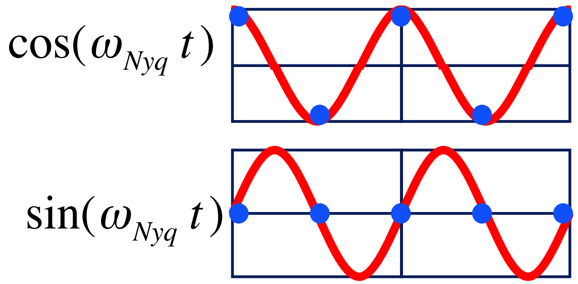
$$t_i = t_0 + i\,\Delta t \qquad i = 1,2,\ldots,N \qquad T = N\,\Delta t$$

Fourier frequencies:

$$\omega_k = k\,\Delta\omega \qquad k = 0,1,\ldots,K_{\max} \qquad \Delta\omega = 2\pi / T$$

Nyquist frequency = 1 cycle / 2 data points

$$\omega_{Nyq} = \frac{2\pi}{2\,\Delta t} = \frac{N\,\pi}{T} = \frac{N}{2}\Delta\omega \quad \Rightarrow \quad K_{\max} = \frac{N}{2}$$

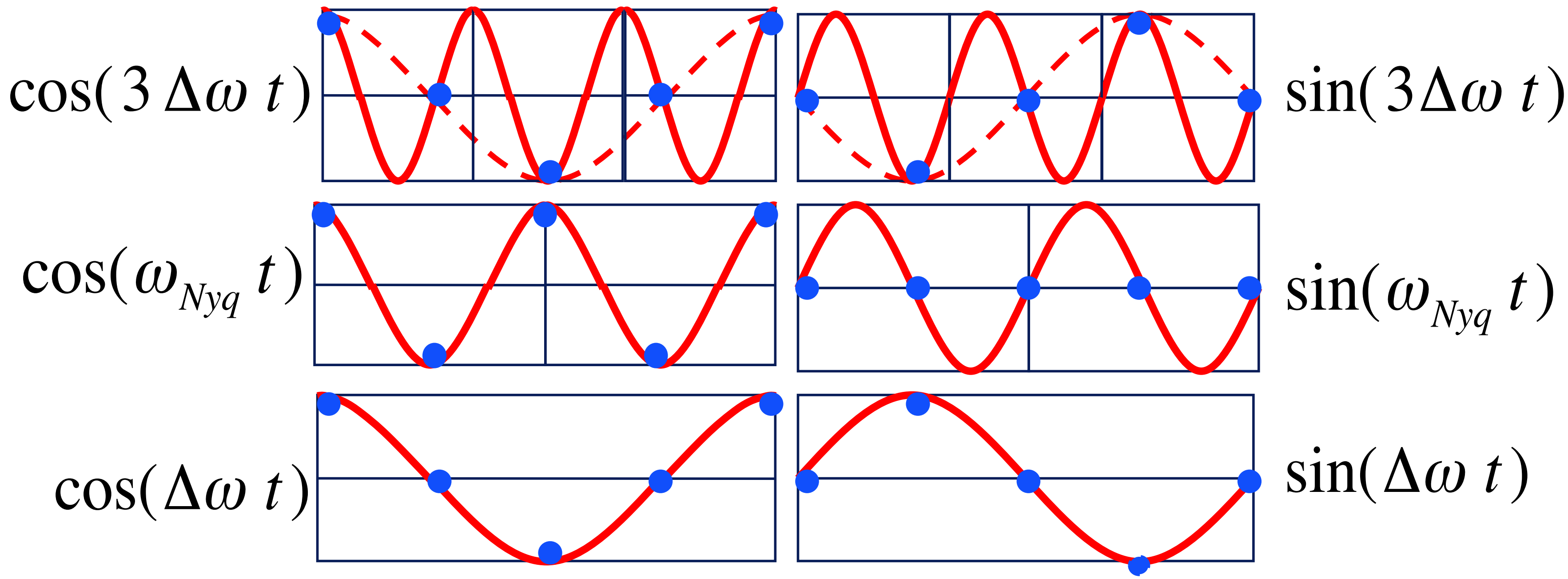


Degrees of freedom: $2(1 + K_{\max}) - 2 = N$

since $\sin(\omega_0\, t_i) = 0 \quad \sin(\omega_{Nyq}\, t_i) = 0$

Exact fit possible !

Aliasing above the Nyquist Frequency

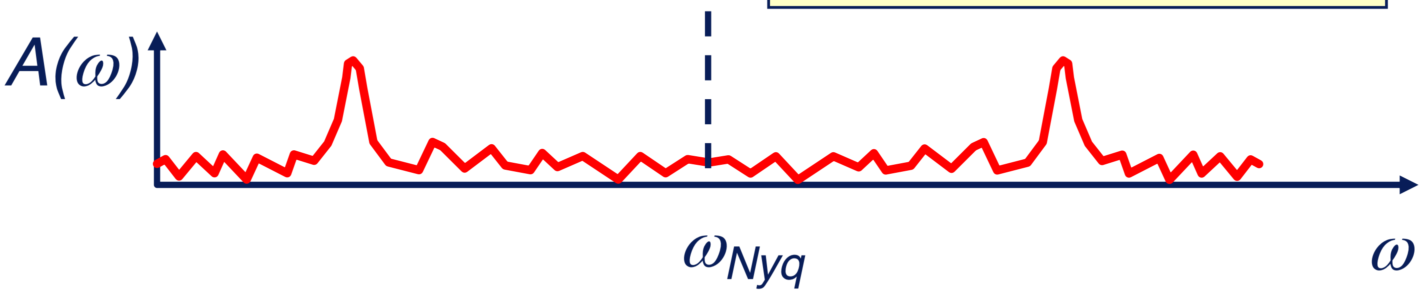


Sampled pattern is the same at $\omega_{Nyq} + k\,\Delta\omega$ and $\omega_{Nyq} - k\,\Delta\omega$.

$$\cos\left[\left(\omega_{Nyq} + k\,\Delta\omega\right) t_i\right] = \cos\left[\left(\omega_{Nyq} - k\,\Delta\omega\right) t_i\right]$$

$$\sin\left[\left(\omega_{Nyq} + k\,\Delta\omega\right) t_i\right] = -\sin\left[\left(\omega_{Nyq} - k\,\Delta\omega\right) t_i\right]$$

Frequencies above Nyquist frequency duplicate those below.



Periodogram

Pure sinusoid signal.

Sampled at N regularly spaced time intervals Δt

The **periodogram**:

Note χ^2 minimum and peak in A at correct ω.

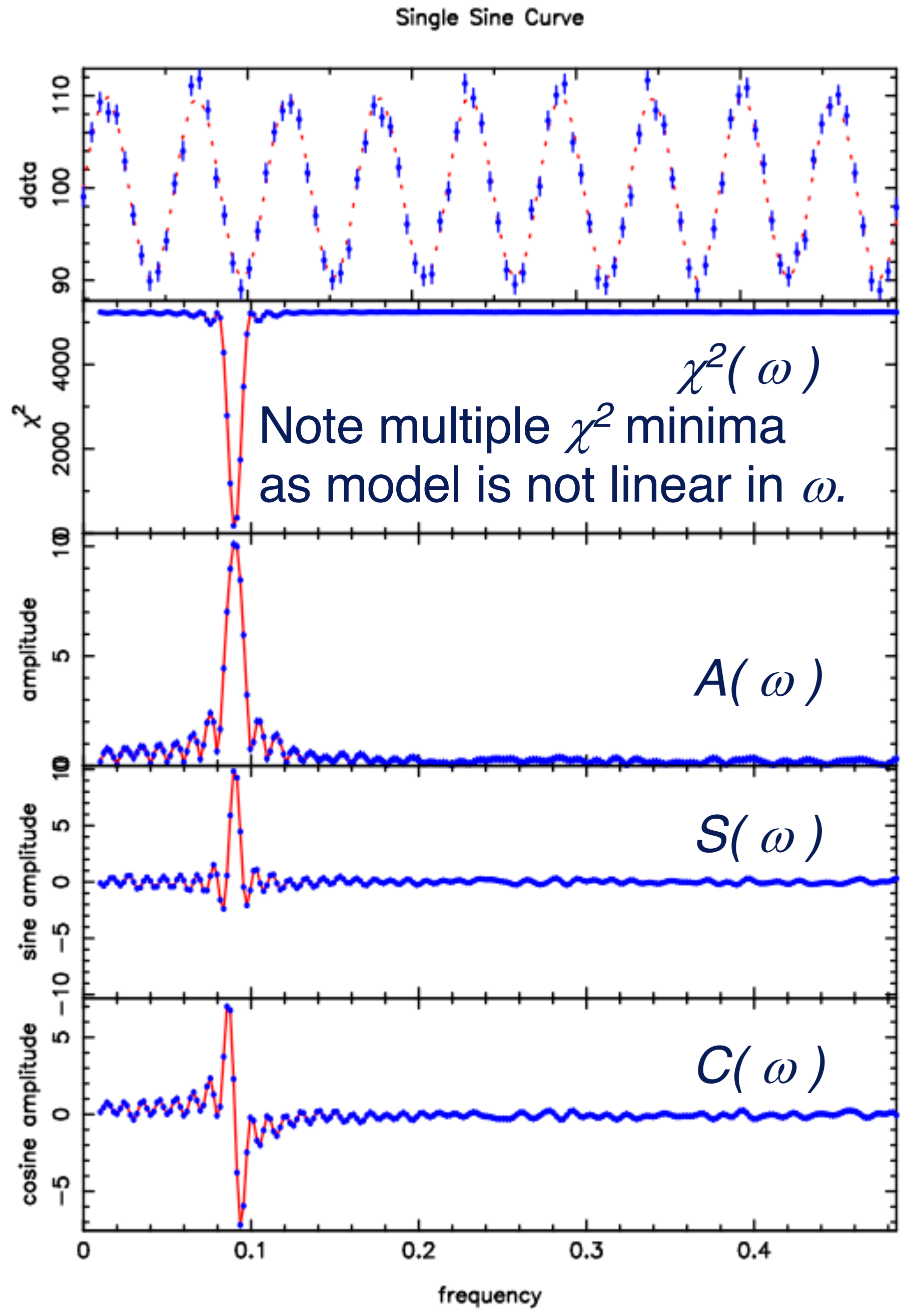
Use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

Sidelobe spacing:

$$\Delta\omega = 2\pi / T = 2\pi / N\,\Delta t$$

Nyquist frequency:

$$\omega_N = (N/2)\,\Delta\omega = N\pi / T = \pi / \Delta t$$



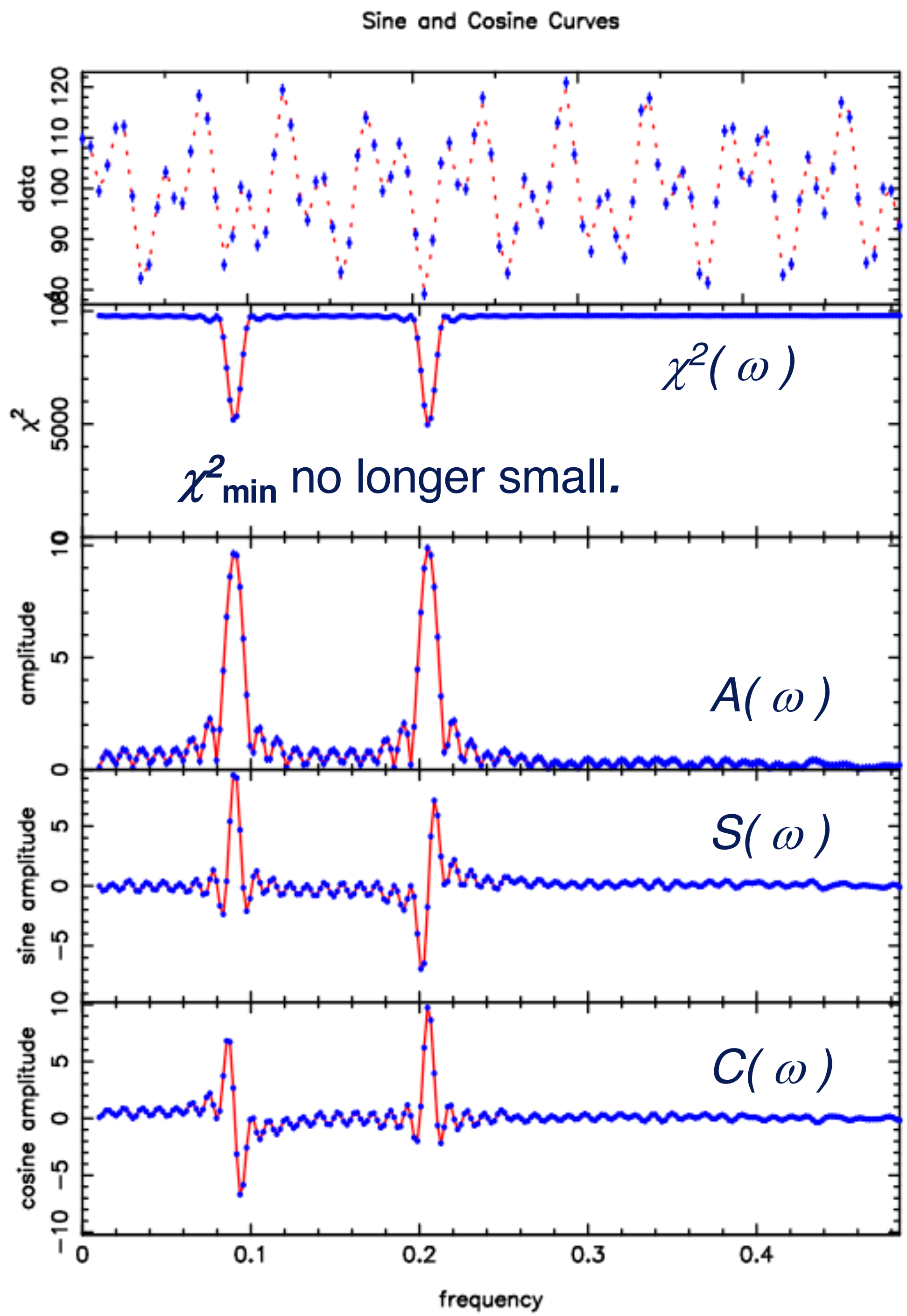
Widely spaced frequencies

Sum of sine and cosine curves at well-separated frequencies.

Periodogram shows two well separated peaks.

χ^2_{min} is high, but can still use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

(This is how we find multiple planets in Doppler data)



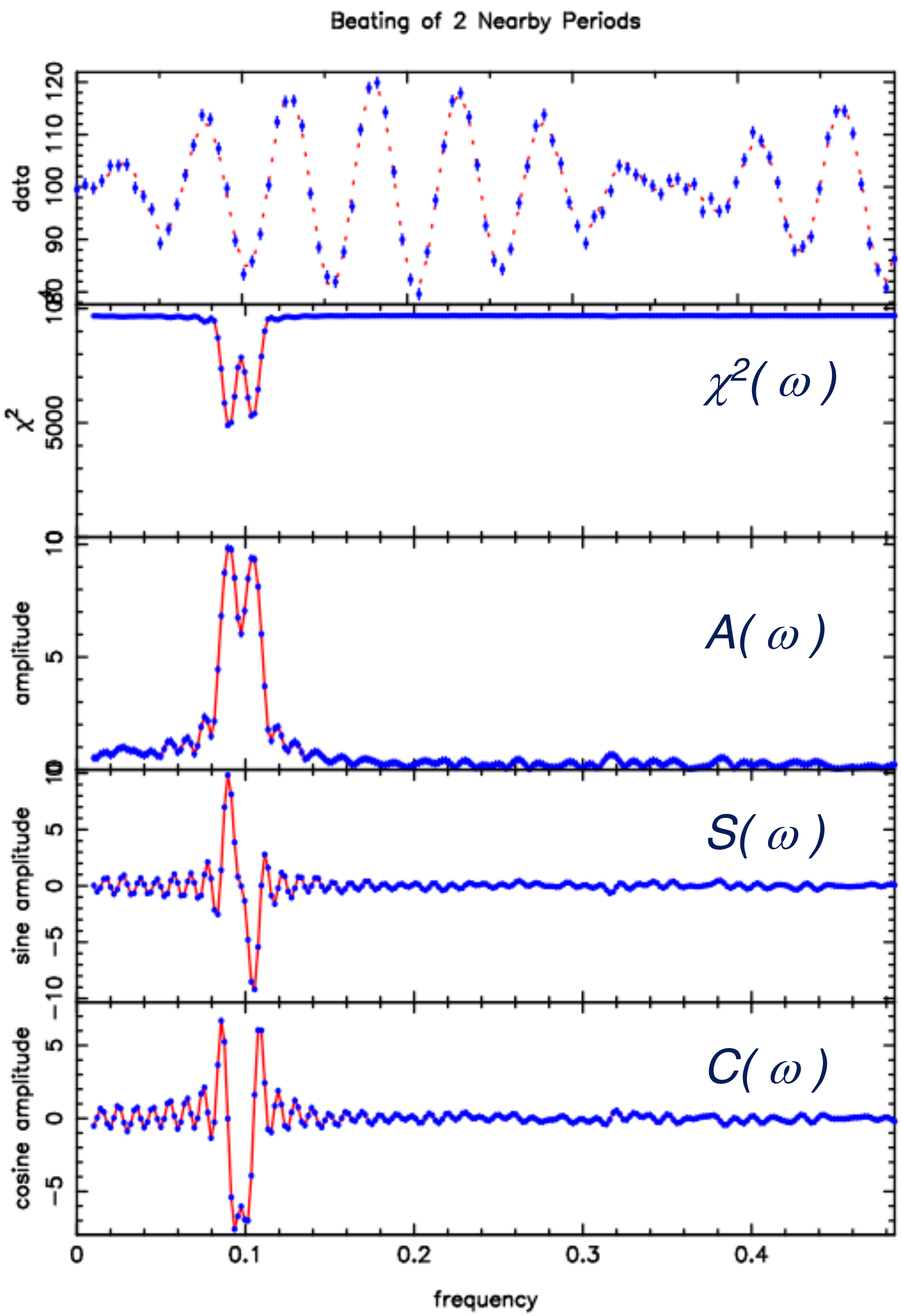
Closely spaced frequencies

Wave trains drift in and out of phase.

Constructive and destructive interference produces "**beating**" in the light curve.

Beat frequency $\omega_B = |\omega_1 - \omega_2|$

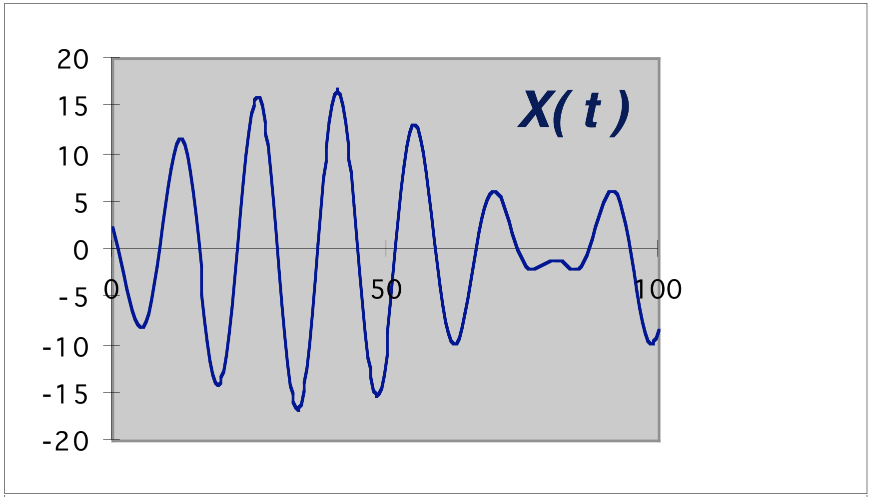
Peaks overlap in periodogram.

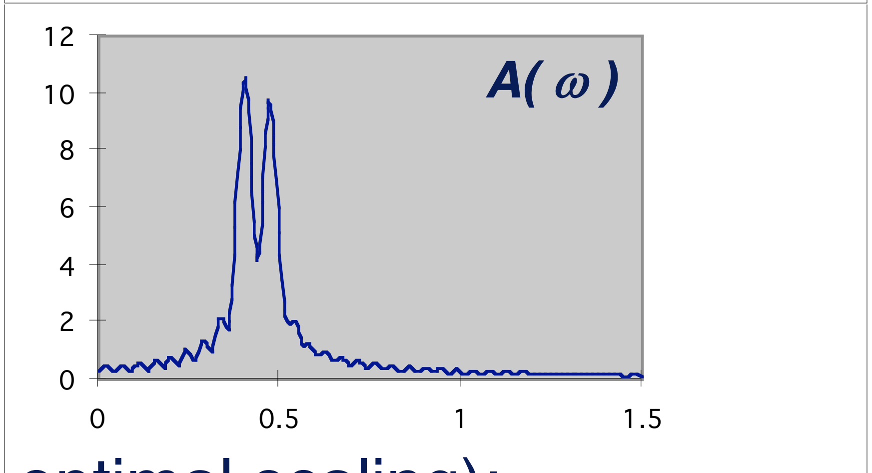


"Pre-whitening"

Disentangle closely-spaced frequencies by "***pre-whitening***" the data.

Fit and subtract strongest period, then fit the next, etc.





Subtract $A_1 \sin(\omega_1 t - \phi_1)$

Fit $A_2 \sin(\omega_2 t - \phi_2)$ to residuals

Subtract $A_2 \sin(\omega_2 t - \phi_2)$

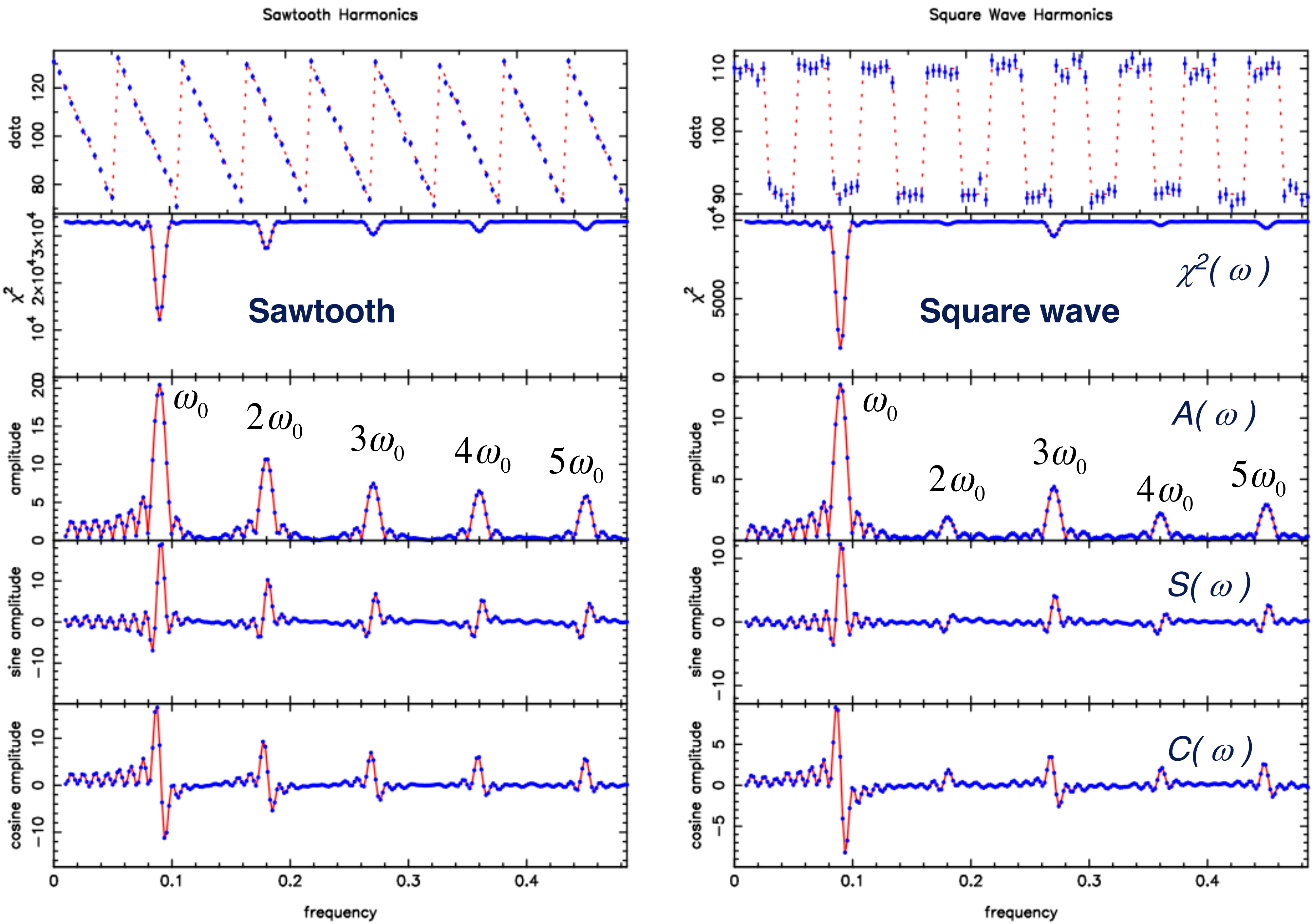
Fit $A_1 \sin(\omega_1 t - \phi_1)$ to residuals

Iterate to convergence

Fits a 7-parameter model (e.g. by iterated optimal scaling):

$$X(t) = X_0 + A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$$
$$= X_0 + S_1 \sin(\omega_1 t) + C_1 \cos(\omega_1 t)$$
$$+ S_2 \sin(\omega_2 t) + C_2 \cos(\omega_2 t)$$

2 non-linear params: ω_1 , ω_2, 5 linear params: X_0, S_1, C_1, S_2, C_2



Sawtooth Harmonics
Square Wave Harmonics
data
χ2
amplitude
sine amplitude
cosine amplitude
frequency
Sawtooth
Square wave
χ2(ω)
A(ω)
S(ω)
C(ω)
ω0
2ω0
3ω0
4ω0
5ω0
0
0.1
0.2
0.3
0.4

Non-sinusoidal Waveforms => Harmonics

- **Fundamental frequency:** ω_0
- **Harmonics** at $\omega = k\ \omega_0$, for $k = 2, 3, \ldots$
  modify the **shape** of the waveform.
- Fit any shape periodic function by including amplitudes for :
  - $\sin(2\omega_0 t)$, $\cos(2\omega_0 t)$
  - $\sin(3\omega_0 t)$, $\cos(3\omega_0 t)$
  - etc

$$X(t) = \hat{X}_0 + \sum_{k=1}^{\infty} \left[\hat{S}_k \sin\left(k\,\omega_0\, t\right) + \hat{C}_k \cos\left(k\,\omega_0\, t\right) \right]$$

$$\hat{A}_k^2 = \hat{S}_k^2 + \hat{C}_k^2, \quad \hat{\phi}_k = \mathrm{atan2}(-\hat{S}_k, \hat{C}_k)$$

- Harmonics are approximately orthogonal (for well-sampled data with uniform phase coverage).
- Add harmonics to the model until their amplitudes become poorly determined – Occam's razor, simplest model that fits.
- Use e.g. the BIC to decide which terms to include/omit.
- Harmonics above the Nyquist frequency will be aliased, by "folding back" across ω_{Nyq}, from ω to $\omega_{\mathrm{Nyq}} - (\omega - \omega_{\mathrm{Nyq}})$.

Data gaps and aliasing

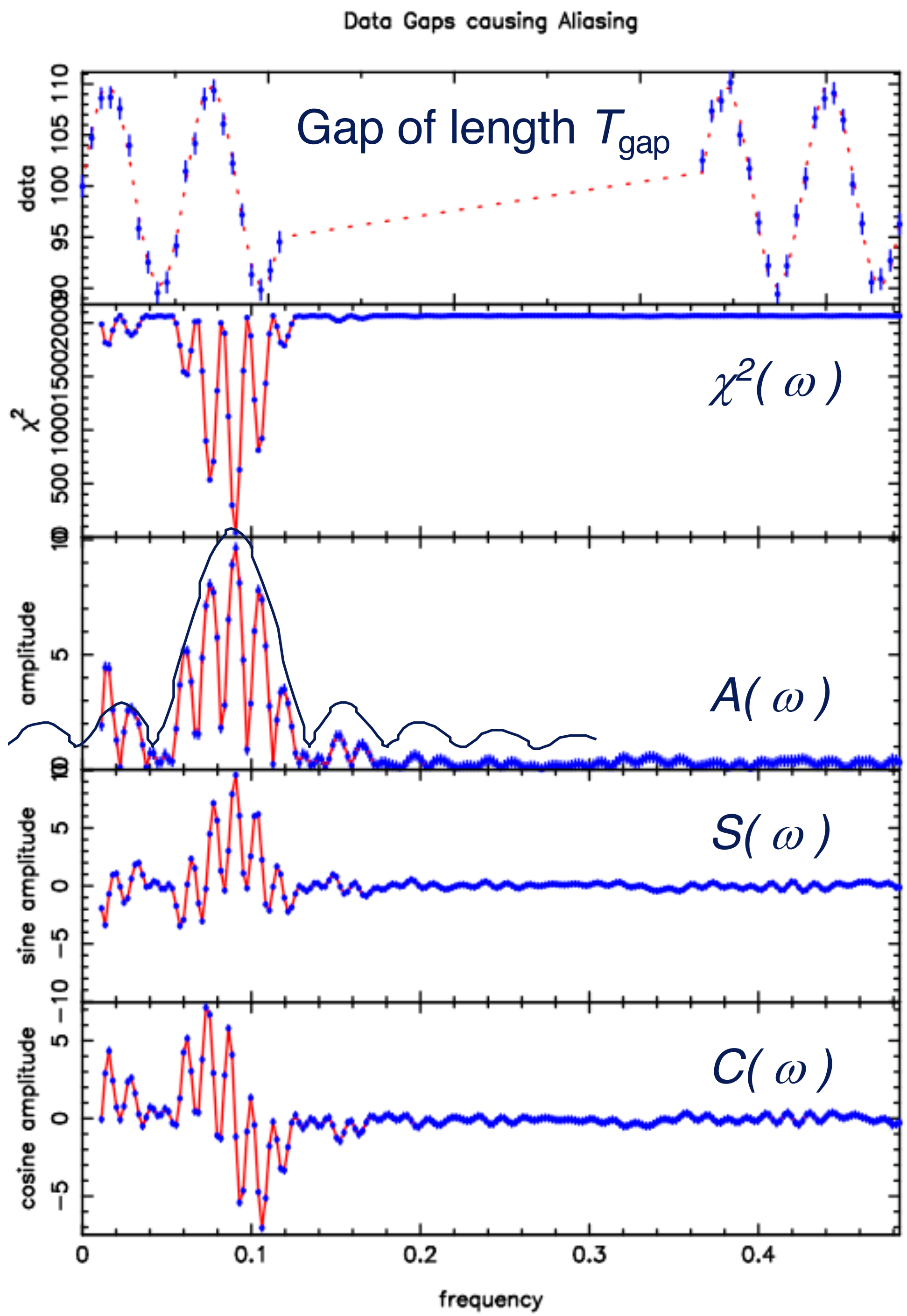
Cycle-count ambiguity:

How many cycles elapse in the gap between two data segments?

- Periodogram has **sidelobes (aliases)** spaced by

$$\Delta\omega = \frac{2\pi}{T_{gap}} \qquad \Delta f = \frac{1 \text{ cycle}}{T_{gap}}$$

- Sidelobes appear within a broader **envelope** determined by duration of data segments.



Fini -- ADA 13