ADA 13 -- 9am Tue 11 Oct 2022

Time Series Analysis, Ephemerides

Fourier Analysis:
Fourier frequencies and basis functions,
Nyquist sampling.

Periodogram analysis (part 1):
sidelobes, aliasing, harmonics

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Timing Analysis - Defining an Ephemeris

Timings: Observed times of a fiducial point in a periodic lightcurve, e.g. mid-eclipse.

$$t_i \pm \sigma_i$$

The Ephemeris:

$t = t_0 + P\,E$ = predicted time

t_0 = epoch of phase 0

P = period

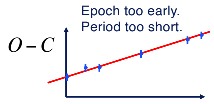
$E = n + \phi$ = cycle number + phase

$O\text{-}C$ = observed - calculated

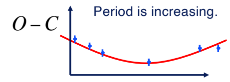
$= t_i - (t_0 + P\,E_i)$

$n_i = \mathrm{NINT}\left[\frac{t_i - t_0}{P}\right]$

phase: $\phi_i = \frac{t_i - t_0}{P} - n_i, \quad 0 < \phi < 1$



Fit a line to correct t_0 and P.



Fit quadratic ephemeris:

$t = t_0 + P_0\,E + B\,E^2$

$P = dt/dE = P_0 + 2B\,E$

$\dot{P} = dP/dt = 2B/P$

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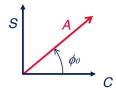
Hunting for Sinusoidal Signals

(e.g. Planet hunting -- circular orbit radial velocity curve)

Search a time series for a sinusoidal oscillation of unknown frequency ω:



- Fit a sinusoid (scale 3 patterns):

$$X(t) = X_0 + A\cos(\omega t + \phi_0)$$
$$= X_0 + C\cos\omega t + S\sin\omega t$$

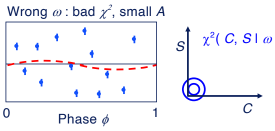
Amplitude: $A^2 = C^2 + S^2$

Phase at $t=0$: $\phi_0 = \tan^{-1}(-S/C)$

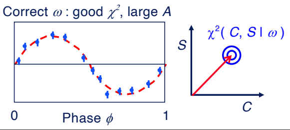
Programming hint: Use `phi=atan2(-S,C)` if you care about which quadrant ϕ ends up in!

- "Fold" data on a trial period $P = 2\pi/\omega$

Wrong ω : bad χ^2, small A



Correct ω : good χ^2, large A



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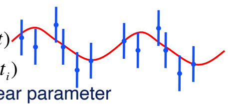
Sinusoid + Background

Data: $X_i \pm \sigma_i$ at $t = t_i$

Model: $X(t) = X_0 + S\sin(\omega t) + C\cos(\omega t)$



3 Patterns: $1, \quad s_i = \sin(\omega t_i), \quad c_i = \cos(\omega t_i)$

3-parameter linear regression + 1 non-linear parameter

Iterated Optimal Scaling:

$$\hat{X}_0 = \frac{\sum \left(X_i - \hat{S}\,s_i - \hat{C}\,c_i\right)/\sigma_i^2}{\sum 1/\sigma_i^2}, \quad \mathrm{Var}\left[\hat{X}_0\right] = \frac{1}{\sum 1/\sigma_i^2}$$

$$\hat{S} = \frac{\sum \left(X_i - \hat{X}_0 - \hat{C}\,c_i\right)s_i/\sigma_i^2}{\sum s_i^2/\sigma_i^2}, \quad \mathrm{Var}\left[\hat{S}\right] = \frac{1}{\sum s_i^2/\sigma_i^2}$$

$$\hat{C} = \frac{\sum \left(X_i - \hat{X}_0 - \hat{S}\,s_i\right)c_i/\sigma_i^2}{\sum c_i^2/\sigma_i^2}, \quad \mathrm{Var}\left[\hat{C}\right] = \frac{1}{\sum c_i^2/\sigma_i^2}$$

Iterate (if patterns not orthogonal).

Variance formulas assume orthogonal parameters, otherwise give too small error bars.

Use the inverse-Hessian matrix, e.g. when phase coverage is not close to uniform.

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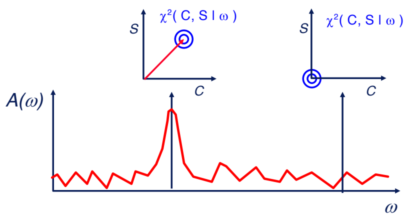
Periodogram : grid scan in frequency

Model is non-linear in ω (or $P = 2\pi/\omega$, or $f = 1/P$).

Use **grid-search**: fit sine curve for a grid of ω values.

Periodogram: plot $A(\omega)$ and/or $\chi^2(\omega)$.



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Periodogram of a finite data train

Purely sinusoidal time variation.

Sampled at N regularly spaced time intervals Δt

The **periodogram**:

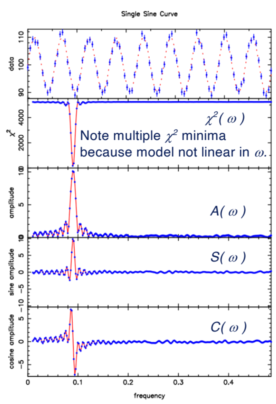
Note χ^2 minimum and peak in A at correct ω.

Use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

Note sidelobes and finite width of peak.

Why not a delta function?

(Spectral leakage)



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Spectral Leakage due to finite timespan T

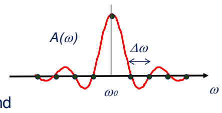
A pure sinusoid at frequency ω_0 "leaks" into adjacent frequencies ω due to the finite timespan T of the data.

$$\hat{A}(\omega) \approx A_0 \frac{\sum (\sin\omega_0 t_i)(\sin\omega t_i)/\sigma_i^2}{\sum (\sin\omega t_i)^2/\sigma_i^2}$$

= Optimal Scaling of the pattern sin(ωt) to fit data varying as A_0 sin($\omega_0 t$).

Special case : **Evenly spaced data ,** at times $t_i = t_0 + i\,\Delta t$ for $i = 1,..N$, and **Equal error bars,** $\sigma_i = \sigma$:

$$\hat{A}(\omega) = A_0 \frac{\sin \pi x}{\pi x} \quad \text{where} \quad x = \frac{\omega - \omega_0}{\Delta\omega}$$



This "Sinc" function has a ***1/x* envelope** and **evenly spaced zeroes** at frequency step

$$\Delta\omega = 2\pi / N\,\Delta t = 2\pi / T .$$

"De-tuning" by $\Delta\omega$ gives an **orthogonal function** with 1 extra cycle per time $T = N\,\Delta t$.

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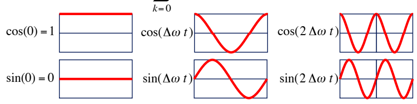
Fourier Frequencies and Basis Functions

$$f(t) = \sum_{k=0}^{K_{max}} \left[S_k \sin(\omega_k t) + C_k \cos(\omega_k t) \right]$$



De-tuning by $\Delta\omega$ gives an orthogonal function with 1 extra cycle during time *T.*

Orthogonal for **evenly-spaced data** with **equal error bars**.

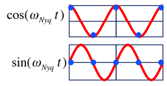
$$t_i = t_0 + i\,\Delta t \quad i = 1, 2, ..., N \qquad T = N\,\Delta t$$

Fourier frequencies:

$$\omega_k = k\,\Delta\omega \quad k = 0, 1, ..., K_{max} \quad \Delta\omega = 2\pi / T$$

Nyquist frequency = 1 cycle / 2 data points

$$\omega_{Nyq} = \frac{2\pi}{2\,\Delta t} = \frac{N\pi}{T} = \frac{N}{2}\Delta\omega \quad \Rightarrow \quad K_{max} = \frac{N}{2}$$



Degrees of freedom: $2(1 + K_{max}) - 2 = N$

since $\sin(\omega_0 t_i) = 0 \quad \sin(\omega_{Nyq} t_i) = 0$

Exact fit possible !

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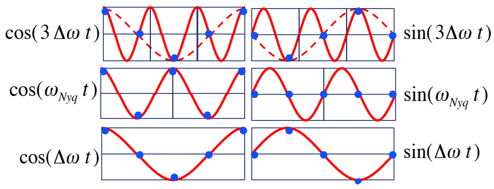
Aliasing above the Nyquist Frequency

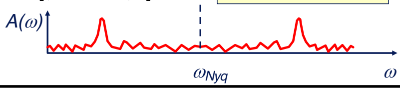


Sampled pattern is the same at $\omega_{Nyq} + k\,\Delta\omega$ and $\omega_{Nyq} - k\,\Delta\omega$.

$$\cos\left[\left(\omega_{Nyq} + k\,\Delta\omega\right) t_i\right] = \cos\left[\left(\omega_{Nyq} - k\,\Delta\omega\right) t_i\right]$$

$$\sin\left[\left(\omega_{Nyq} + k\,\Delta\omega\right) t_i\right] = -\sin\left[\left(\omega_{Nyq} - k\,\Delta\omega\right) t_i\right]$$

Frequencies above Nyquist frequency duplicate those below.



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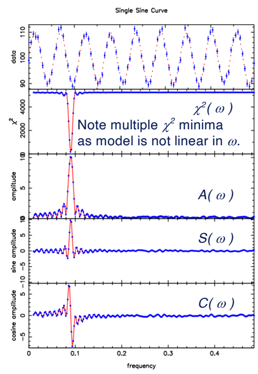
Periodogram

Pure sinusoid signal.
Sampled at *N* regularly spaced time intervals Δt

The **periodogram**:
Note χ^2 minimum and peak in *A* at correct ω.
Use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

Sidelobe spacing:
$\Delta\omega = 2\pi / T = 2\pi / N\,\Delta t$

Nyquist frequency:
$\omega_N = (N / 2)\,\Delta\omega = N\pi / T = \pi / \Delta t$



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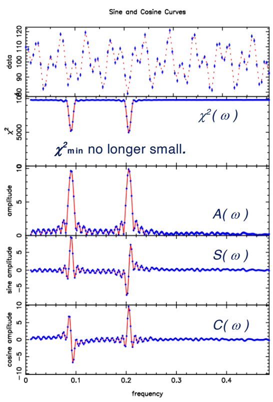
Widely spaced frequencies

Sum of sine and cosine curves at well-separated frequencies.

Periodogram shows two well separated peaks.

χ^2_{min} is high, but can still use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

(This is how we find multiple planets in Doppler data)



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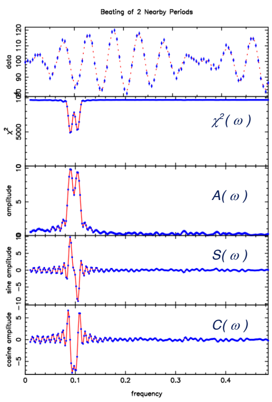
Closely spaced frequencies

Wave trains drift in and out of phase.

Constructive and destructive interference produces "**beating**" in the light curve.

Beat frequency $\omega_B = |\omega_1 - \omega_2|$

Peaks overlap in periodogram.



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"Pre-whitening"

Disentangle closely-spaced frequencies by "***pre-whitening***" the data.

Fit and subtract strongest period, then fit the next, etc.

- Subtract $A_1 \sin(\omega_1 t - \phi_1)$
- Fit $A_2 \sin(\omega_2 t - \phi_2)$ to residuals
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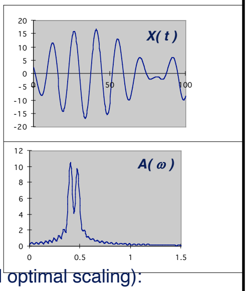
- Fit $A_1 \sin(\omega_1 t - \phi_1)$ to residuals
- Iterate to convergence



Fits a 7-parameter model (e.g. by iterated optimal scaling):

$$X(t) = X_0 + A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$$
$$= X_0 + S_1 \sin(\omega_1 t) + C_1 \cos(\omega_1 t) + S_2 \sin(\omega_2 t) + C_2 \cos(\omega_2 t)$$

2 non-linear params: ω_1, ω_2, 5 linear params: X_0, S_1, C_1, S_2, C_2

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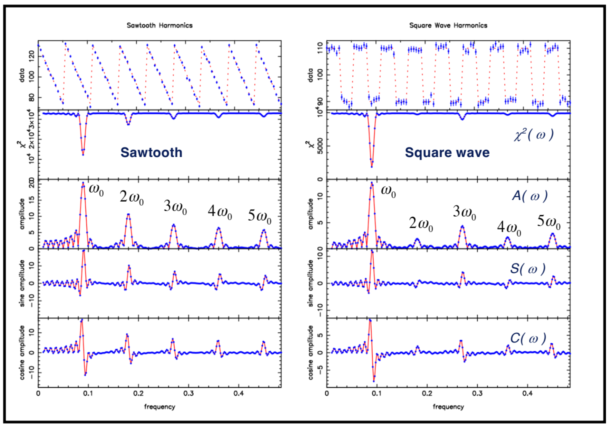

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Non-sinusoidal Waveforms => Harmonics

- **Fundamental frequency:** ω_0
- **Harmonics** at $\omega = k\,\omega_0$, for $k = 2, 3, \ldots$ modify the **shape** of the waveform.
- Fit any shape periodic function by including amplitudes for :
  - $\sin(2\omega_0 t)$, $\cos(2\omega_0 t)$
  - $\sin(3\omega_0 t)$, $\cos(3\omega_0 t)$
  - etc

$$X(t) = \hat{X}_0 + \sum_{k=1}^{\infty} \left[\hat{S}_k \sin(k\omega_0 t) + \hat{C}_k \cos(k\omega_0 t) \right]$$
$$\hat{A}_k^2 = \hat{S}_k^2 + \hat{C}_k^2, \quad \hat{\phi}_k = \mathrm{atan2}(-\hat{S}_k, \hat{C}_k)$$

- Harmonics are approximately orthogonal (for well-sampled data with uniform phase coverage).
- Add harmonics to the model until their amplitudes become poorly determined – Occam's razor, simplest model that fits.
- Use e.g. the BIC to decide which terms to include/omit.
- Harmonics above the Nyquist frequency will be aliased, by "folding back" across ω_{Nyq}, from ω to $\omega_{Nyq} - (\omega - \omega_{Nyq})$.

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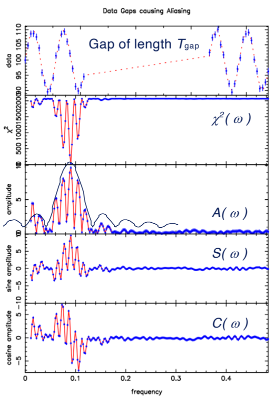
Data gaps and aliasing

Cycle-count ambiguity:
How many cycles elapse in the gap between two data segments?

- Periodogram has **sidelobes (aliases)** spaced by

$$\Delta\omega = \frac{2\pi}{T_{gap}} \quad \Delta f = \frac{1\ \mathrm{cycle}}{T_{gap}}$$

- Sidelobes appear within a broader **envelope** determined by duration of data segments.



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