ADA 14 -- 9am Thu 13 Oct 2022

Periodogram Analysis (continued)
Quasi-Periodic Oscillations
Dynamic Power Spectra (GW detection)
White/Red Noise (Wavelets, Splines)

Periodogram of a Sinusoid + Spike

Single high value is sum of cosine curves all in phase at time t_0:

$$X(t) = \mu + A\sin(\omega_0\, t) + \Delta\, \delta\left(t - t_0\right) \pm \sigma$$

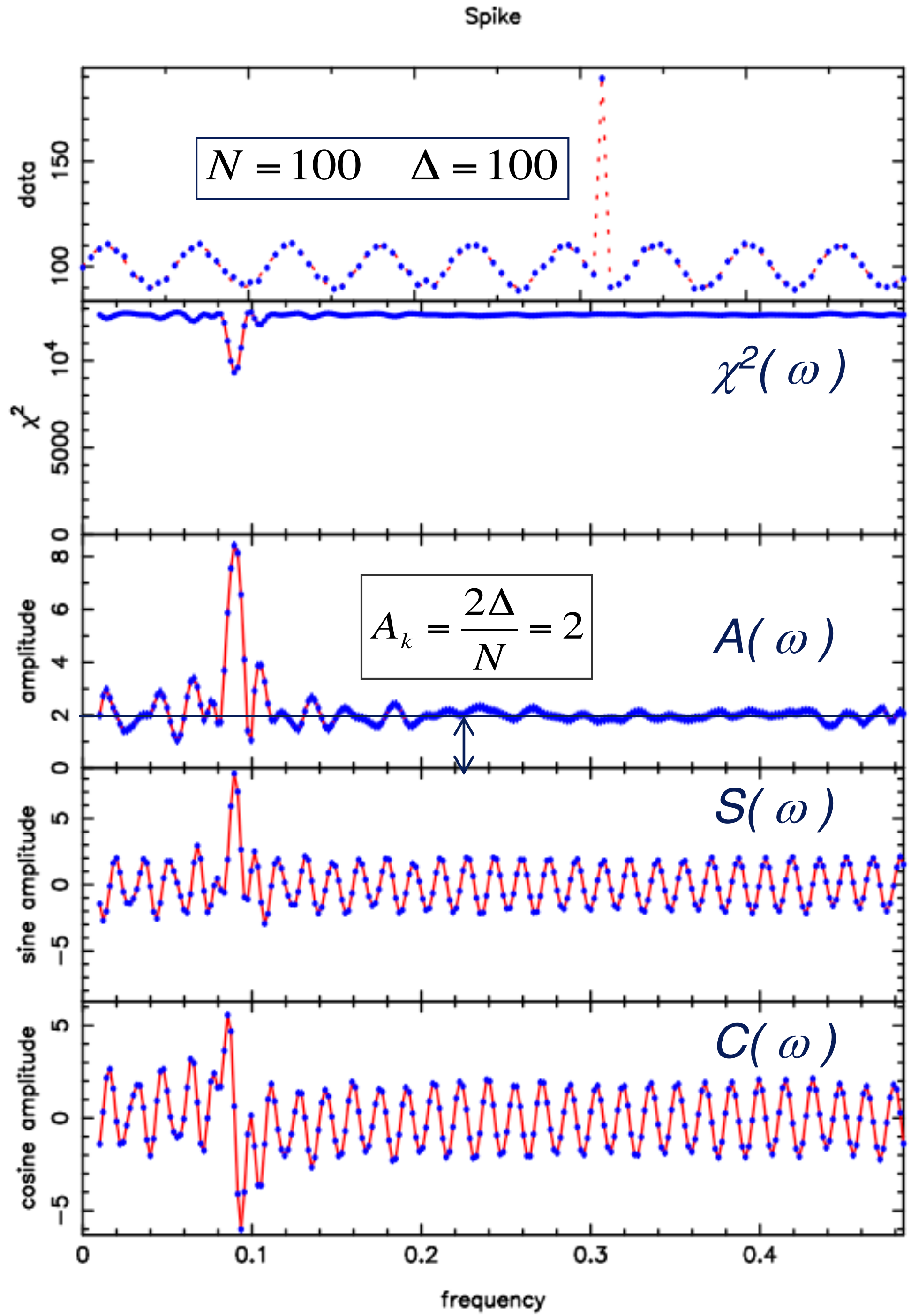
This spike raises the amplitude uniformly at all frequencies.

$$S_k = \frac{\Delta \sin(\omega_k t_0)/\sigma^2}{\sum_{i=1}^{N} \sin^2(\omega_k t_i)/\sigma^2} = \frac{2\Delta}{N}\sin(\omega_k t_0)$$

$$C_k = \frac{\Delta \cos(\omega_k t_0)/\sigma^2}{\sum_{i=1}^{N} \cos^2(\omega_k t_i)/\sigma^2} = \frac{2\Delta}{N}\cos(\omega_k t_0)$$

$$A_k^2 = S_k^2 + C_k^2 = \left(\frac{2\Delta}{N}\right)^2$$

$$\text{Note}: \ \left\langle \sin^2 \right\rangle = \left\langle \cos^2 \right\rangle = 1/2$$



Periodogram of a Sinusoid + White Noise

- **White Noise** is generated by sampling a Gaussian random number at each time.

$$X_i \sim G(\mu, \sigma^2)$$

- Or, use a Gaussian random number for each sine and cosine amplitude.

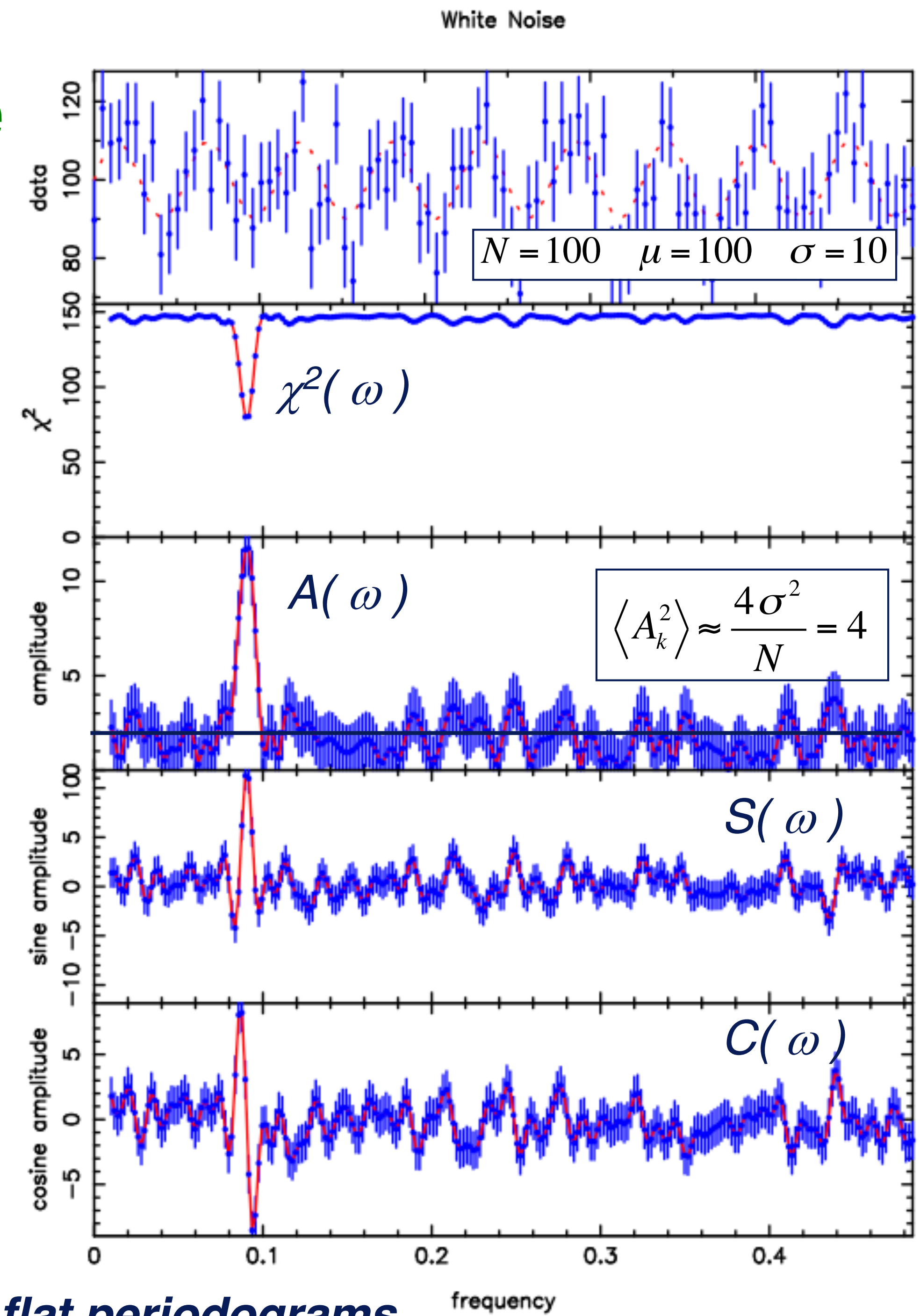
$$C_0 = \mu, \quad C_k, S_k \sim G\left(0, 2\sigma^2 / N\right)$$

$$C_k = \frac{\sum_{i=1}^{N} (X_i - \mu)\cos(\omega_k t_i) / \sigma_i^2}{\sum_{i=1}^{N} \cos^2(\omega_k t_i) / \sigma_i^2} \qquad \langle C_k \rangle = 0$$

$$\langle C_k^2 \rangle = \mathrm{Var}\left[C_k\right] = \frac{1}{\sum_{i=1}^{N} \cos^2(\omega_k t_i) / \sigma_i^2} = \frac{2\sigma^2}{N}$$

$$A_k^2 \equiv C_k^2 + S_k^2 \sim \frac{2\sigma^2}{N} \chi_2^2 \qquad \langle A_k^2 \rangle = \frac{4\sigma^2}{N}$$

$$\text{Parseval's theorem:} \quad \left\langle \sum_{k=1}^{N/2} A_k^2 \right\rangle = 2\sigma^2$$



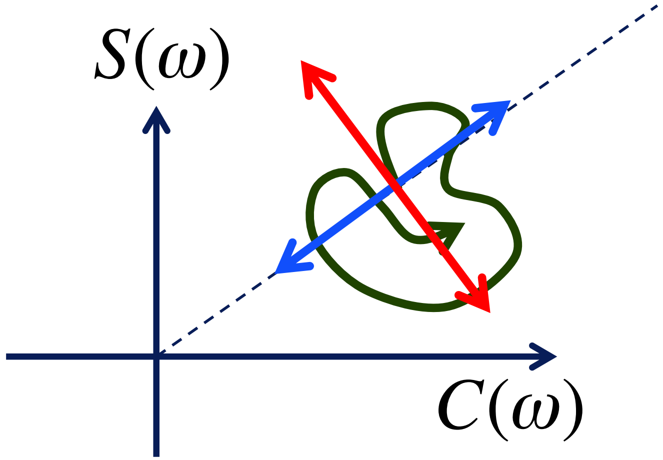
Note: Both white noise and a spike have flat periodograms.

Unstable or "Quasi-Periodic" Oscillations

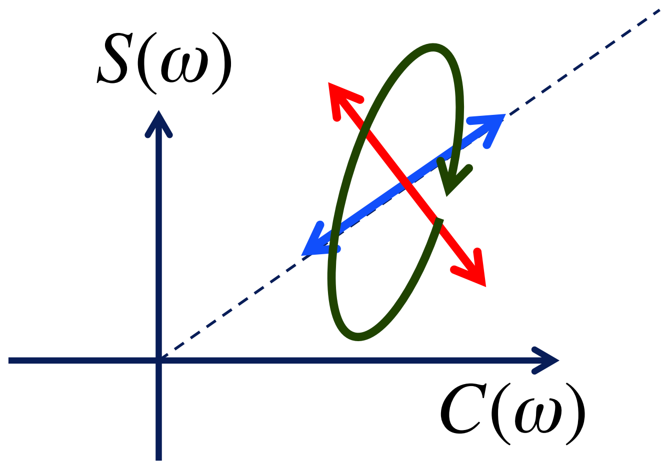
$$X(t) = [A_0 + A(t)] \sin(\omega_0 t + \Phi_0 + \Phi(t))$$

***Amplitude modulation**: $A(t)$*

***Phase modulation**: $\Phi(t)$*



Special case:



Unstable oscillation
has a **broader periodogram peak.**

Amplitude drifts -> sidelobes

Phase drifts equivalent to frequency ω changing with time.

$$A(t) = A \sin \Omega t + B \cos \Omega t$$

$$\Phi(t) = \alpha \sin \Omega t + \beta \cos \Omega t$$

Amplitude Modulation

Set $A_0 = 1$ and $\Phi_0 = 0$.

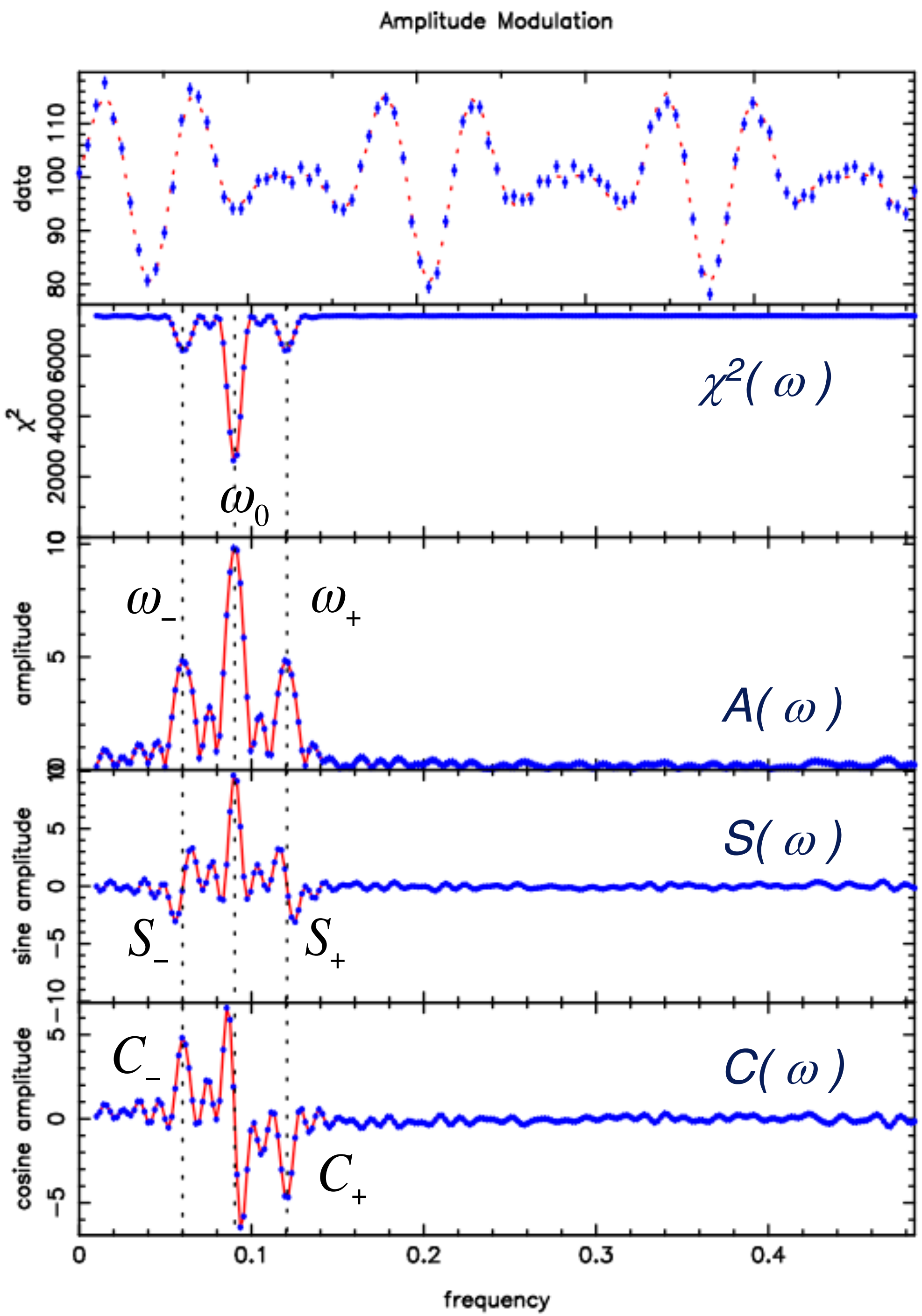
Oscillation frequency ω_0 with slow amplitude modulation at lower frequency Ω:

$$X(t) = (1 + A\sin\Omega t + B\cos\Omega t)\sin\omega_0 t$$
$$= \sin(\omega_0 t) + (A/2)[\cos(\omega_- t) - \cos(\omega_+ t)]$$
$$+ (B/2)[\sin(\omega_- t) + \sin(\omega_+ t)]$$

Note: **sidelobes** at $\omega_\pm = \omega_0 \pm \Omega$.

Sine amplitudes in phase. $S_+ = S_-$

Cosine amps anti-phased. $C_+ = -C_-$



Phase Modulation

Oscillation frequency ω_0 with slow phase modulation at lower frequency Ω :

$$X(t) = \sin(\omega_0 t + \alpha \sin \Omega t + \beta \cos \Omega t)$$

Note: $\sin(x + \Delta x) \approx \sin x + \Delta x \cos x$:

$$X(t) \approx \sin(\omega_0 t) + \left(\alpha \sin \Omega t + \beta \cos \Omega t\right) \cos \omega_0 t$$

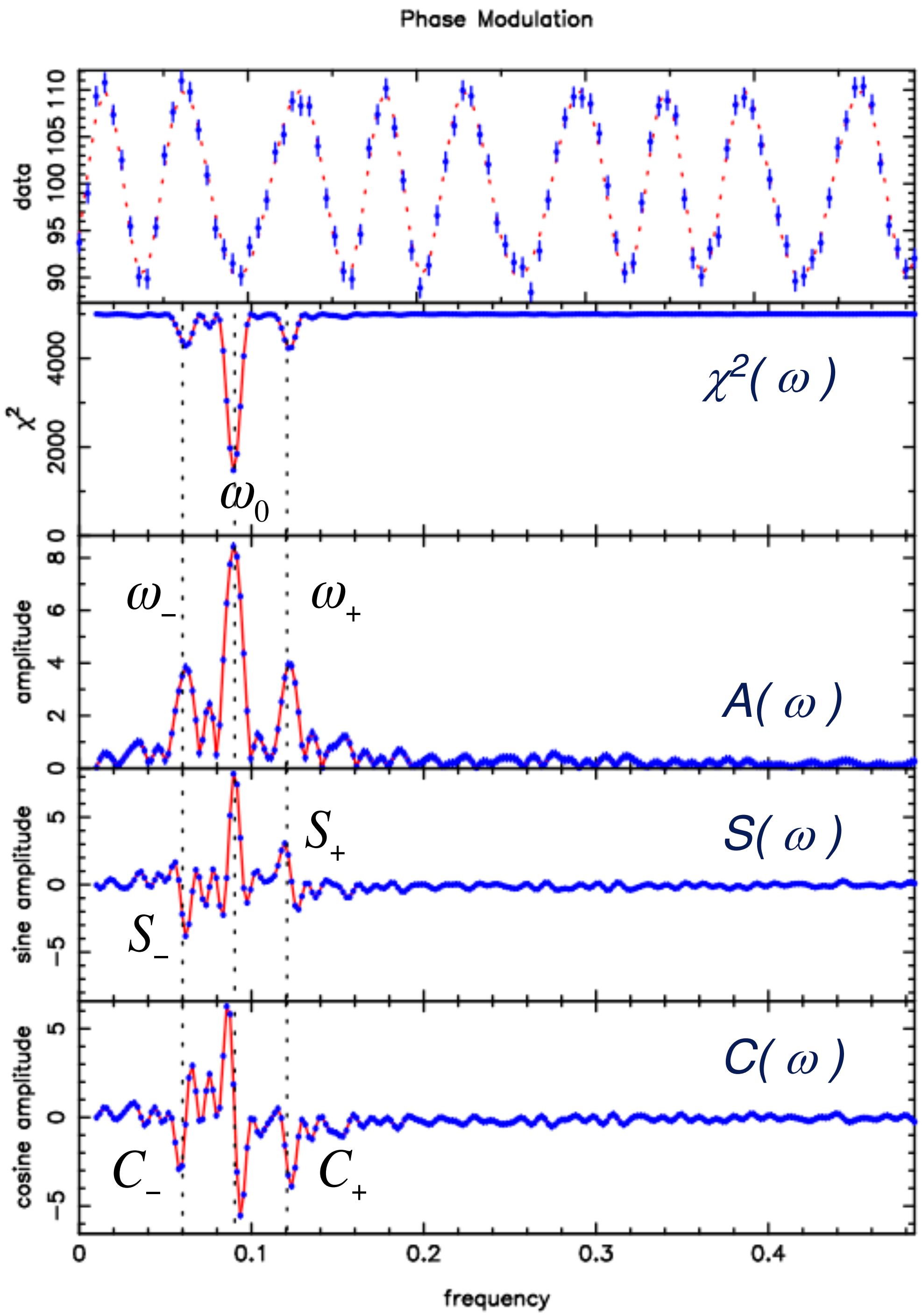
$$= \sin(\omega_0 t) + (\alpha / 2)\left[-\sin(\omega_- t) + \sin(\omega_+ t)\right]$$

$$+ (\beta / 2)\left[\cos(\omega_- t) + \cos(\omega_+ t)\right]$$

Again, **sidelobes** at $\omega_\pm = \omega_0 \pm \Omega$ but now with

Sine amps anti-phased: $S_+ = -S_-$

Cosine amps in phase: $C_+ = C_-$



Phase relations for Sidelobes

Both Amplitude and Phase Modulation:

$$X(t) = (1 + A\sin\Omega t + B\cos\Omega t)\sin(\omega_0 t + \alpha\sin\Omega t + \beta\cos\Omega t)$$

$$\approx \sin(\omega_0 t) + S_- \sin(\omega_- t) + C_- \cos(\omega_- t)$$

$$+ S_+ \sin(\omega_+ t) + C_+ \cos(\omega_+ t)$$

$$\omega_\pm \equiv \omega_0 \pm \Omega, \quad S_\pm = \frac{B \pm \alpha}{2}, \quad C_\pm = \frac{\beta \mp A}{2}$$

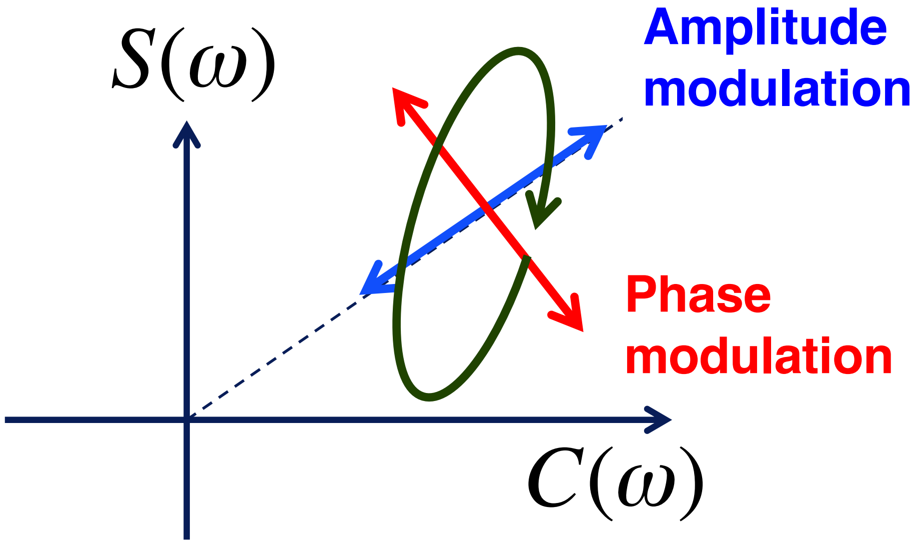
Amplitude and Phase Modulation Spectra:

$$A(\Omega) = C_-(\Omega) - C_+(\Omega)$$

$$B(\Omega) = S_-(\Omega) + S_+(\Omega)$$

$$\alpha(\Omega) = S_+(\Omega) - S_-(\Omega)$$

$$\beta(\Omega) = C_+(\Omega) + C_-(\Omega)$$



Dynamic Power Spectrum

For periodic oscillations with amplitude and phase that vary with time.

Data: $X_i \pm \sigma_i$ at $t = t_i$

Model: $\mu(t) = X_0(t) + S(t)\sin(\omega t) + C(t)\cos(\omega t)$

3 Patterns: 1, $s_i = \sin(\omega t_i)$, $c_i = \cos(\omega t_i)$

Like Running Optimal Average, but including Sin and Cos amplitudes in the fit to each time window.

Iterated Optimal Scaling:

$$\hat{X}_0(t) = \frac{\sum \left(X_i - \hat{S}\, s_i - \hat{C}\, c_i \right) w_i(t)}{\sum w_i(t)},$$

$$\hat{S}(t) = \frac{\sum \left(X_i - \hat{X}_0 - \hat{C}\, c_i \right) s_i\, w_i(t)}{\sum s_i^2 w_i(t)},$$

$$\hat{C}(t) = \frac{\sum \left(X_i - \hat{X}_0 - \hat{S}\, s_i \right) c_i\, w_i(t)}{\sum c_i^2 w_i(t)},$$

$$\hat{A}^2(t) = \hat{C}^2(t) + \hat{S}^2(t)$$

Iterate (patterns not orthogonal).

$$w_i(t) = \frac{G(t - t_i)}{\sigma_i^{\,2}}$$

$$G(t) = \exp\left\{ -\frac{t^2}{2\Delta^2} \right\}$$

Time-resolution set by parameter Δ.

Dynamic Power Spectrum

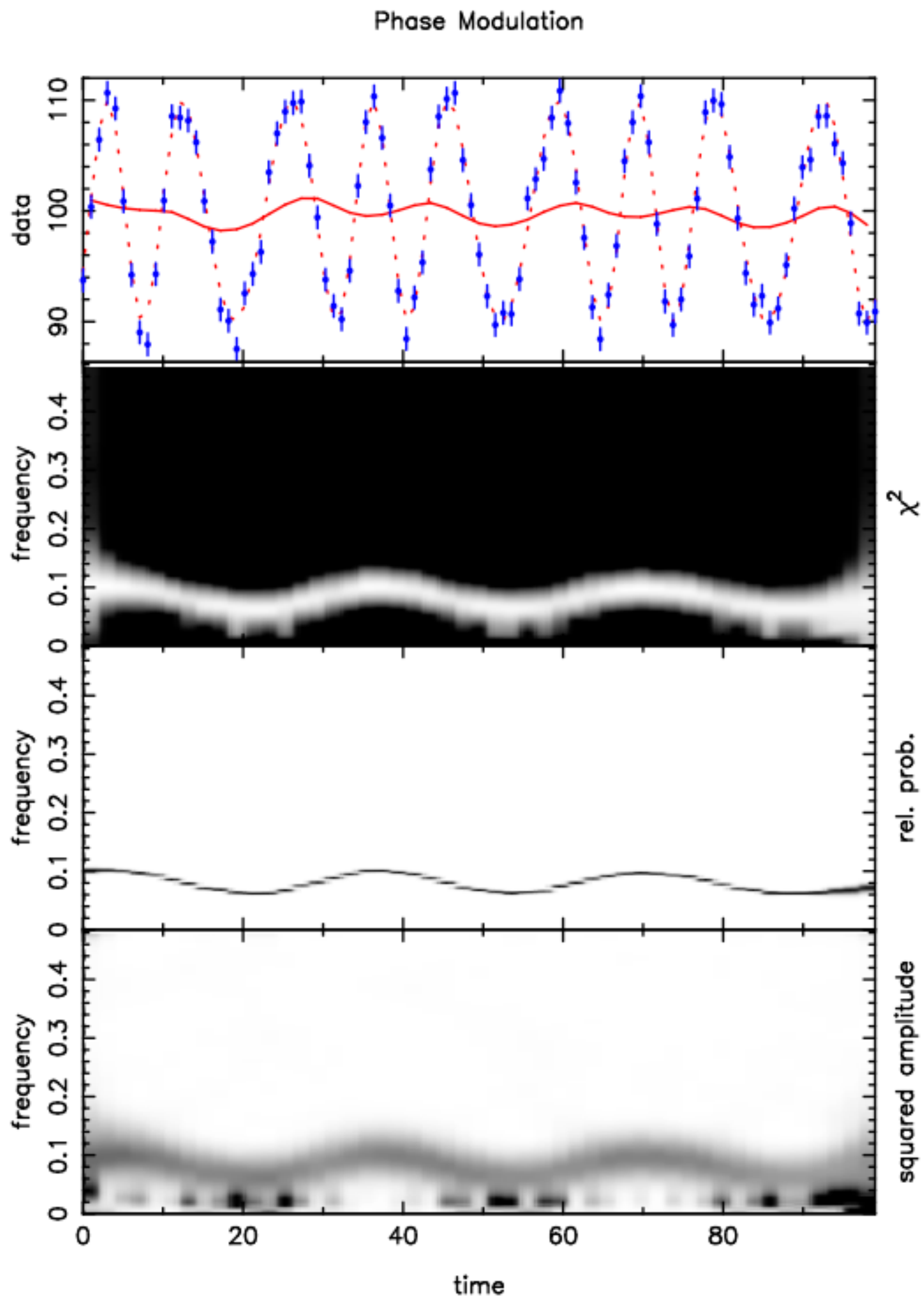
Phase modulation is equivalent to a wandering frequency.

Badness-of-Fit: $\chi^2(\,\omega, t\,)$

Probability: $P \sim \exp\{\,-\,\chi^2 / 2\,\}$

Power density: $A^2(\,\omega, t\,)$

Note : the probability peak is much sharper than power density peak.

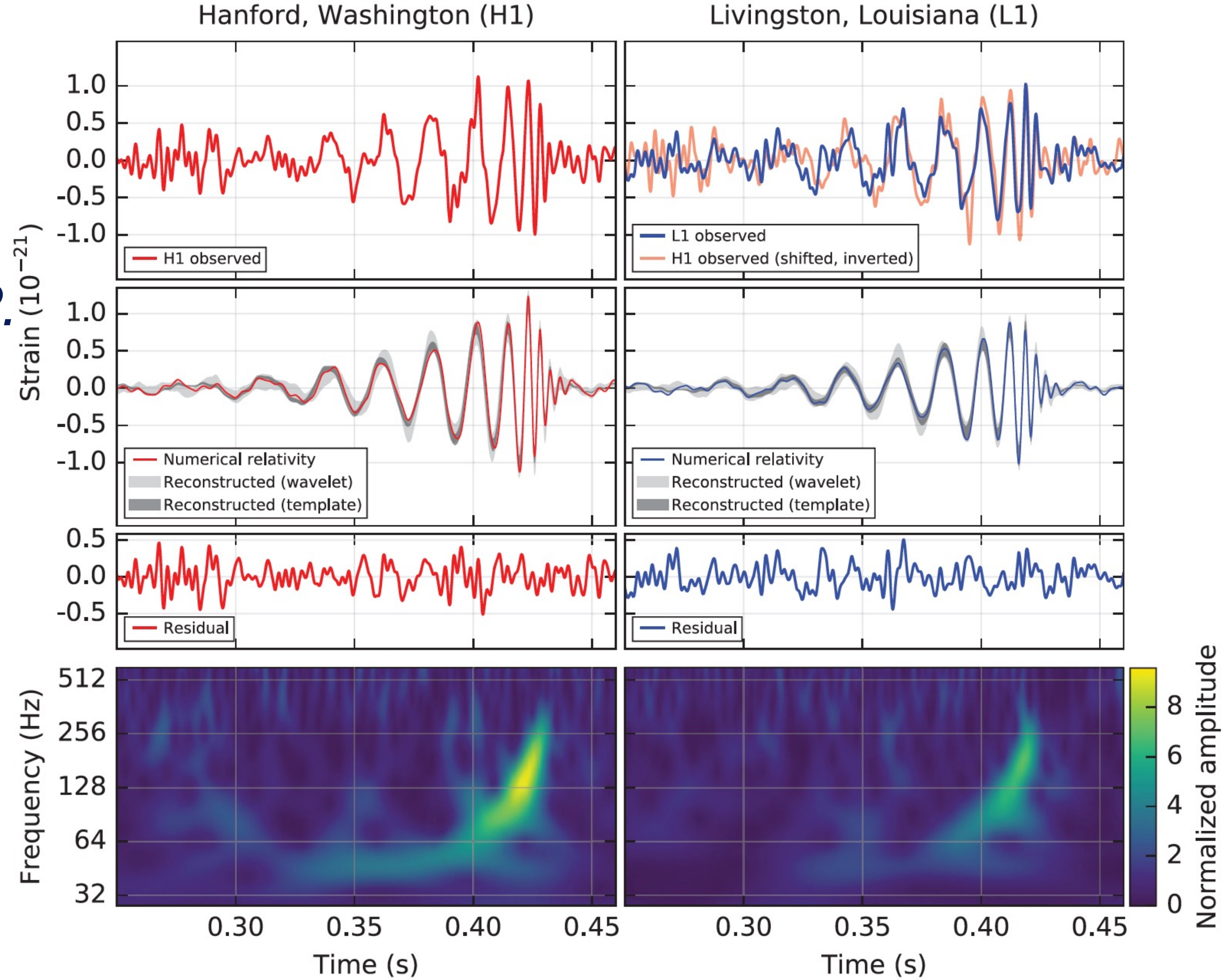


Example of Dynamic Power Spectra

Gravitational Waves *from a* ***Binary Black Hole Merger***

Abbott et al. (2016)
Phys Rev L 116, 1102.

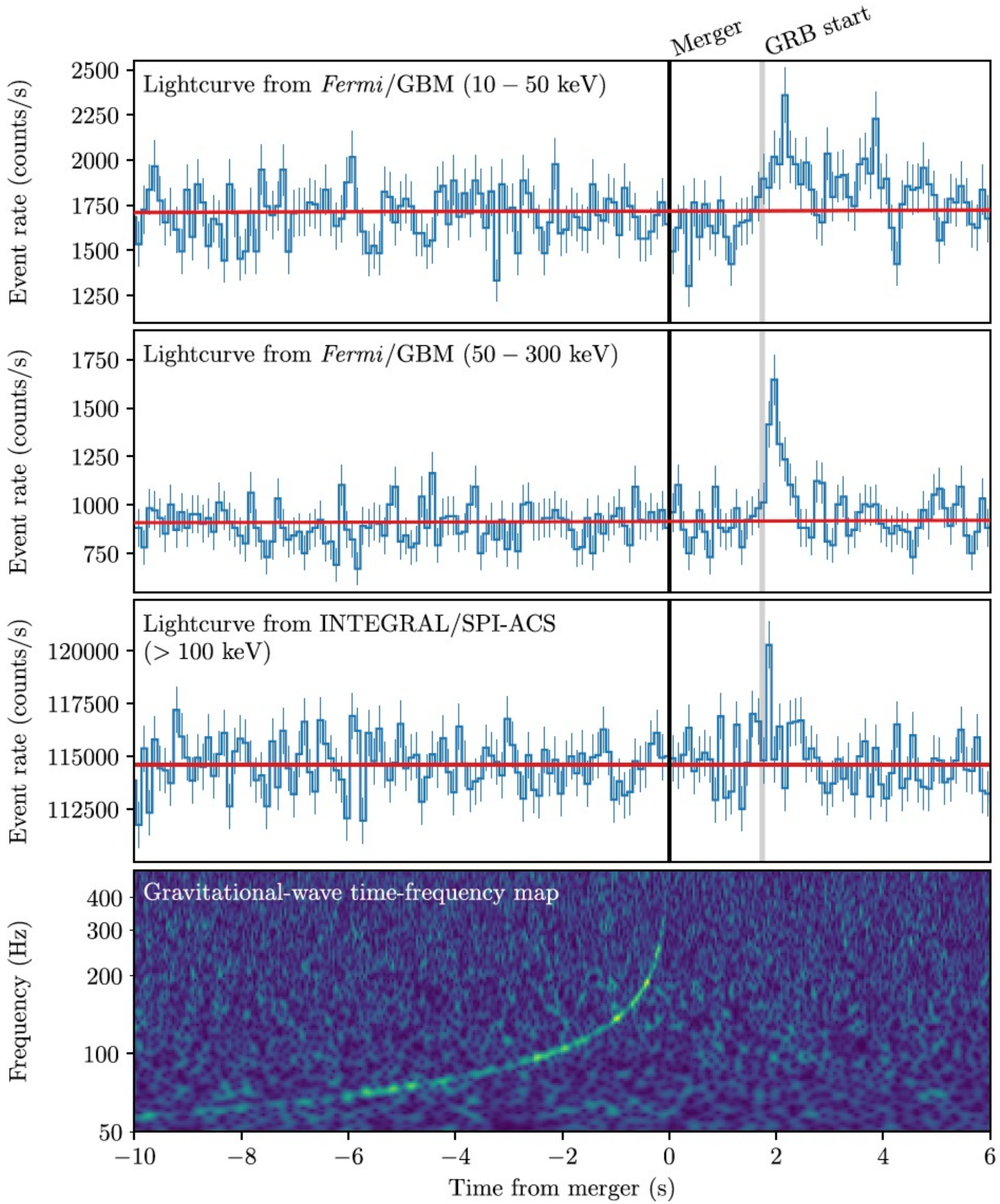
2017 Nobel Prize :
R.Weiss (MIT),
K.Thorne, B.Barish
(Caltech)



Example 2: Dynamic Power Spectrum

Gravitational Waves *and*
Gamma Rays *from a*
Binary Neutron Star Merger*:*
GW170817 and
GRB 170817A.

Abbott et al. (2017)
ApJL 848, L13.



Summary: Fourier Analysis

model: $\mu(t) = \mu_0 + \sum_k c_k \cos \omega_k t + s_k \sin \omega_k t$

even spacing: $t_i = t_0 + i\,\Delta t \qquad T = N\,\Delta t \qquad i = 1,2,\ldots,N$

Fourier frequencies: $\omega_k = k\,\Delta\omega = 2\pi/P_k \quad P_k = T\,/\,k \quad k = 0,1,\ldots,K_{\max} = N\,/\,2$

Nyquist frequency: $\omega_{Nyq} = \pi\,/\,\Delta t = 2\pi/P_{Nyq} \quad P_{Nyq} = 2\,\Delta t$

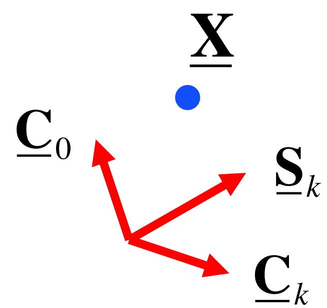
Orthogonal basis: $\underline{\mathbf{C}}_k = \cos \omega_k \underline{\mathbf{t}} \quad \underline{\mathbf{S}}_k = \sin \omega_k \underline{\mathbf{t}}$

Model: $\underline{\mu} = \mu_o \underline{\mathbf{C}}_0 + \sum_{k=1} \left(c_k \underline{\mathbf{C}}_k + s_k \underline{\mathbf{S}}_k \right)$

Exact fit possible by using N parameters to fit N data points.

Badness-of-fit: $\chi^2 = \left\| \underline{\mathbf{X}} - \underline{\mu} \right\|^2$

Optimal fit: $\hat{\mu}_0 = \dfrac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_0}{\underline{\mathbf{C}}_0 \bullet \underline{\mathbf{C}}_0} \quad \hat{c}_k = \dfrac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_k}{\underline{\mathbf{C}}_k \bullet \underline{\mathbf{C}}_k} \quad \hat{s}_k = \dfrac{\underline{\mathbf{X}} \bullet \underline{\mathbf{S}}_k}{\underline{\mathbf{S}}_k \bullet \underline{\mathbf{S}}_k}$



Power spectrum: $P(\omega_k)\,\Delta\omega = \hat{A}_k^2 \equiv \hat{c}_k^2 + \hat{s}_k^2$

Decomposes lightcurve into frequency components.

Wavelet Analysis - Wavelet Basis Functions

Fourier basis isolates in **frequency** but not in time.

Delta basis isolates in **time** but not in **frequency**.

Wavelet basis isolates in both **frequency and time**.

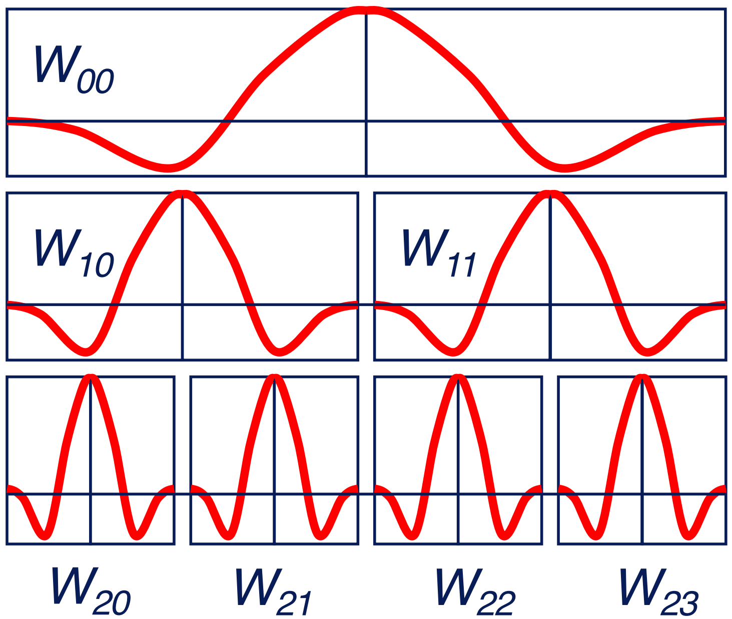
Exact fit possible by using N parameters to fit N data points.

$$W_{k\,j}(x) = W\left[2^k(x-j)\right]$$

$$j = 0, \ldots, \left(2^k - 1\right)$$

Many wavelet shapes possible. e.g. "Mexican Hat" wavelet:

$$W(x) \equiv (1-x^2)\exp(-x^2/2)$$



At each new level, k -> k+1:

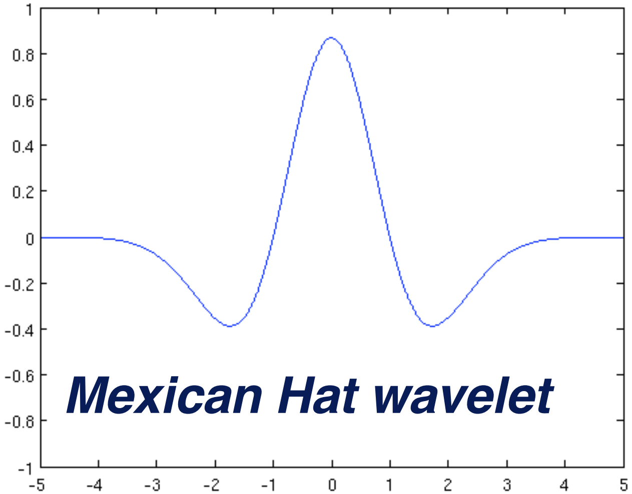
Double the wavelet frequency.

Double the number of wavelets.

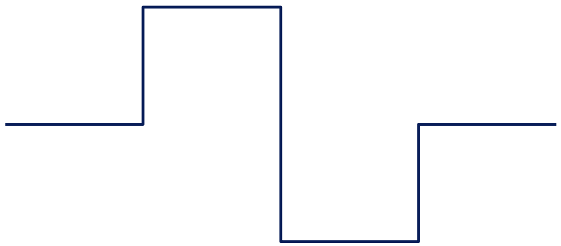
Complete orthogonal basis.

(Used e.g. for data compression)

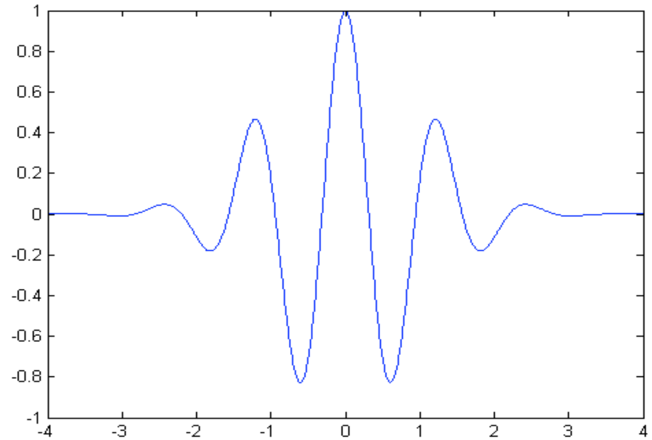
Various Wavelet Shapes

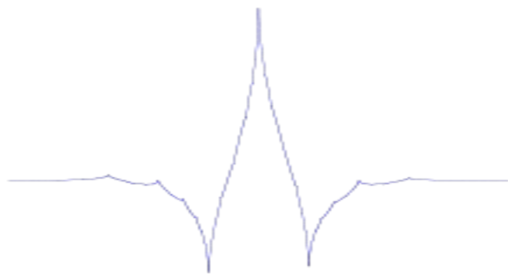


Haar wavelet

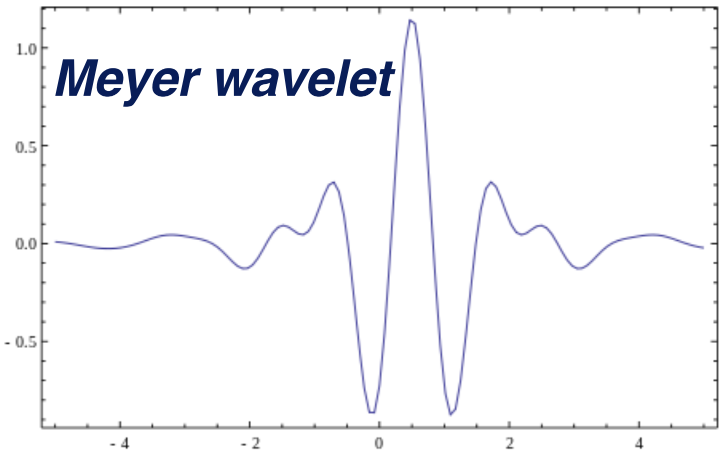


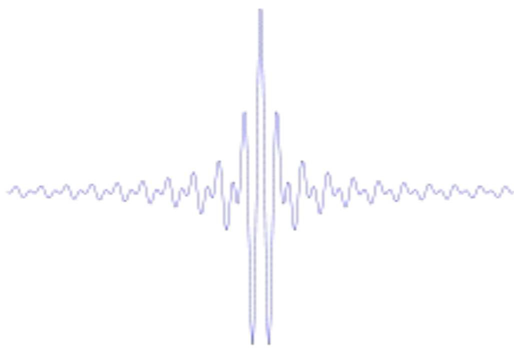
Morlet wavelet



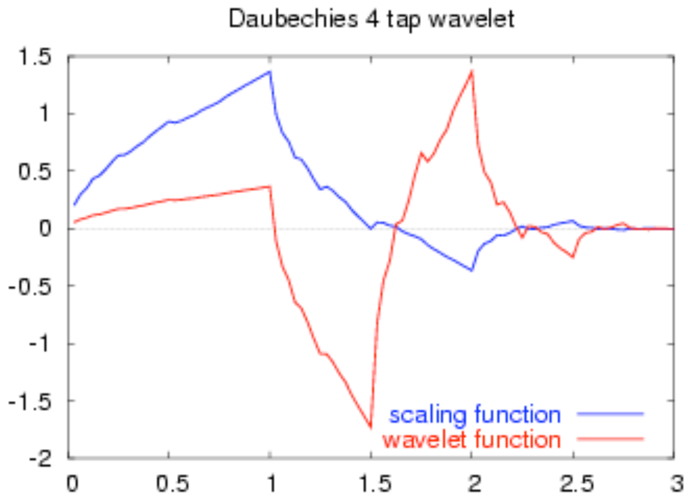


Coiflet wavelet



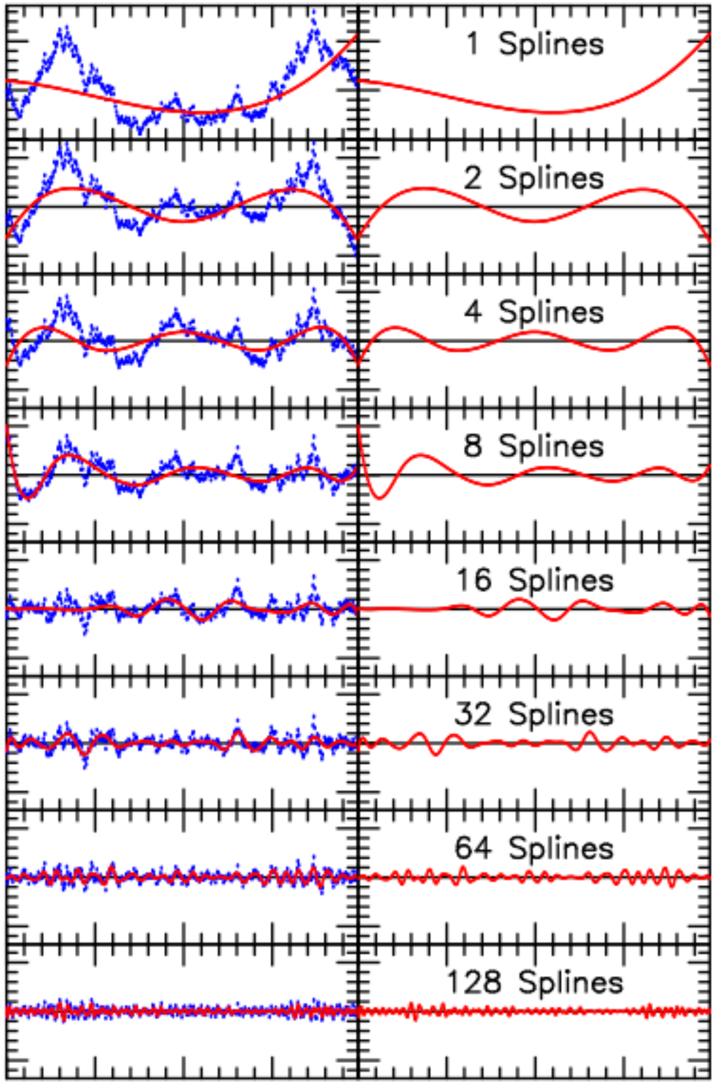


Shannon wavelet



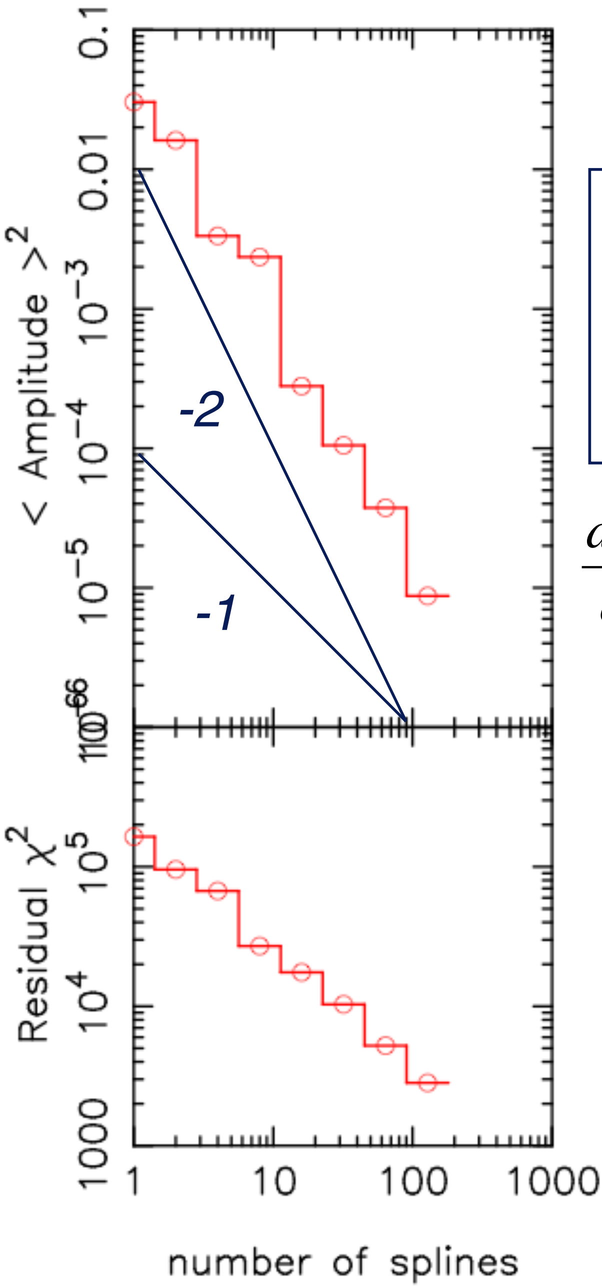
Spline Decomposition - Red Noise

Fit and subtract sequence of cubic splines.



Orthogonal spline basis.

Red Noise: Random walk



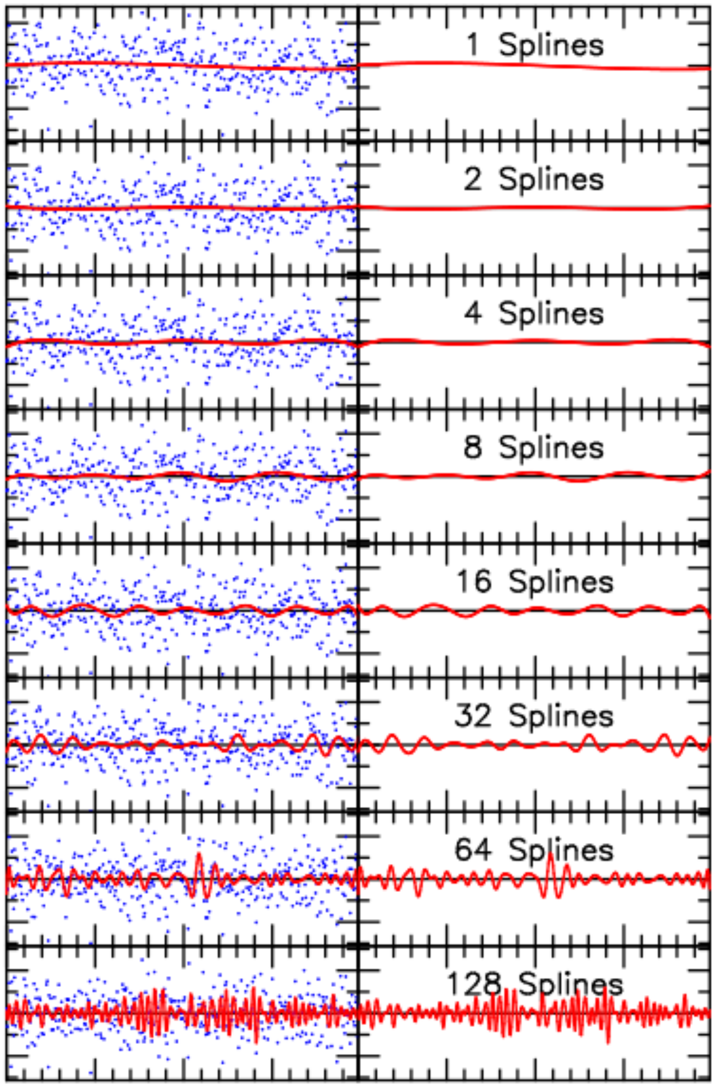
Red noise power spectrum.

Power law

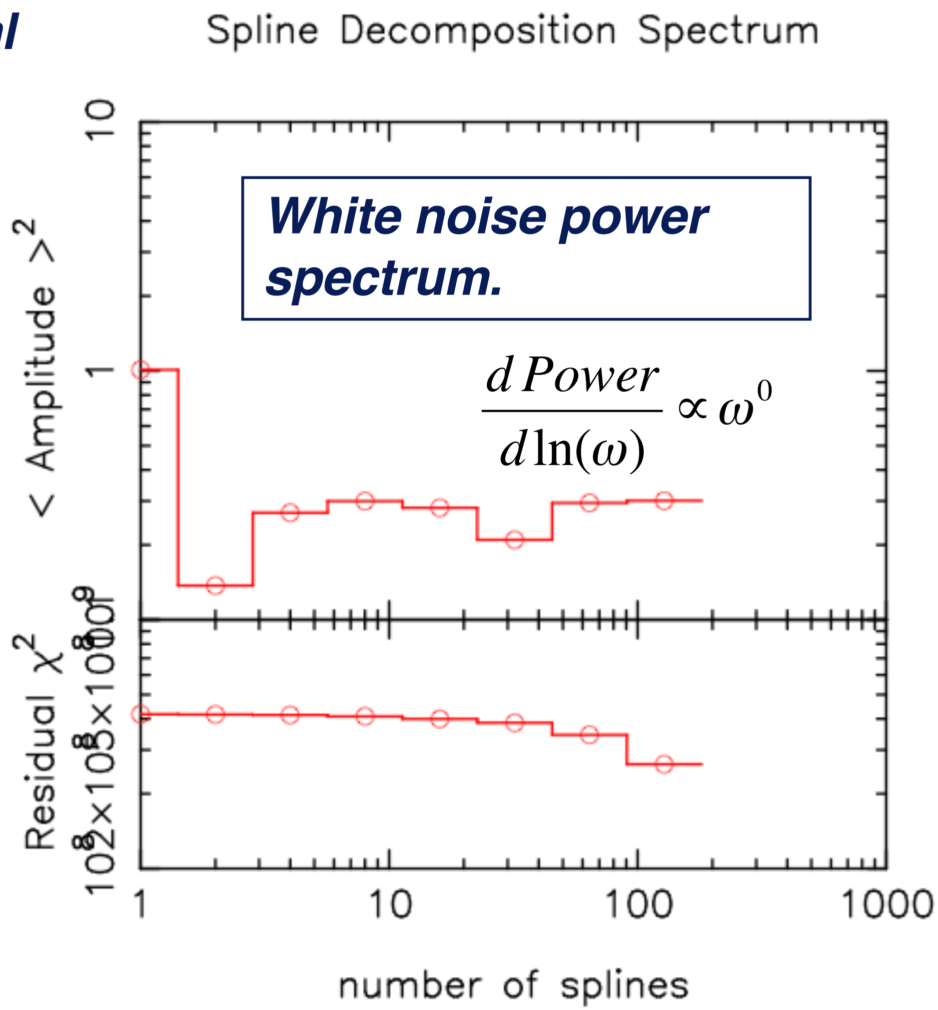
$$\frac{d\,Power}{d\ln(\omega)} \propto \omega^{-2}$$

Spline Decomposition - White Noise

Fit and subtract sequence of cubic splines.



Orthogonal spline basis.



White Noise: Independent noise at each time

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