Steady-State Accretion Disc

- **thickness** $$\frac{H}{R} \approx \frac{c_S}{V_\theta} \qquad V_\theta = \left(\frac{G\,M}{R} \right)^{1/2}$$
- **surface density** $$\Sigma \equiv \int_{-\infty}^{\infty} \rho \; dz = \sqrt{2\pi} \; \rho_0 \; H$$
  - (kg/m^2)
- **accretion rate** $$\dot{M} = 2\pi \; R \; (-V_R)\,\Sigma$$
  - (kg / s)

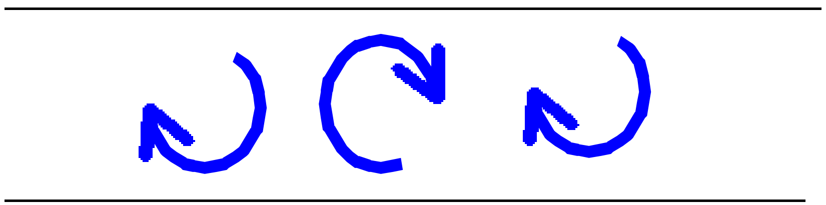
$$= 3\pi \; \nu \; \Sigma \left(1 - \left(\frac{R_*}{R} \right)^{1/2} \right)$$

- **inflow velocity** $$V_R \approx -\frac{3}{2}\frac{\nu}{R}$$
- **viscosity** $$\nu \equiv \alpha \; c_S \; H$$
  - (m^2 / s)
  - -- alpha model (hides uncertain physics)

Anomalous Viscosity

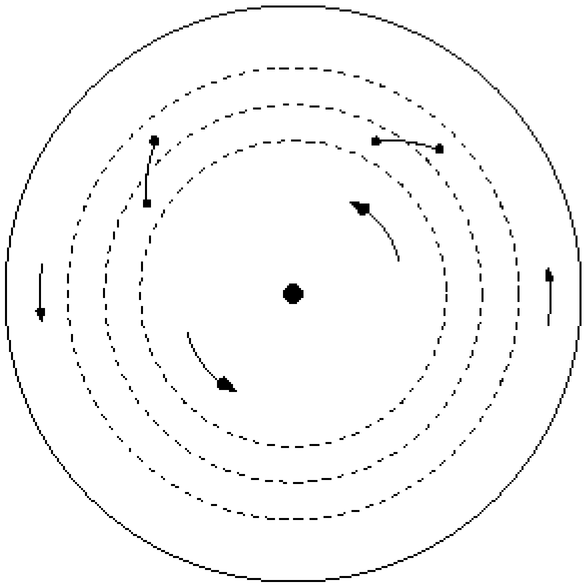
- **Viscosity arises from turbulent eddies**
- **largest eddie size ~ H**
- **largest eddie velocity ~ sound speed**

$$n = a \; c_S \; H$$



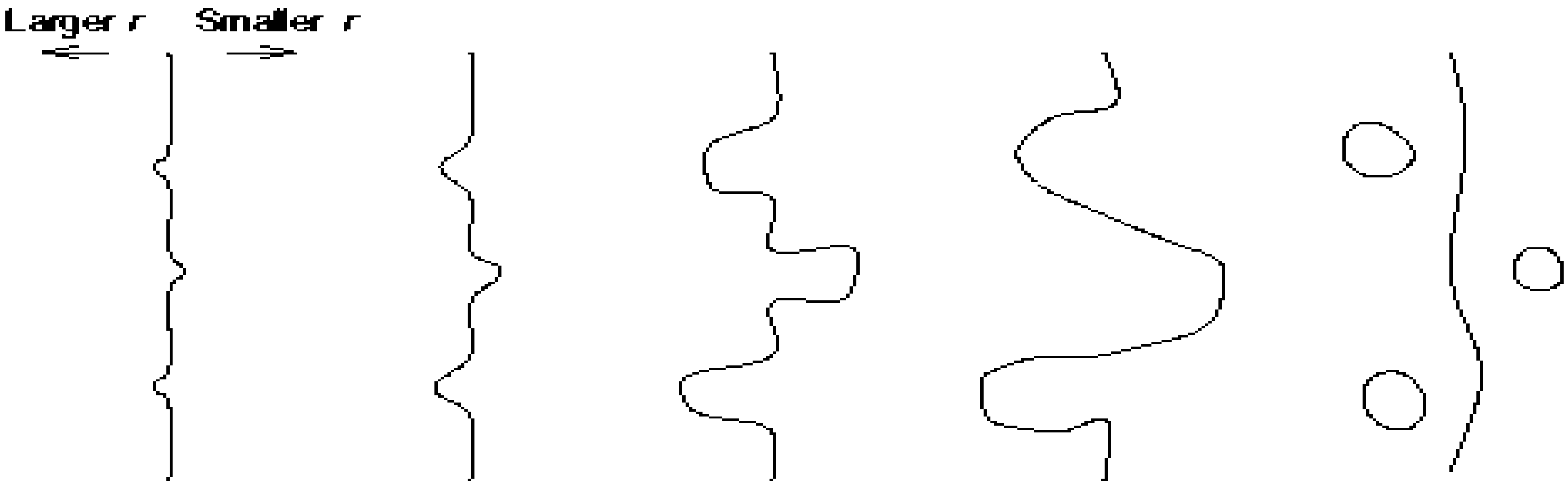
- **alpha = dimensionless scale factor**
- **0 < alpha < 1**

Magneto-Rotational Instability

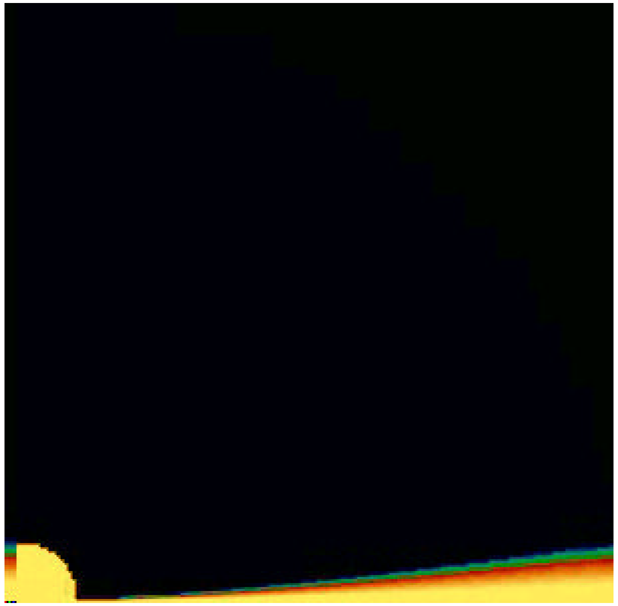


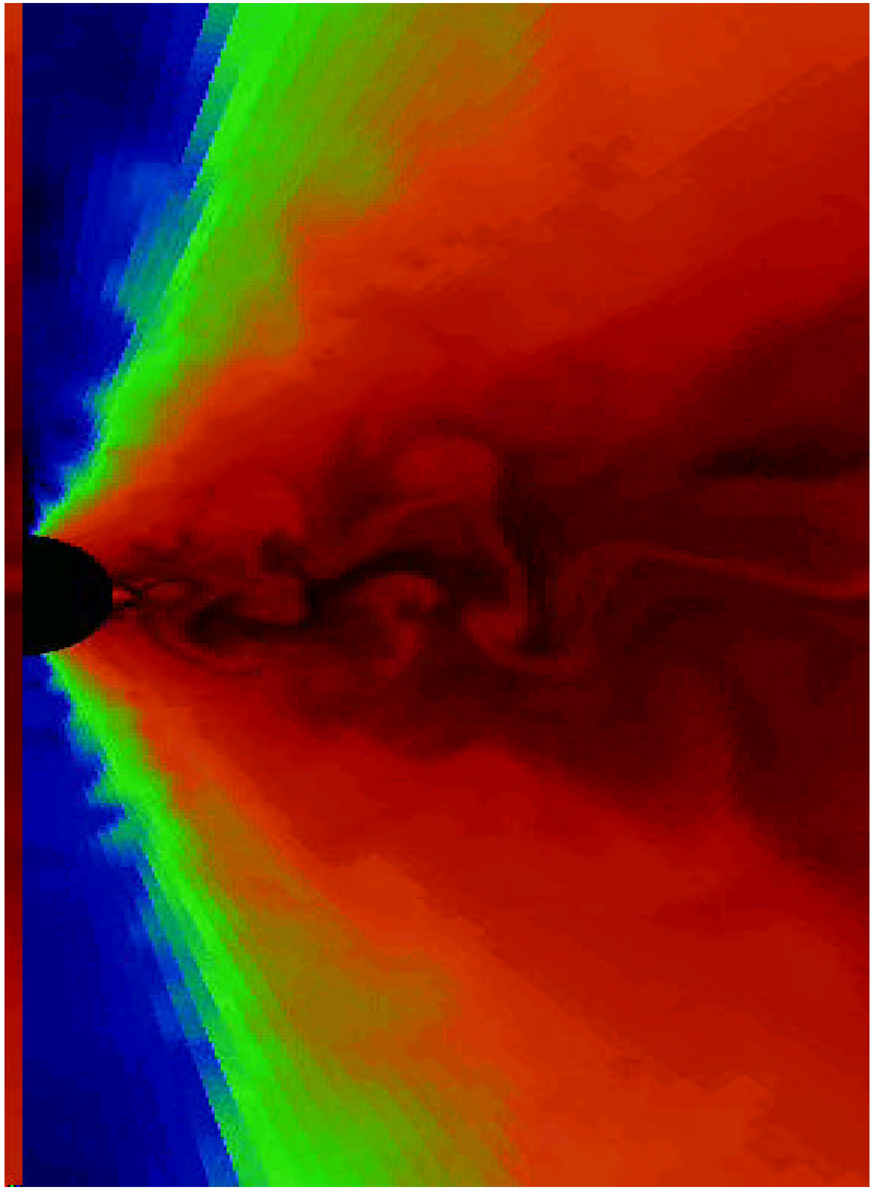
Magnetic fields link different annuli

generate MHD turbulence



Magneto-Rotational Instability





Timescales

- **Dynamical timescale -- Kepler orbit time**

$$t_{dyn} \sim \frac{R}{V_\theta} \sim \Omega^{-1}$$

- **Vertical hydrostatic equilibrium timescale**

$$t_{hse} \sim \frac{H}{c_S} \sim \Omega^{-1}$$

- **Radial inflow time**

$$t_{visc} \sim \frac{R}{-V_R} \sim \frac{R^2}{\nu} \sim \frac{R^2}{\alpha \; c_S \; H} \sim \alpha^{-1} \left(\frac{R}{H} \right)^2 \Omega^{-1}$$

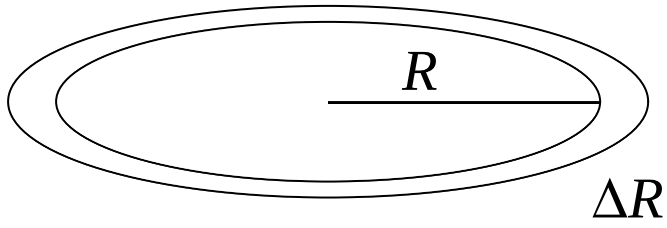
Steady-State Disc T(R)

- **Energy (kinetic + potential) per mass**

$$\varepsilon = \frac{V_\theta^{\,2}}{2} + \Phi_{grav} = \frac{G\,M}{2\,R} - \frac{G\,M}{R} = -\frac{1}{2}\frac{G\,M}{R} \qquad \frac{d\varepsilon}{dR} = \frac{1}{2}\frac{G\,M}{R^2}$$

- **energy radiated = energy released**

$$\sigma\,T^4 \times 2\pi\,R\,\Delta R \times 2 = \dot{M}\,\frac{d\varepsilon}{dR}\,\Delta R$$



- **Temperature profile**

$$T^4 = \frac{G\,M\,\dot{M}}{8\pi\,\sigma\,R^3} \qquad T \propto \left(M\,\dot{M}\right)^{1/4} R^{-3/4}$$

- **Disc Luminosity**

$$L_{disk} = 2\int_{R_*}^{\infty} 2\pi\,R\,dR\,\sigma\,T^4 = \frac{G\,M\,\dot{M}}{2}\int_{R_*}^{\infty}\frac{dR}{R^2} = \frac{G\,M\,\dot{M}}{2\,R_*}$$

Note: 1/2 of energy remains as kinetic energy

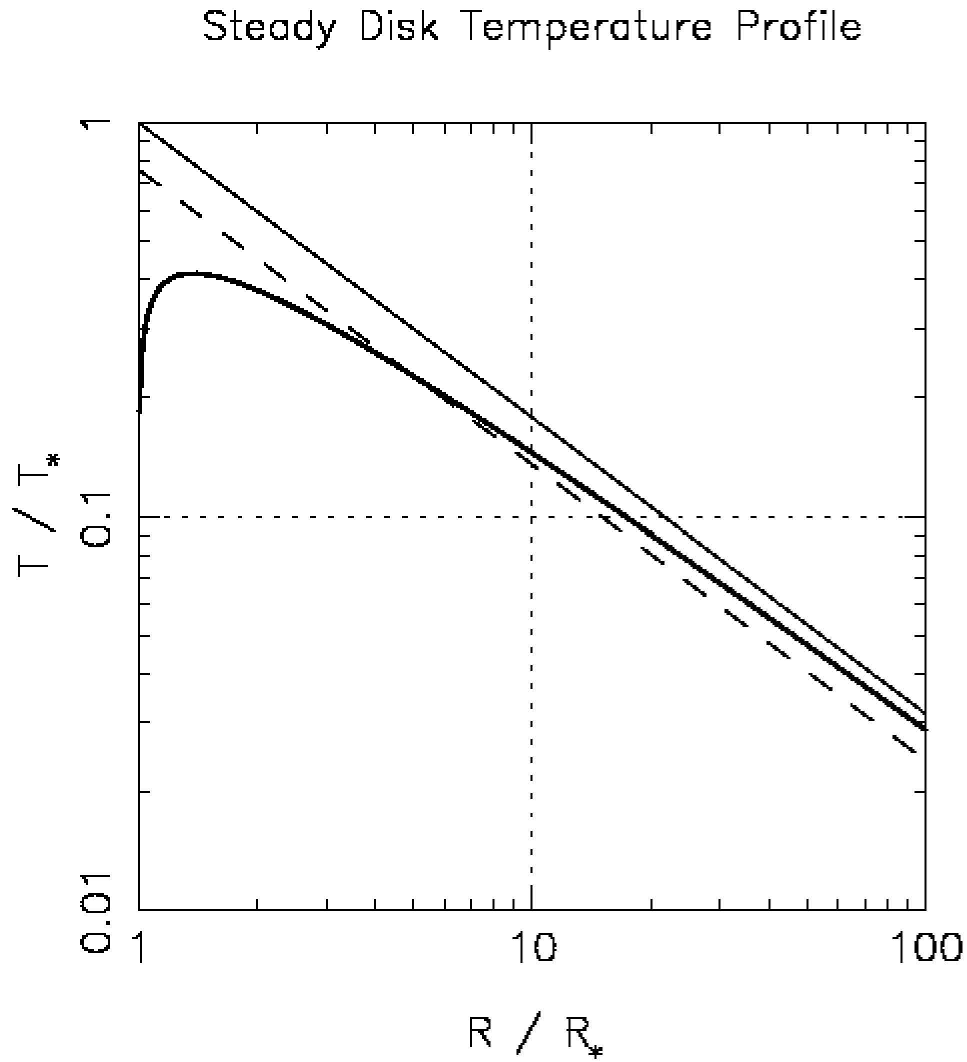
Temperature Profile

- **Include work done by viscous torques**

$$T^4 = \frac{3\,G\,M\,\dot{M}}{8\pi\,\sigma\,R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

$$= T_*^{\,4}\left(\frac{R}{R_*}\right)^{-3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

$$\boxed{T_* = \left(\frac{3\,G\,M\,\dot{M}}{8\pi\,\sigma\,R_*^{\,3}}\right)^{1/4}}$$



Inner disc luminosity lower.

Outer disk luminosity 3x higher.

max T occurs outside min R

Maximum Temperature

- **Including work done by viscous torques**

$$\left(\frac{T}{T_*}\right)^4 = \left(\frac{R}{R_*}\right)^{-3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right] \qquad T_* = \left(\frac{3\,G\,M\,\dot{M}}{8\pi\,\sigma\,R_*^{\,3}}\right)^{1/4}$$

- **Maximum temperature**

$$\frac{d}{dR}\left[T^4\right] = T_*^{\,4}\,\frac{d}{dR}\left[\frac{1}{R^3} - \frac{R_*^{\,1/2}}{R^{5/2}}\right]$$

$$= T_*^{\,4}\left[-\frac{3}{R^4} + \frac{5}{2}\frac{R_*^{\,1/2}}{R^{7/2}}\right] \rightarrow \boxed{\frac{R}{R_*} = \left(\frac{6}{5}\right)^2 = 1.44}$$

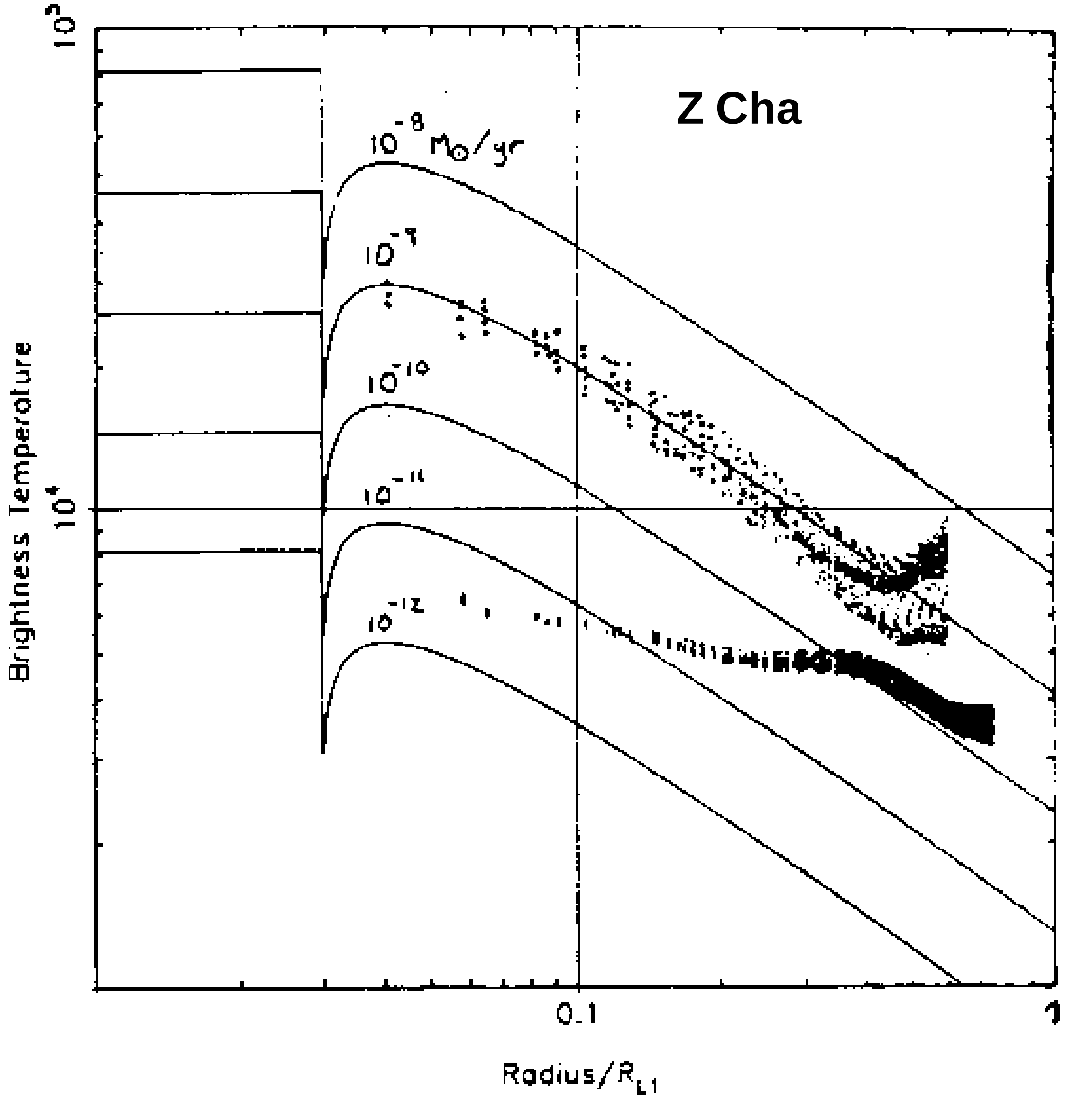
$$\boxed{\frac{T_{\max}}{T_*} = \left(\frac{5}{6}\right)^{3/2}\left(1-\frac{5}{6}\right)^{1/4} = \frac{5^{3/2}}{6^{7/4}} \cong 0.486}$$

Observed Temperature Profile

Eclipse lightcurve shapes used to map temperature profile of disk.

Outburst disk matches steady disk theory

Quiescent disk does not (optically thin)



Disc Luminosity

- **Including work done by viscous torques**

$$T^4 = \frac{3\,G\,M\,\dot{M}}{8p\ s\ R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

- **Disc luminosity**

$$L_{disk} = 2\int_{R_*}^{\infty} 2p\ R\,dR\ s\ T^4$$

$$= \frac{G\,M\,\dot{M}}{2}\int_{R_*}^{\infty} dR\left(\frac{1}{R^2} - \frac{R_*^{\,1/2}}{R^{5/2}}\right)$$

$$= \frac{G\,M\,\dot{M}}{2}\left[-\frac{1}{R} + \frac{3}{2}\frac{R_*^{\,1/2}}{R^{3/2}}\right]_{R_*}^{\infty}$$

$$= \frac{G\,M\,\dot{M}}{2\,R_*}$$

Same as before.

Energy is conserved but re-distributed in radius by viscous torques.

Blackbody Disc Spectrum

$$F_n = \int B_n(T(R)) \frac{2p\ R\,dR \cos i}{D^2}$$

$$B_n(T) = \frac{2\,h\,n^3}{c^2\left(e^x - 1\right)} = \frac{2\,k^3\,T^3}{c^2\,h^2}\frac{x^3}{e^x - 1} \qquad x \equiv \frac{h\,n}{k\,T}$$

$$\frac{T}{T_*} = \left(\frac{R}{R_*}\right)^{-3/4} = \left(\frac{x}{x_*}\right)^{-1} \quad \rightarrow \quad B_n(l, T) = \frac{2\,k^3\,T_*^{\,3}}{c^2\,h^2}\frac{x_*^{\,3}}{e^x - 1}$$

$$\frac{R}{R_*} = \left(\frac{x}{x_*}\right)^{4/3} \qquad \frac{dR}{R_*} = \frac{4}{3}\left(\frac{x}{x_*}\right)^{1/3}\left(\frac{dx}{x_*}\right)$$

Blackbody Disc Spectrum

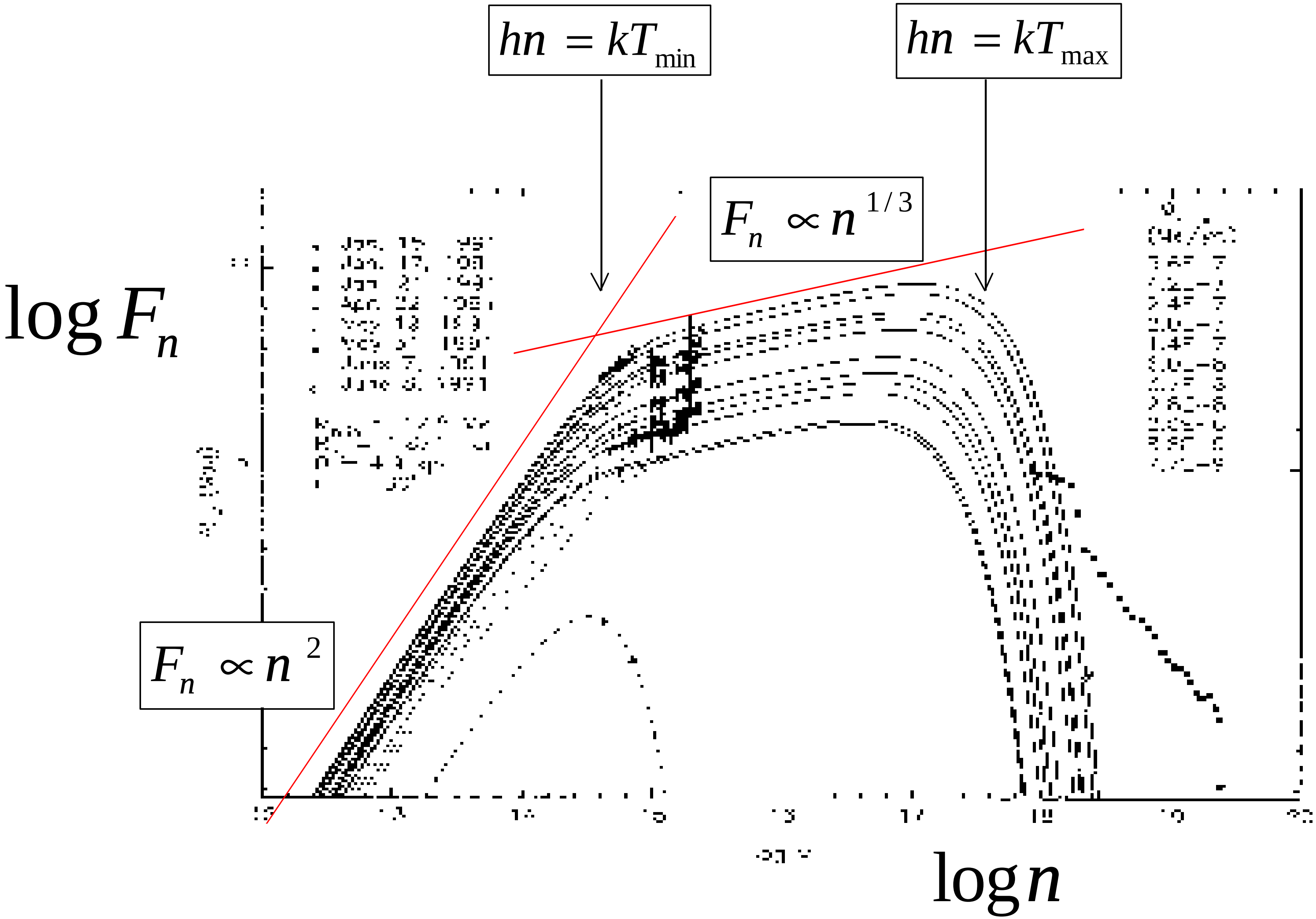
$$F_\nu = \int_{R_*}^{\infty} B_\nu(T(R)) \, \frac{2\pi \, R \, dR \cos i}{D^2}$$

$$= \int_{x_*}^{\infty} \frac{2\left(k\,T_*\right)^3}{\left(c\,h\right)^2} \frac{{x_*}^3}{e^x - 1} \frac{2\pi \, {R_*}^2 \cos i}{D^2} \frac{4}{3} \left(\frac{x}{x_*}\right)^{5/3} \left(\frac{dx}{x_*}\right)$$

$$= \frac{16\pi}{3} \frac{{R_*}^2 \cos i}{D^2} \frac{\left(k\,T_*\right)^3}{\left(h\,c\right)^2} \left(\frac{h\,\nu}{k\,T_*}\right)^{1/3} \int_{x_*}^{\infty} \frac{x^{5/3} dx}{e^x - 1}$$

$$F_\nu \propto \nu^{1/3}$$

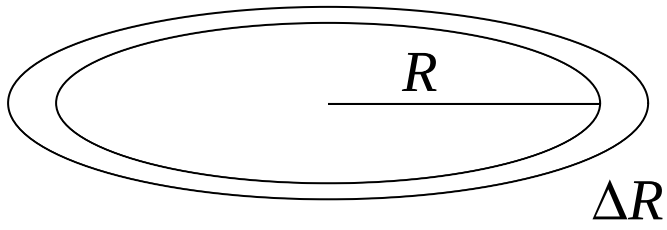
Blackbody Disc Spectrum



Surface Density Evolution

$$\Delta m = 2p \; R \; \Delta R \; \Sigma$$

$$\Delta \ell = 2p \; R \; \Delta R \; \Sigma \; R^2 \; \Omega$$



mass conservati on :

$$\frac{\partial}{\partial t}\left(2p \; R \; \Delta R \; \Sigma\right) = V_R \; 2p \; R \; \Sigma \; \Big|_{R+\Delta R}^{R}$$

$$R \; \dot{\Sigma} + \frac{\partial}{\partial R}\left(R \; \Sigma \; V_R \right) = 0$$

angular momentum :

$$R \frac{\partial}{\partial t}\left(\Sigma \; R^2 \; \Omega \right) + \frac{\partial}{\partial R}\left(R \; \Sigma \; V_R \; R^2 \; \Omega \right) = \frac{1}{2p} \frac{\partial G}{\partial R}$$

viscous torque : $G(R,t) = 2p \; R \, n \; \Sigma \; R^2 \; \dfrac{\partial \Omega}{\partial R}$

surface density evolution (diffusion) :

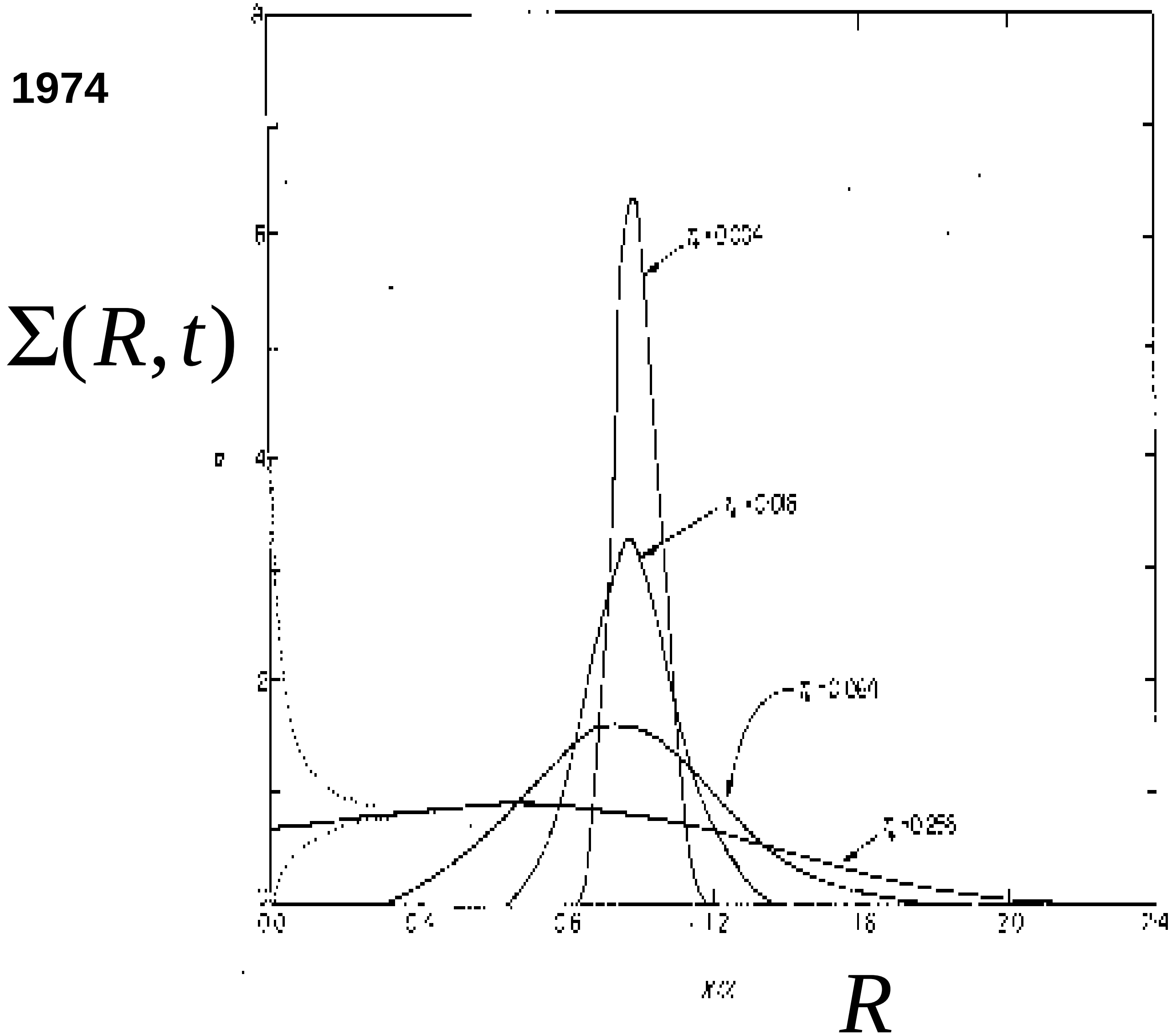
$$\dot{\Sigma} = \frac{3}{R} \frac{\partial}{\partial R}\left(R^{1/2} \frac{\partial}{\partial R}\left[n \; \Sigma \; R^{1/2} \right] \right)$$

simple diffusion :

$$\dot{\Sigma} = n \; \frac{\partial^2 \Sigma}{\partial \; R^2}$$

Ring spreads to form a disc

Pringle, Lynden-Bell 1974
MNRAS 168 603



Surface density evolution

diffusion in radius