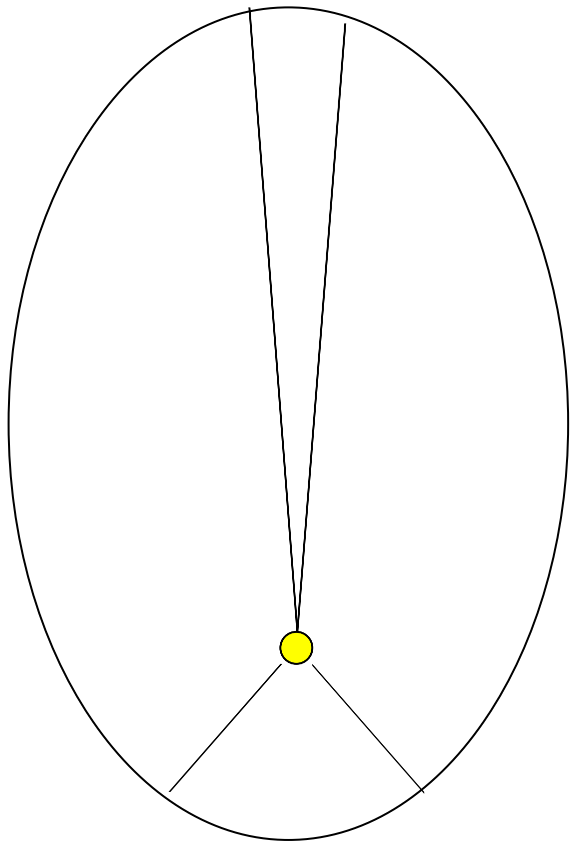
Kepler -- kinematics

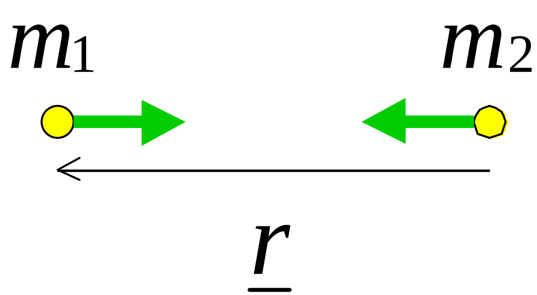
- **Empirical fit to Tycho Brahe's planet motion observations**

- **Kepler's laws:**
- **1. Elliptical orbits**
  Sun at focus
- **2. Equal areas in**
  equal times
- **3.** $$P^2 \propto a^3$$



Newton -- dynamics

$$\underline{F}_1 = -\underline{F}_2$$



$$\underline{F} = m\,\underline{\ddot{r}}$$

$$\underline{F}_1 = -\frac{G\,m_1\,m_2}{r^2}\,\underline{\hat{r}}$$

The *N*-body problem

$$\Phi(\underline{r},t) = -\sum_{i=1}^{N} \frac{G\,m_i}{\left|\underline{r}-\underline{r}_i\right|} = \text{gravitational potential}$$

$$\underline{f} = -\underline{\nabla}\Phi = \text{acceleration}$$

$$\text{equation of motion:}\quad m\,\ddot{\underline{r}} = \underline{F} = m\,\underline{f} = -m\,\underline{\nabla}\Phi$$

$$\underline{J} \equiv \underline{r}\times(m\,\dot{\underline{r}}) = \text{angular momentum}$$

$$\dot{\underline{J}} = \underline{r}\times(m\,\ddot{\underline{r}}) + \dot{\underline{r}}\times(m\,\dot{\underline{r}})$$

$$= \underline{r}\times\underline{F} \equiv \underline{G} = \text{torque}$$

0

$$\text{Note: If } \underline{F} \parallel \underline{r} \text{ then } \underline{J} = \text{constant}$$

$$\text{and } \underline{r}, \dot{\underline{r}} \text{ (both} \perp \underline{J}\text{) define orbital plane.}$$

Energy of orbit

$$\text{Kinetic Energy :} \quad K \equiv \frac{1}{2} m \left| \underline{\dot{r}} \right|^2 = \frac{1}{2} m (\underline{\dot{r}} \bullet \underline{\dot{r}})$$

$$\dot{K} = m \, \underline{\dot{r}} \bullet \underline{\ddot{r}} = \underline{\dot{r}} \bullet \underline{F} = -m \, \underline{\dot{r}} \bullet \underline{\nabla} \Phi$$

$$\text{Potential Energy :} \quad W \equiv m \Phi$$

$$\dot{W} = m \dot{\Phi} + m \left(\frac{d\Phi}{dx} \frac{dx}{dt} + \frac{d\Phi}{dy} \frac{dy}{dt} + \frac{d\Phi}{dz} \frac{dz}{dt} \right)$$

$$= m \dot{\Phi} + m \nabla \Phi \bullet \underline{\dot{r}}$$

$$\text{Total Energy :} \quad E = K + W$$

$$\dot{E} = \dot{K} + \dot{W} = \left(-m \underline{\dot{r}} \bullet \nabla \Phi \right) + \left(m \dot{\Phi} + m \nabla \Phi \bullet \underline{\dot{r}} \right) = m \dot{\Phi}$$

$$\therefore \quad \text{Energy is constant if } \dot{\Phi} = 0$$

Orbits in spherical potentials

If $\Phi(r)$ depends on $|\underline{r}|$, not $\underline{r}$, then

$$\underline{f} = -\underline{\nabla}\Phi = -\frac{d\Phi}{dr}\hat{\underline{r}}$$

then

$$\dot{\underline{J}} = \underline{r} \times m\underline{f} = -m\frac{d\Phi}{dr}\underline{r} \times \hat{\underline{r}} = 0$$

$\rightarrow$ angular momentum about origin is conserved!

$\therefore$ orbital motion is in a plane

$\underline{r}$, $\dot{\underline{r}}$ in plane,

specific angular momentum

$\underline{L} \equiv \underline{J}/m$ perpendicular to plane

Kinematics

position $\underline{r}(t)$

velocity $\dot{\underline{r}} = \dot{r}\,\hat{\underline{r}} + r\dot{q}\,\hat{\underline{q}}$

speed $V = \dot{r}^2 + r^2\,\dot{q}^2$

unit vectors rotate with q along the trajectory

$$\frac{d\hat{\underline{r}}}{dt} = \frac{d\hat{\underline{r}}}{dq}\,\frac{dq}{dt} = \hat{\underline{q}}\,\dot{q}$$

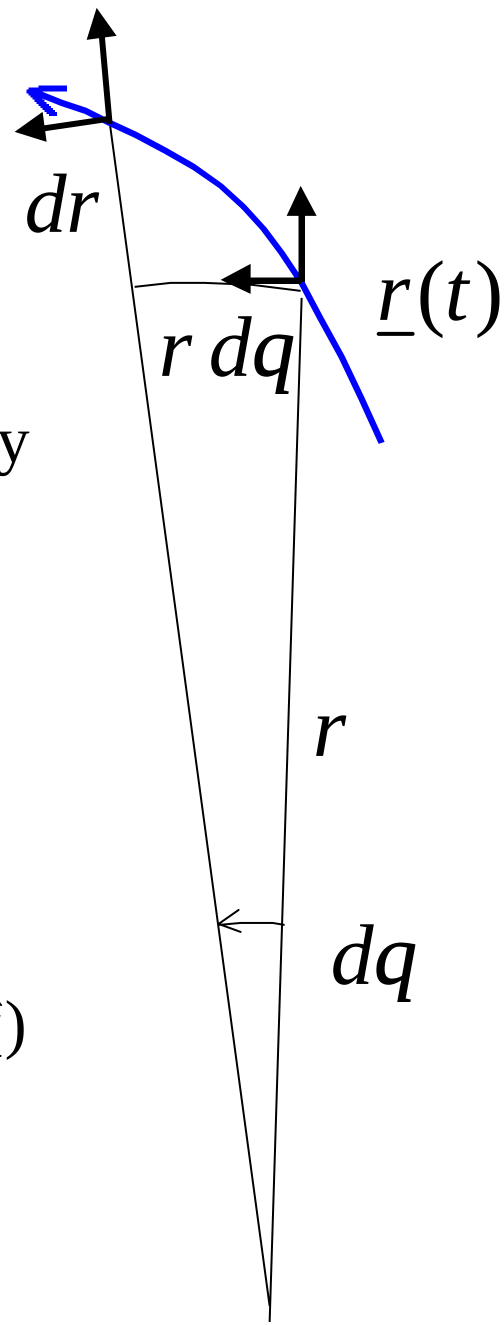
$$\frac{d\hat{\underline{q}}}{dt} = \frac{d\hat{\underline{q}}}{dq}\,\frac{dq}{dt} = -\hat{\underline{r}}\,\dot{q}$$

acceleration

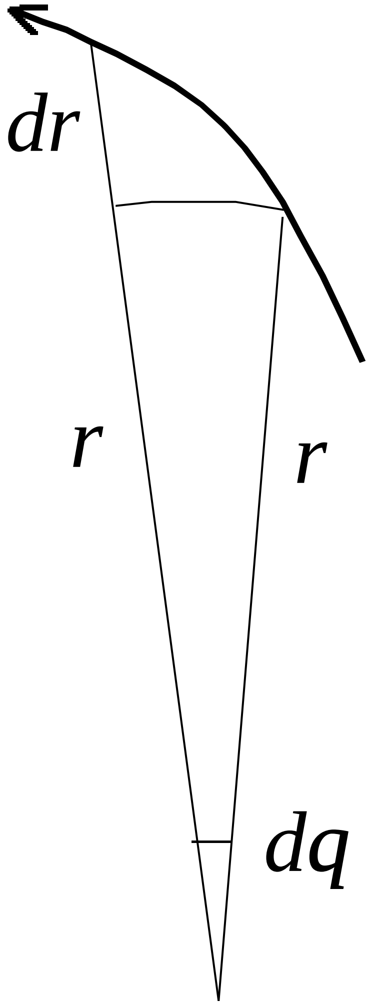
$$\ddot{\underline{r}} = \ddot{r}\,\hat{\underline{r}} + \dot{r}\,(\dot{q}\,\hat{\underline{q}}) + \dot{r}\,\dot{q}\,\hat{\underline{q}} + r\,\ddot{q}\,\hat{\underline{q}} + r\,\dot{q}\,(-\dot{q}\,\hat{\underline{r}})$$

$$= \left(\ddot{r} - r\,\dot{q}^2\right)\hat{\underline{r}} + \left(2\,\dot{r}\,\dot{q} + r\ddot{q}\right)\hat{\underline{q}}$$

should be 0 if *F* parallel to *r*



Kepler's 2nd Law



area swept out by $\underline{r}$ in time dt

$$dA = \tfrac{1}{2}\left(r^2\, dq + r\, dr\, dq\right)$$

$$\dot{A} = \tfrac{1}{2} r^2 \dot{q}$$

$$= \tfrac{1}{2}\, \underline{r} \times \underline{\dot{r}} = \frac{J}{2m} = \text{constant}$$

$$\text{Note}: \frac{1}{r}\frac{d}{dt}\left(r^2 \dot{q}\right) = 2\dot{r}\dot{q} + r\ddot{q} = 0$$

transverse component of acceleration $= 0$

Conic Sections

$$r = \frac{\ell}{1 + e\cos q}$$

$e =$ eccentricity

> 1 hyperbola

$= 1$ parabola

< 1 ellipse

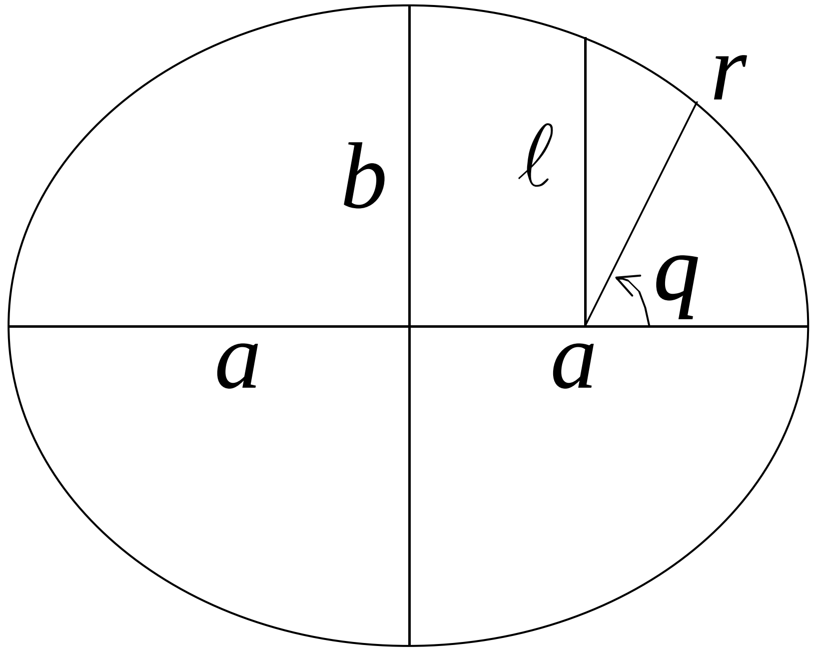
$= 0$ circle

$\ell =$ semi - latus rectum

$a =$ semi - major axis

$b =$ semi - minor axis

$$= a\left(1 - e^2\right)^{1/2}$$



$q = 0 \qquad r = \dfrac{\ell}{1+e} \quad = a(1-e)$ periastron

$q = 90^\circ \qquad r = \ell \quad = a(1-e^2)$

$q = 180^\circ \qquad r = \dfrac{\ell}{1-e} \quad = a(1+e)$ apastron