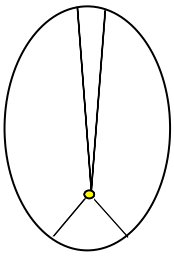
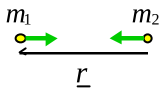
Kepler -- kinematics

- Empirical fit to Tycho Brahe's planet motion observations
- Kepler's laws:
- 1. Elliptical orbits Sun at focus
- 2. Equal areas in equal times
- 3. $P^2 \propto a^3$



Newton -- dynamics

$$\underline{F}_1 = -\underline{F}_2$$



$$\underline{F} = m\,\ddot{\underline{r}}$$

$$\underline{F}_1 = -\frac{G\,m_1\,m_2}{r^2}\,\hat{\underline{r}}$$

The *N*-body problem

$$\Phi(\underline{r},t) = -\sum_{i=1}^{N} \frac{G\,m_i}{|\underline{r}-\underline{r}_i|} = \text{gravitational potential}$$

$$\underline{f} = -\underline{\nabla}\Phi = \text{acceleration}$$

$$\text{equation of motion:} \quad m\ddot{\underline{r}} = \underline{F} = m\,\underline{f} = -m\underline{\nabla}\Phi$$

$$\underline{J} \equiv \underline{r}\times(m\dot{\underline{r}}) \;= \text{angular momentum}$$

$$\dot{\underline{J}} = \underline{r}\times(m\ddot{\underline{r}}) + \dot{\underline{r}}\times(m\dot{\underline{r}})$$

$$= \underline{r}\times\underline{F} \equiv \underline{G} \;= \text{torque} \quad 0$$

Note: If $\underline{F} \parallel \underline{r}$ then $\underline{J} = \text{constant}$

and $\underline{r}, \dot{\underline{r}}$ (both $\perp \underline{J}$) define orbital plane.

Energy of orbit

Kinetic Energy : $K \equiv \frac{1}{2}m|\dot{\underline{r}}|^2 = \frac{1}{2}m(\dot{\underline{r}}\bullet\dot{\underline{r}})$

$\dot{K} = m\,\dot{\underline{r}}\bullet\ddot{\underline{r}} = \dot{\underline{r}}\bullet\underline{F} = -m\dot{\underline{r}}\bullet\underline{\nabla}\Phi$

Potential Energy : $W \equiv m\,\Phi$

$$\dot{W} = m\dot{\Phi} + m\left(\frac{d\Phi}{dx}\frac{dx}{dt} + \frac{d\Phi}{dy}\frac{dy}{dt} + \frac{d\Phi}{dz}\frac{dz}{dt}\right)$$

$$= m\dot{\Phi} + m\nabla\Phi\bullet\dot{\underline{r}}$$

Total Energy : $E = K + W$

$$\dot{E} = \dot{K} + \dot{W} = \left(-m\dot{\underline{r}}\bullet\nabla\Phi\right) + \left(m\dot{\Phi} + m\nabla\Phi\bullet\dot{\underline{r}}\right) = m\dot{\Phi}$$

$\therefore$ Energy is constant if $\dot{\Phi} = 0$

Orbits in spherical potentials

If $\Phi(r)$ depends on $|\underline{r}|$, not $\underline{r}$, then

$$\underline{f} = -\underline{\nabla}\Phi = -\frac{d\Phi}{dr}\hat{\underline{r}}$$

then

$$\dot{\underline{J}} = \underline{r}\times m\,\underline{f} = -m\frac{d\Phi}{dr}\underline{r}\times\hat{\underline{r}} = 0$$

$\rightarrow$ angular momentum about origin is conserved !

$\therefore$ orbital motion is in a plane

$\underline{r}, \dot{\underline{r}}$ in plane,

specific angular momentum

$\underline{L} \equiv \underline{J}/m$ perpendicular to plane

Kinematics

position $\underline{r}(t)$

velocity $\dot{\underline{r}} = \dot{r}\,\hat{\underline{r}} + r\dot{q}\,\hat{\underline{q}}$

speed $V = \dot{r}^2 + r^2\dot{q}^2$

unit vectors rotate with q along the trajectory

$$\frac{d\hat{\underline{r}}}{dt} = \frac{d\hat{\underline{r}}}{dq}\frac{dq}{dt} = \hat{\underline{q}}\,\dot{q}$$

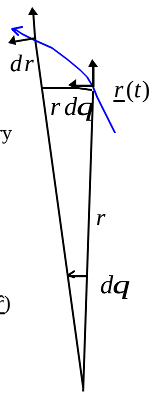
$$\frac{d\hat{\underline{q}}}{dt} = \frac{d\hat{\underline{q}}}{dq}\frac{dq}{dt} = -\hat{\underline{r}}\,\dot{q}$$

acceleration

$$\ddot{\underline{r}} = \ddot{r}\,\hat{\underline{r}} + \dot{r}\,(\dot{q}\,\hat{\underline{q}}) + \dot{r}\,\dot{q}\,\hat{\underline{q}} + r\,\ddot{q}\,\hat{\underline{q}} + r\,\dot{q}\,(-\dot{q}\,\hat{\underline{r}})$$

$$= \left(\ddot{r} - r\,\dot{q}^2\right)\hat{\underline{r}} + \left(2\,\dot{r}\dot{q} + r\ddot{q}\right)\hat{\underline{q}}$$

should be 0 if $\underline{F}$ parallel to $\underline{r}$



Kepler's 2nd Law

area swept out by $\underline{r}$ in time dt

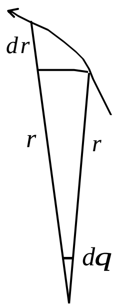
$$dA = \tfrac{1}{2}\left(r^2 dq + r\,dr\,dq\right)$$

$$\dot{A} = \tfrac{1}{2} r^2 \dot{q}$$

$$= \tfrac{1}{2}\underline{r} \times \dot{\underline{r}} = \frac{J}{2m} = \text{constant}$$

Note: $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{q}\right) = 2\dot{r}\dot{q} + r\ddot{q} = 0$

transverse component of acceleration $= 0$



Conic Sections

$r = \dfrac{\ell}{1 + e\cos q}$

e = eccentricity

> 1 hyperbola

$= 1$ parabola

< 1 ellipse

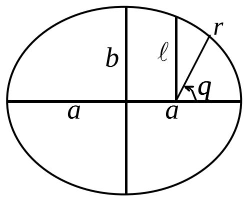
$= 0$ circle

ℓ = semi-latusrectum

a = semi-major axis

b = semi-minor axis

$= a\left(1 - e^2\right)^{1/2}$



$q = 0$	$r = \dfrac{\ell}{1+e}$	$= a(1-e)$	periastron
$q = 90^\circ$	$r = \ell$	$= a(1-e^2)$	
$q = 180^\circ$	$r = \dfrac{\ell}{1-e}$	$= a(1+e)$	apastron