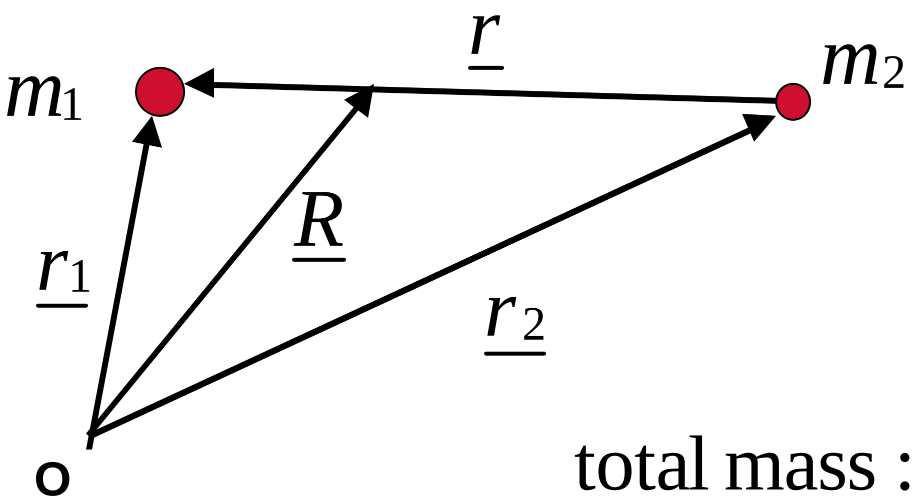
The Two-Body Problem

- **Newtonian gravity**
- **2 point masses**
  - **good approx for stars**



$$\underline{r} = \underline{r}_1 - \underline{r}_2$$

$$\hat{\underline{r}} = \underline{r} / r$$

$$\text{total mass}: \quad M = m_1 + m_2$$

  - **O: origin**

$$\text{centre of mass}: \quad \underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{M}$$

Two-Body motion

- **Newtons laws of motion + law of gravitation**

Equations of motion:

$$\underline{F}_1 = m_1 \ddot{\underline{r}}_1 = \frac{-G m_1 m_2}{r^2} \hat{\underline{r}} \qquad \underline{F}_2 = m_2 \ddot{\underline{r}}_2 = -\underline{F}_1$$

add the equations $\quad m_1 \ddot{\underline{r}}_1 + m_2 \ddot{\underline{r}}_2 = 0$

integrate dt $\quad m_1 \dot{\underline{r}}_1 + m_2 \dot{\underline{r}}_2 = \underline{A}$

again $\quad m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{A}\, t + \underline{B}$

definition of $\underline{R}$ $\quad M\, \underline{R} = \underline{A}\, t + \underline{B}$

- $\therefore$ **Centre of mass moves with constant velocity**
  - **(unless acted on by an external force)**

Relative Motion

- **Important for eclipses**
- **Subtract two equations of motion**

$$\ddot{\underline{r}} = \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = \frac{-G\,M}{r^2}\,\hat{\underline{r}}$$

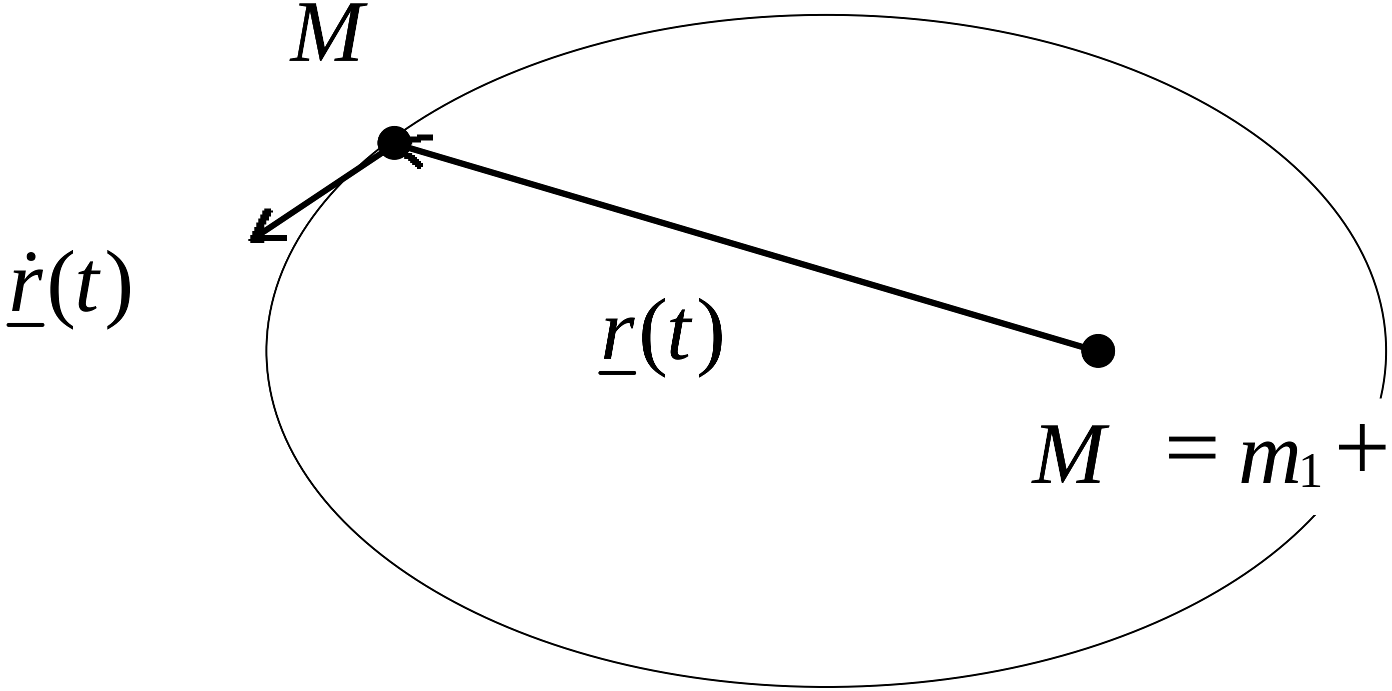
- **Multiply by $\mu = m_1 m_2 / M$ = "reduced mass"**

$$\mu\ \ddot{\underline{r}} = \frac{-G\ M\ \mu}{r^2}\,\hat{\underline{r}} = \frac{-G\ m_1\ m_2}{r^2}\,\hat{\underline{r}}$$

- **Relative orbit is as if:**
  - **orbiter has reduced mass μ = reduced mass**
  - **stationary central mass is M = total mass.**

The Relative Orbit

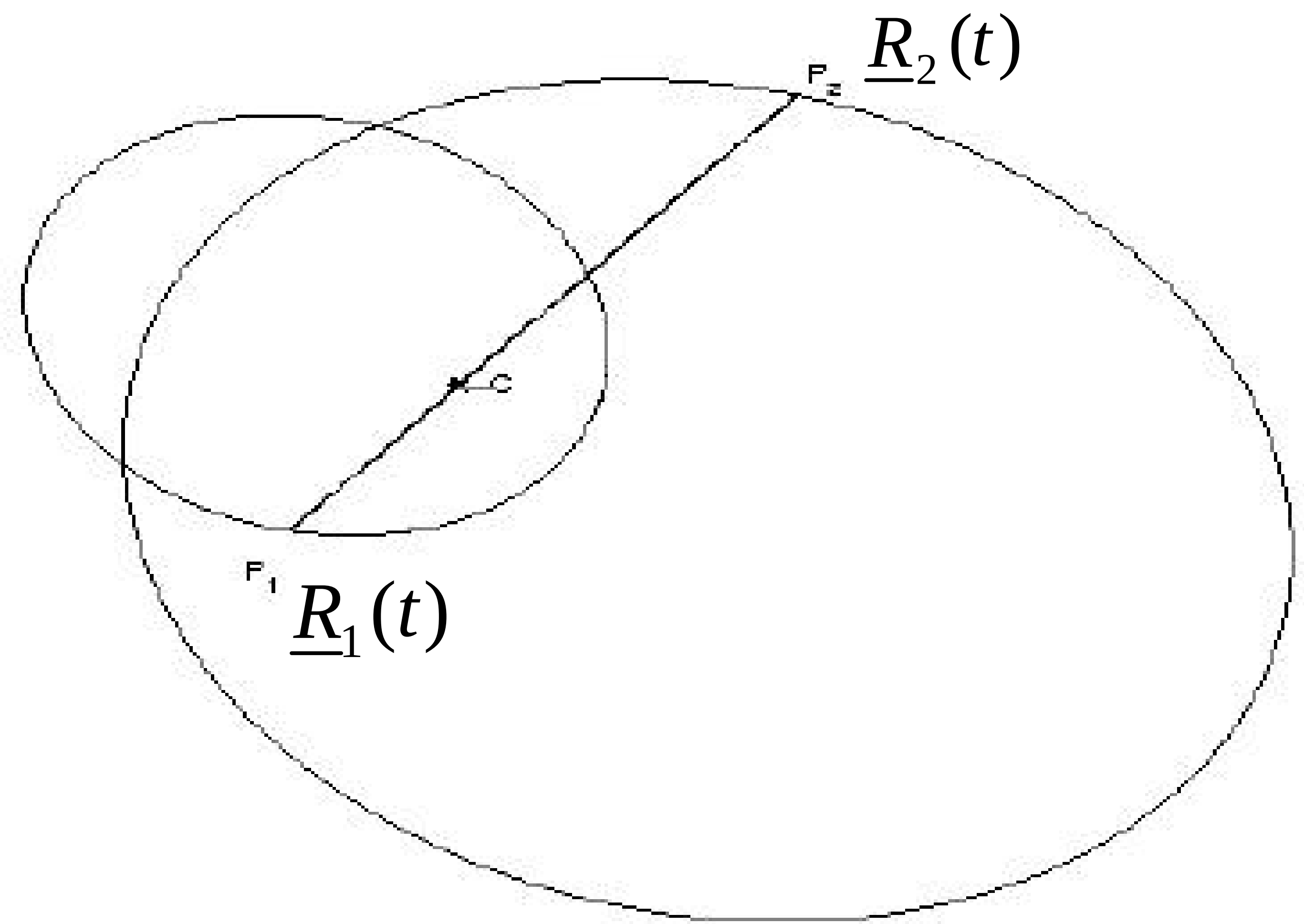
$$m = \frac{m_1 m_2}{M}$$



$$M = m_1 + m_2$$

Important for Eclipses

2 Barycentric Orbits



Important for Radial Velocities

Barycentric Orbits

- **Important for radial velocity curves**

$$m_1 \underline{R_1} + m_2 \underline{R_2} = 0 \quad \text{centre of mass frame}$$

$$\underline{r} = \underline{R_1} - \underline{R_2} = \frac{m_1 + m_2}{m_2} \underline{R_1} = \frac{M}{m_2} \underline{R_1} = -\frac{M}{m_1} \underline{R_2}$$

equations of motion :

$$\ddot{R}_1 = -\frac{G\, m_2}{r^3} \underline{r} \qquad \ddot{R}_2 = -\frac{G\, m_1}{r^3} \left(-\underline{r}\right)$$

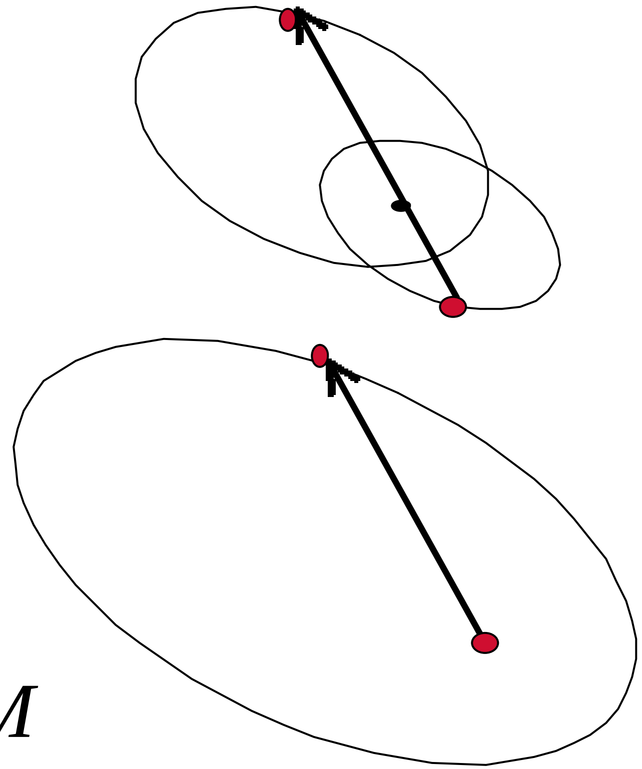
– **Eliminate r^3 using $r = f(R_1)$ and $r = f(R_1)$:**

$$\ddot{R}_1 = -\frac{G\, m_2^{\,3}}{M^2} \frac{\underline{R_1}}{R_1^{\,3}} \qquad \ddot{R}_2 = -\frac{G\, m_1^{\,3}}{M^2} \frac{\underline{R_2}}{R_2^{\,3}}$$

– **the acceleration of each star relative to the centre of mass**

Relative vs Barycentric orbits

- (*a, e, P, v*)   relative orbit
- (*a ,e, P ,v*)$_{1,2}$ barycentric orbits
- m_1 and m_2 on straight line thru C



$$P_1 = P_2 = P \qquad e_1 = e_2 = e$$

$$a_1 = a\ m_2 / M \qquad a_2 = a\ m_1 / M$$

$$a = a_1 + a_2 \qquad M = m_1 + m_2$$

$$a_1 : a_2 : a = V_1 : V_2 : V = m_2 : m_1 : M$$

Kepler :

$$\frac{4p^2}{GP^2} = \frac{M}{a^3} = \frac{m_2^3 / M^2}{a_1^3} = \frac{m_1^3 / M^2}{a_2^3}$$

Orbital Speed

$$\text{orbital speed}: \quad V^2 = \dot{r}^2 + r^2\dot{q}^2$$

$$\text{conic section}: \quad r = \frac{\ell}{1 + e\cos q}$$

$$\frac{d}{dt}\left[1 + e\cos q = \frac{\ell}{r}\right] \Rightarrow -e\,(\sin q\,)\,\dot{q} = -\frac{\ell}{r^2}\dot{r}$$

$$\text{specific angular momentum}: \ r^2\dot{q} = L$$

$$\therefore \quad \dot{r} = \frac{r^2\dot{q}}{\ell}e\sin q = \frac{L}{\ell}e\sin q$$

$$r\,\dot{q} = \frac{L}{r} = \frac{L}{\ell}(1 + e\cos q\,)$$

Orbital Speed

$$V^2 = \left(\frac{L}{\ell}\right)^2 \left[e^2 \sin^2\theta + (1 + e\cos\theta)^2\right]$$

$$= \left(\frac{L}{\ell}\right)^2 \left[e^2 + 1 + 2e\cos\theta\right]$$

$$= \left(\frac{L}{\ell}\right)^2 \left[2(e\cos\theta + 1) + e^2 - 1\right]$$

$$V^2 = \frac{L^2}{\ell}\left[\frac{2}{r} - \frac{1 - e^2}{\ell}\right]$$

$$\ell = \frac{L^2}{G\,M}$$

$$\frac{L}{\ell} = \frac{G\,M}{L}$$

Orbital Speed

$$V^2 = \frac{L^2}{\ell}\left[\frac{2}{r} - \frac{1-e^2}{\ell}\right]$$

ellipse : $\ell = a(1-e^2)$ $\quad L^2 = GM\,\ell$

$$V^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

ellipse	$e < 1$	$\ell = a(1-e^2)$	$V^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$
circle	$e = 0$	$\ell = a$	$V^2 = \frac{GM}{a}$
parabola	$e = 1$		$V^2 = \frac{2GM}{r}$
hyperbola	$e > 1$	$\ell = a(1-e^2)$	$V^2 = GM\left[\frac{2}{r} + \frac{1}{a}\right]$

Energy of Orbit

Kinetic energy:

$$KE = T = \frac{1}{2} m_1 V_1^{\,2} + \frac{1}{2} m_2 V_2^{\,2} = \frac{m_1 m_2}{2M} V^2$$

orbital speed: $V^2 = G M \left(\frac{2}{r} - \frac{1}{a} \right)$

Potential energy:

$$PE = W = -\int_r^\infty \frac{G m_1 m_2}{r^2} dr = -\frac{G m_1 m_2}{r}$$

Total energy: $E = T + W = -\frac{G m_1 m_2}{2a} < 0$

Binding energy: $-E > 0$

Angular momentum of the orbit

- **Angular momentum vector J, defines orbital plane**
  - $\underline{J} = m_1\underline{L}_1 + m_2\underline{L}_2$ and $L^2 = G\,M\,a\,(1-e^2)$
    and $L_1^2 = G\,(m_2^3/M^2)\,a_1\,(1-e^2)$
    and $a_1/a = m_2/M$
    - same for $\underline{L}_2$
  - hence

$$L_1 = \frac{m_2^2}{M^2} L; \quad L_2 = \frac{m_1^2}{M^2} L$$

therefore

$$J^2 = \frac{G\, m_1^2 m_2^2}{M} a\,(1-e^2)$$

and the final expression for J is

$$J = \frac{2p\ a^2\ m_1\ m_2}{P\ M}\sqrt{1-e^2}$$

Orbital Angular momentum

- **Given masses *m1,m2* and Energy *E*,**
  - **the angular momentum *J* determines the shape of the orbit**
  - **ie the eccentricity (or the conic section parameter *l*)**

- **For given *E*,**
  - **circular orbits have maximum *J***
  - ***J* decreases as *e* → 1**
  - **orbit becomes rectilinear ellipse**

- **relation between *E*, and *J* very important in**

  determining when systems interact

  mass exchange and orbital evolution