Perturbed Two-Body motion

- **“close” to 2-body motion**
  - **planetary systems**
  - **multi-star systems (triples, quadruples, ...)**
  - **close binaries (non-spherical stars)**
  - **general relativity corrections (Mercury, binary pulsars)**
- **Perturbed potential**

$$\ddot{O} = \ddot{O}_0 + S$$

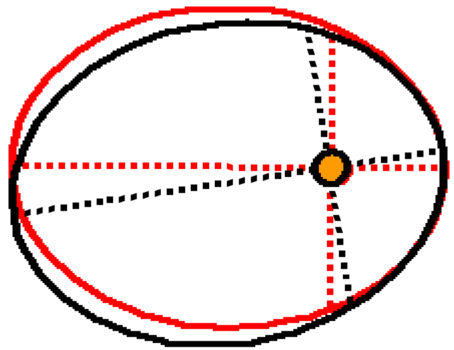
point mass potential ($\ddot{O}_0$)

small corrections (S)

the ***<u>disturbing function</u>***

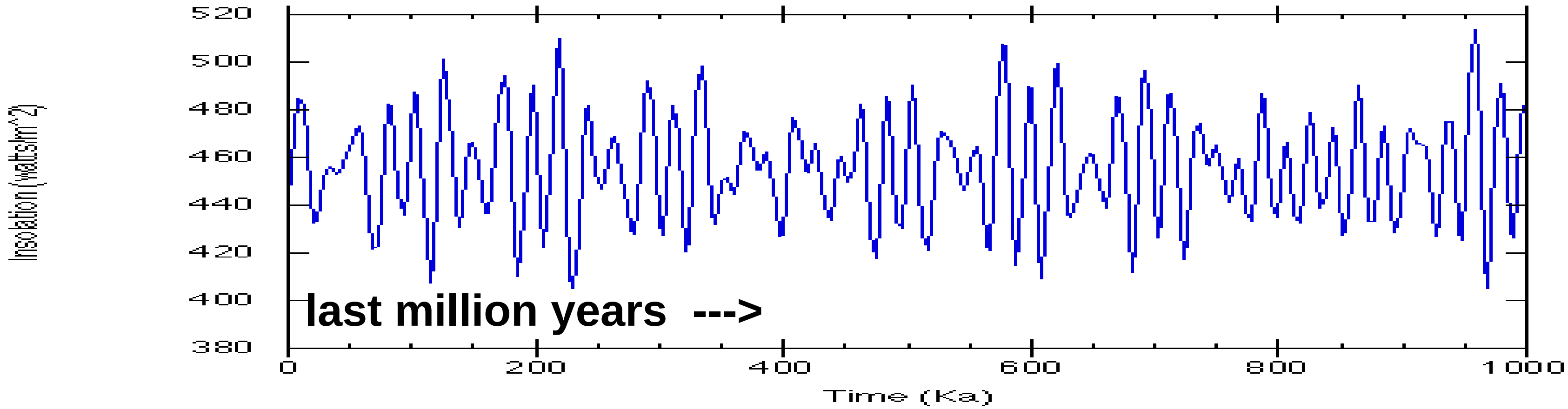
Lagrange’s Planetary Equations

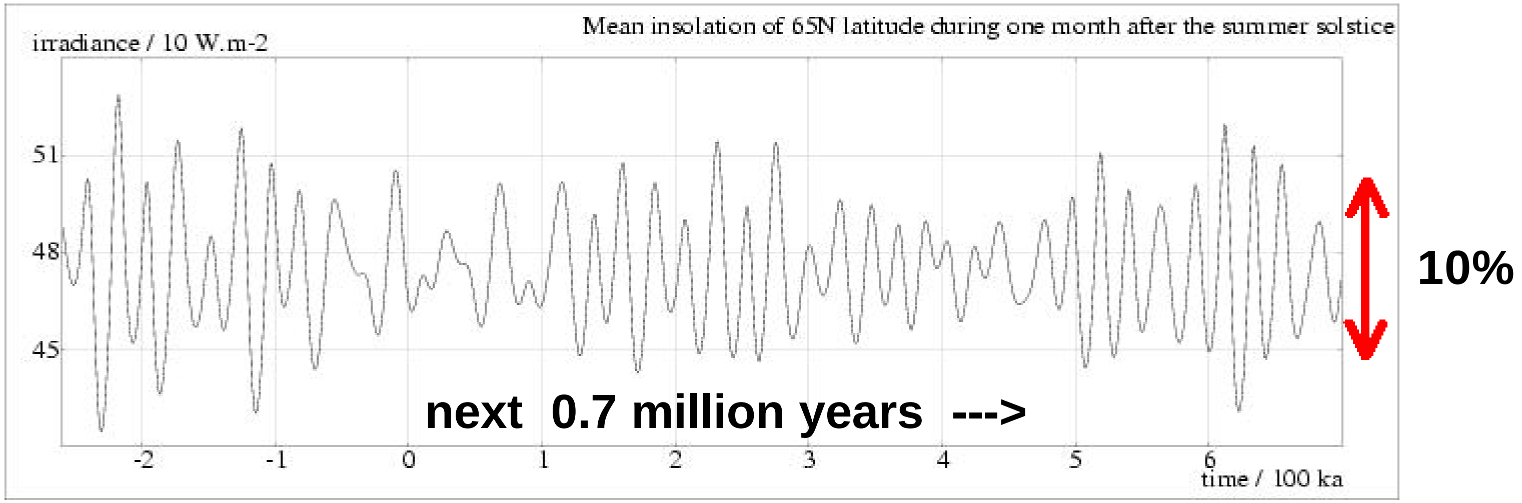
- **Developed for Solar System Planets**
  - **main potential due to the Sun**
  - **secondary potentials from other planets**
  - **slow <u>secular</u> and/or <u>periodic</u> changes to orbital elements due to orbit-averaged effect of *S***
- **Milankovitch Cycles**
  - **effects on Earth’s climate**
- **Apsidal Motion**
  - **precession of the orbit in its own plane**
  - **e.g. Mercury (43 arcsec / century)**
  - **tests General Relativity**
  - **observable in binaries (few deg / year)**
  - **tests stellar structure theory**



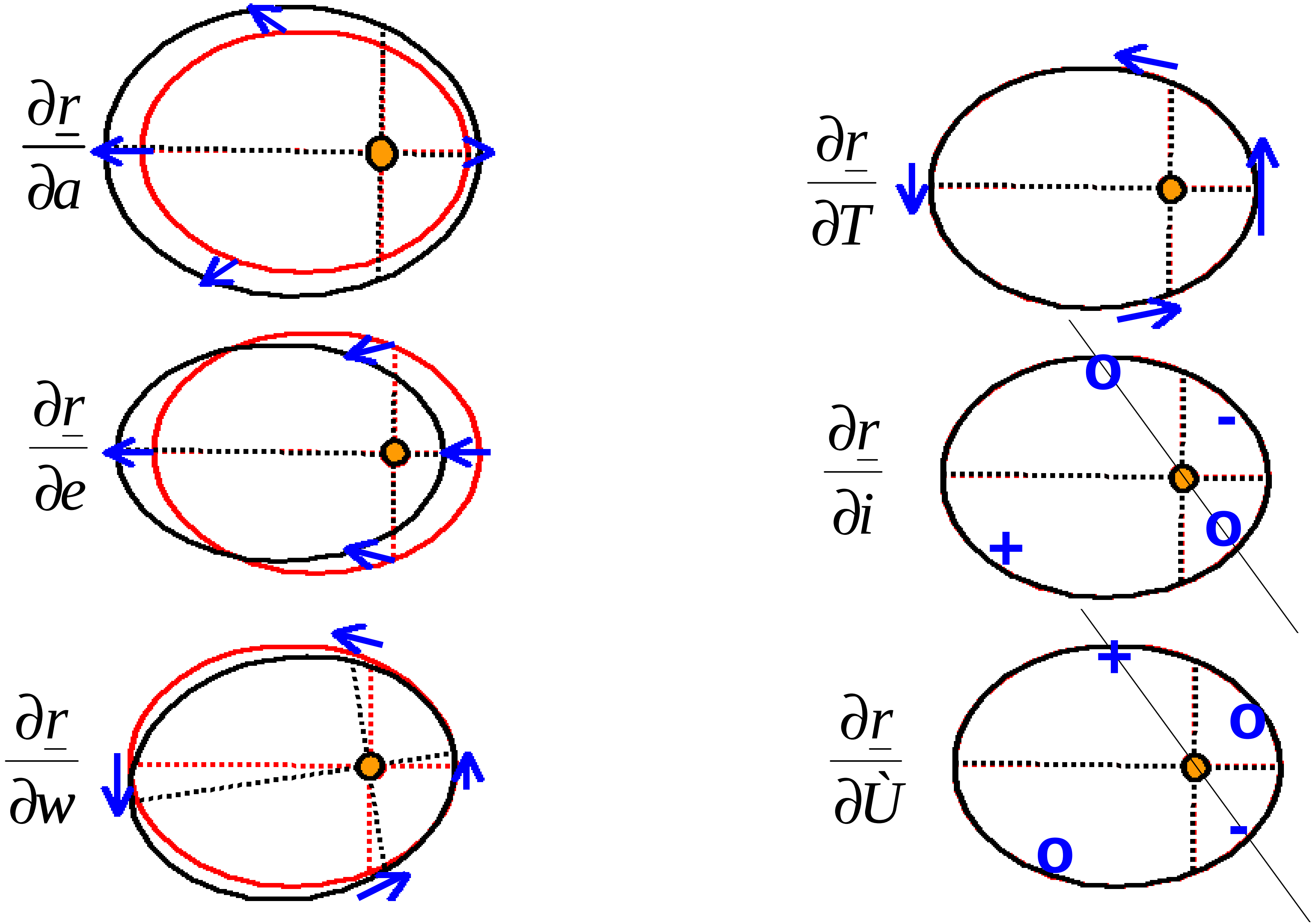
Milankovitch Cycles

- **Northern hemisphere insolation, including**
  - **rotation axis precession (23 ka), tilt (41 ka)**
  - **orbit eccentricity (100 ka)**





changes in 6 orbital elements



Lagrange's Equations

Orbit depends on time and 6 orbital elements

$$\underline{r}(t, a, e, i, \omega, \Omega, T) \qquad 4\pi^2 G M P^2 = a^3$$

$$\dot{\underline{r}} = \frac{\partial \underline{r}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i}\,\dot{e}_i \quad \text{where} \quad e_i = (a, e, i, \omega, \Omega, T)$$

$$\frac{\partial^2 \underline{r}}{\partial t^2} + \frac{d}{dt}\left[\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i}\,\dot{e}_i\right] = -\underline{\nabla}\Phi_0 - \underline{\nabla}S$$

2 - body orbit slow changes in orbital elements

$$\frac{d}{dt}\left[\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i}\,\dot{e}_i\right] \cdot \frac{\partial \underline{r}}{\partial e_k} = -\underline{\nabla}S \cdot \frac{\partial \underline{r}}{\partial e_k}$$

$$\sum_i M_{ik}\dot{e}_i = -\int \frac{\partial S}{\partial e_k}\,dt \quad \rightarrow \quad \boxed{\dot{e}_i = -\sum_k M_{ik}^{-1} \int \frac{\partial S}{\partial e_k}\,dt}$$

Lagrange’s planetary equations

$$\dot{a} = \frac{2}{n\,a}\frac{\partial S}{\partial c}; \qquad \dot{e} = \frac{1}{n\,a^2 e}\left[\,(1-e^2)\frac{\partial S}{\partial c} - \sqrt{1-e^2}\,\frac{\partial S}{\partial w}\,\right]$$

$$\dot{c} = -\frac{(1-e^2)}{n\,a^2 e}\frac{\partial S}{\partial e} - \frac{2}{n\,a}\frac{\partial S}{\partial a}; \qquad \dot{\grave{U}} = \frac{1}{n\,a^2\sqrt{1-e^2}\,\sin i}\frac{\partial S}{\partial i}$$

$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial S}{\partial e} - \frac{\cot i}{n\,a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

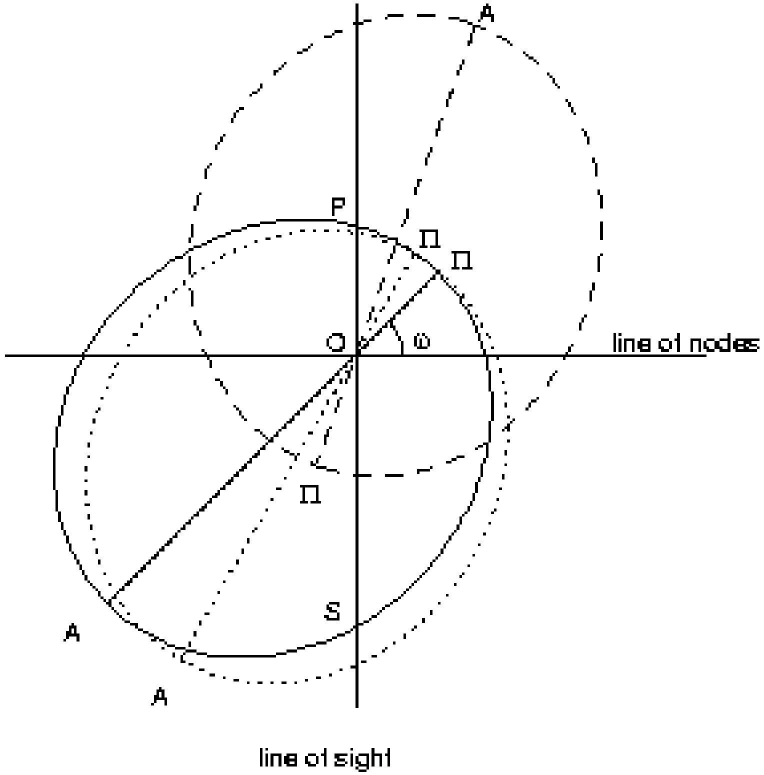
$$\dot{i} = \frac{1}{n\,a^2\sqrt{1-e^2}}\left[\,\cot i\,\frac{\partial S}{\partial w} - \mathrm{cosec}\,i\,\frac{\partial S}{\partial \grave{U}}\,\right]$$

where $n^2 a^3 = G\,M$ and $c = -n\,T$,

$n = 2p\,/\,P$ is the mean daily motion

apsidal motion

– **longitude of periastron, w , changes as the orbit precesses**



$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial S}{\partial e} - \frac{\cot i}{n\,a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

relativistic apsidal motion

- **Mercury perihelion precession**
- **binary neutron stars (pulsars)**

gravitational potential (Newton + small GR correction)

$$\Phi = -\frac{G\,M}{r}\left[\,1+\left(\frac{V}{c}\right)^2+\dots\,\right] \qquad V = L/r$$

$$L^2 = G\,M\,\ell = G\,M\,a\,(1-e^2)$$

disturbing function :

$$S = -\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^3}$$

small inward acceleration

$$-\frac{\partial S}{\partial r} = -\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^4}$$

relativistic apsidal motion

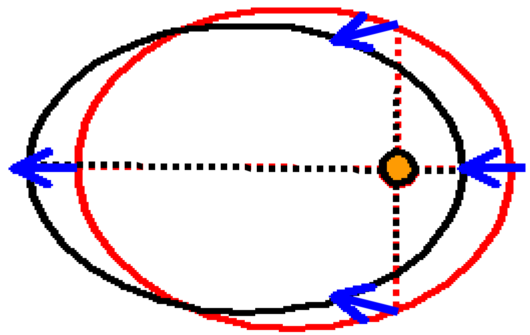
$$S = -\frac{G^2 M^2 a (1-e^2)}{c^2 r^3}$$

$$\dot{w} = \frac{\sqrt{1-e^2}}{n a^2 e} \frac{\partial S}{\partial e} - \frac{\cot i}{n a^2 \sqrt{1-e^2}} \frac{\partial S}{\partial i}$$

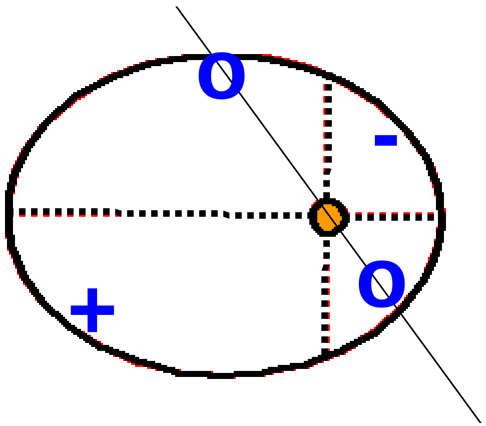
$\frac{\partial S}{\partial i} \rightarrow 0$

$$\frac{\partial S}{\partial e} \equiv \left\langle \nabla S \cdot \frac{\partial \underline{r}}{\partial e} \right\rangle$$

$$\frac{\partial \underline{r}}{\partial e}$$



$$\frac{\partial \underline{r}}{\partial i}$$



orbit average of disturbing function :

$$\langle S \rangle = -\left\langle \frac{G^2 M^2 a(1-e^2)}{c^2 r^3} \right\rangle = -\left(\frac{GM}{ca}\right)^2 \frac{1}{(1-e^2)^{1/2}}$$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{P}\int_0^P \frac{dt}{r^3} = \frac{1}{P}\int_0^{2\pi} \frac{1}{r}\frac{d\theta}{r^2\dot{\theta}}$$

$$= \frac{1}{P}\int_0^{2\pi} \frac{(1+e\cos\theta)}{\ell}\frac{d\theta}{L}$$

$$= \frac{2\pi}{P\,\ell\,L}$$

$$= \frac{1}{a^3\left(1-e^2\right)^{3/2}}$$

$$r^2\dot{\theta} = L = \sqrt{G M \ell}$$

$$r = \frac{\ell}{1+e\cos\theta}$$

$$\ell = a\left(1-e^2\right)$$

$$\frac{2\pi}{P\,L} = \sqrt{\frac{GM}{a^3}}\frac{1}{\sqrt{G M \ell}}$$

$$= \frac{1}{a^2\sqrt{1-e^2}}$$

An incorrect calculation

apsidal motion :

$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial S}{\partial e}$$

$$= \frac{G^2\,M^2}{n\,c^2\,a^4\left(1-e^2\right)}$$

$$= \frac{2p\,G\,M}{P\,c^2\,a\left(1-e^2\right)}$$

Factor of 3 too small :{

disturbing function :

$$\langle S\rangle = -\left\langle \frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^3}\right\rangle$$

$$= -\left(\frac{G\,M}{c\,a}\right)^2 \frac{1}{(1-e^2)^{1/2}}$$

$$\frac{\partial}{\partial e}\langle S\rangle = \left(\frac{G\,M}{c\,a}\right)^2 \frac{e}{(1-e^2)^{3/2}}$$

wrong because

$$\frac{\partial}{\partial e}\langle S\rangle \neq \left\langle \underline{\nabla} S \cdot \frac{\partial \underline{r}}{\partial e}\right\rangle$$

Ron's derivation

Why this power ?

$$\frac{\partial r}{\partial e} = \frac{\frac{\partial}{\partial e}\left\langle r^{-3}\right\rangle}{\frac{\partial}{\partial r}\left\langle r^{-3}\right\rangle} = \frac{3\,a^3\,e\,(1-e^2)^{-5/2}}{a^3\,(-3\,r^{-4})}$$

$$\frac{\partial S}{\partial e} = \underline{\nabla} S \cdot \frac{\partial \underline{r}}{\partial e} = \frac{\partial S}{\partial r}\,\frac{\partial r}{\partial e}$$

$$= \frac{-3\,G^2\,M^2\,a\,(1-e^2)}{c^2\,r^4} \times \frac{-e\,r^4}{a^3\,(1-e^2)^{5/2}}$$

$$= \left(\frac{G\,M}{c\,a}\right)^2 \frac{3\,e}{(1-e^2)^{3/2}}$$

- **put into equation for apsidal motion**

Relativistic apsidal motion

$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial S}{\partial e}$$

with $\quad \dfrac{\partial S}{\partial e} = \left(\dfrac{G\,M}{c\,a}\right)^2 \dfrac{3\,e}{(1-e^2\,)^{3/2}}$

and $\quad n^2 a^3 = G\,M$

gives $\quad \boxed{\dot{w} = \dfrac{6p\;G\,M}{P\,c^2 a\,(1-e^2\,)}}$

Mass determinations

- **combine apsidal motion and mass function to determine the total mass of the system**
- **measure $d\omega / dt$ over many years**
- **measure $f(m_2)$, and $a_1 \sin(i)$ from pulsar timing**

$$\dot{\omega} = \frac{6\pi \; G \; M}{P \; c^2 a \; (1 - e^2 \;)}$$

– **since a / M = a_1 / m_2,**

$$\dot{\omega} = \frac{6\pi \; G \; (\; m_2 \sin i \;)}{P \; c^2 (1 - e^2 \;)(\; a_1 \sin i \;)}$$

– **solve for $m_2 \sin(i)$, then use $f(m_2) = m_2^3 \sin^3(i) / M^2$**

$$\therefore \quad M = \frac{\left(\; m_2 \sin i \; \right)^{3/2}}{\left(\; f(m_2) \; \right)^{1/2}}$$

Aspherical stars and apsidal motion

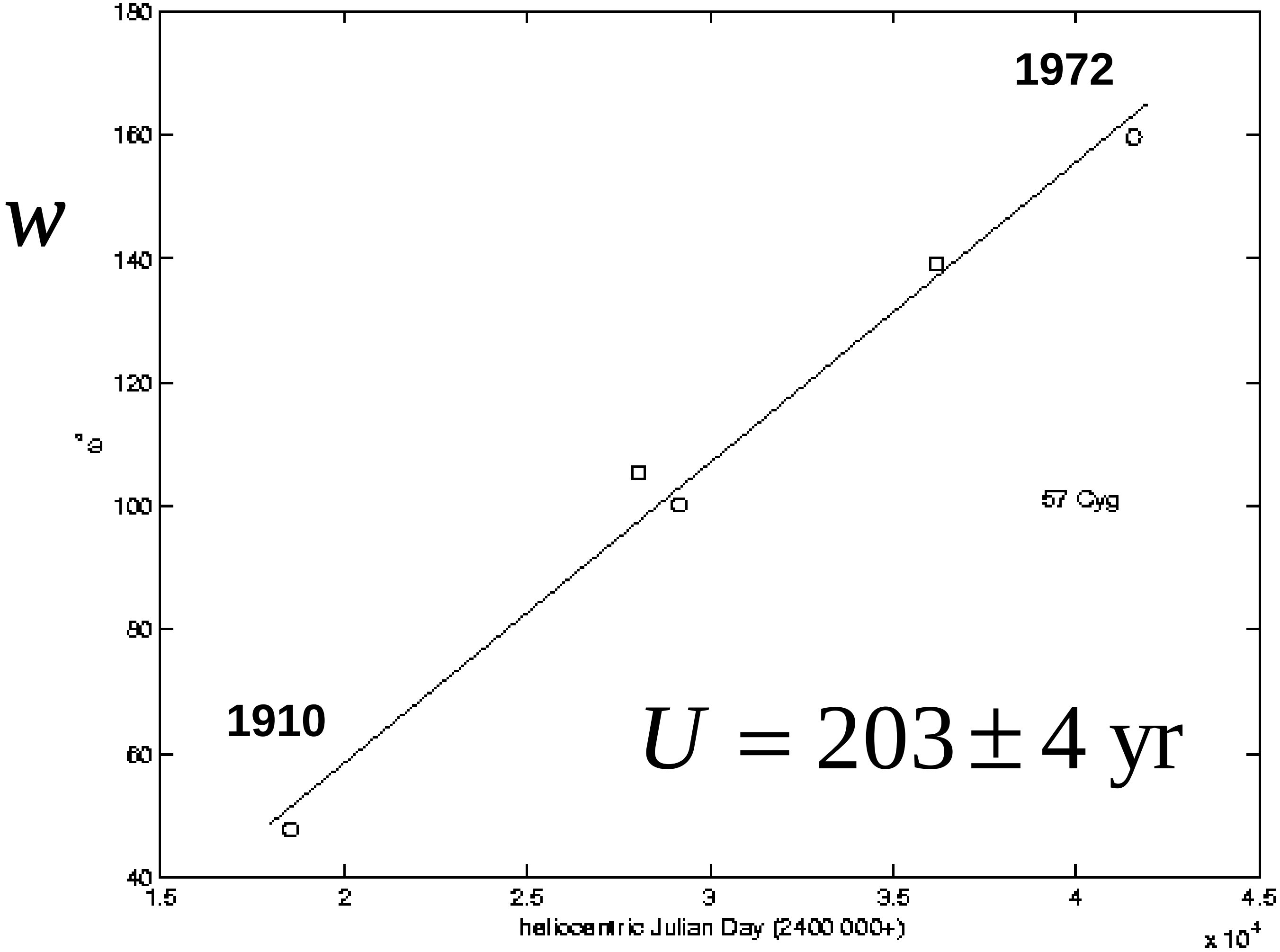
- **For non-spherical stars**
  - **potential not that of 2 point masses**

- **observe w(t) from radial velocity curves**
- **or from eclipses**
  - **changing time of eclipse minima**

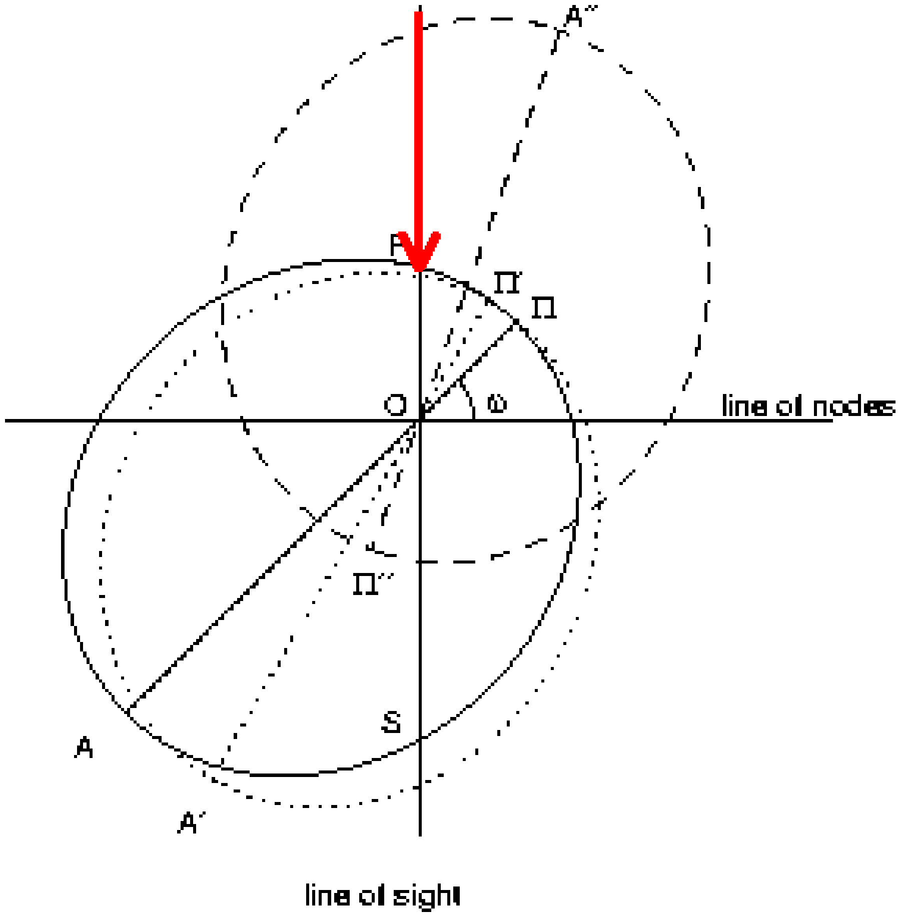
- **Relate apsidal period to orbital period, *U/P***
  - **see handout**
  - **measureable for *U* ~100s of years**

- **tell us about internal structure of stars**
  - **need gravitational potential of non-spherical bodies**

57 Cyg - radial velocity orbit



eclipse times for eccentric orbit



eclipses (for $i = 90^\circ$) at

$$q + w = 90^\circ, 270^\circ$$

eclipses at minima of

$$d(q) = \frac{1-e^2}{1+e\cos q}\sqrt{1-\sin^2 i \sin^2(q+w)}$$

minimum projected separation between stars

eclipse times with apsidal motion

advance of periastron :

$$w(t) = w_0 + \dot{w}\ (t - t_0)$$

eclipse times :

$$t_n - t_0 = n\ P + \frac{P}{2p}\left(w_0 - w_n \right) + \frac{e\ P}{p}\left(\cos w_0 - \cos w_n \right) + \ldots$$

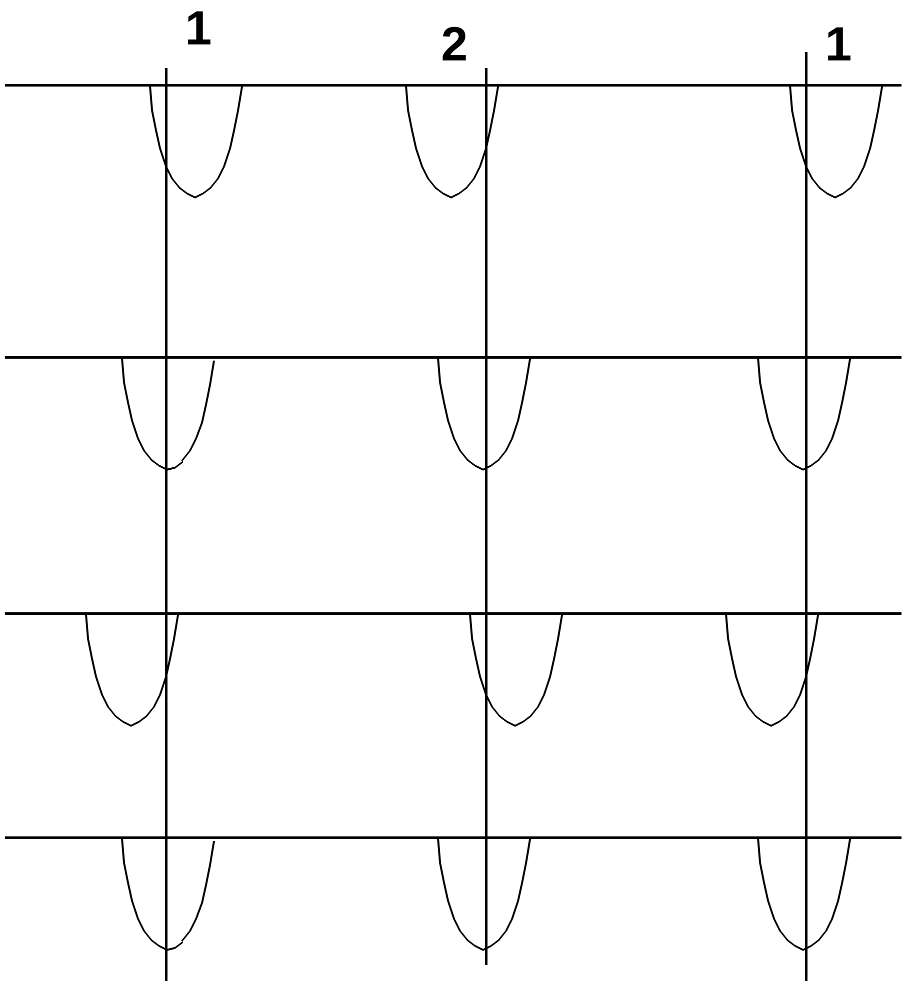
apparent period :

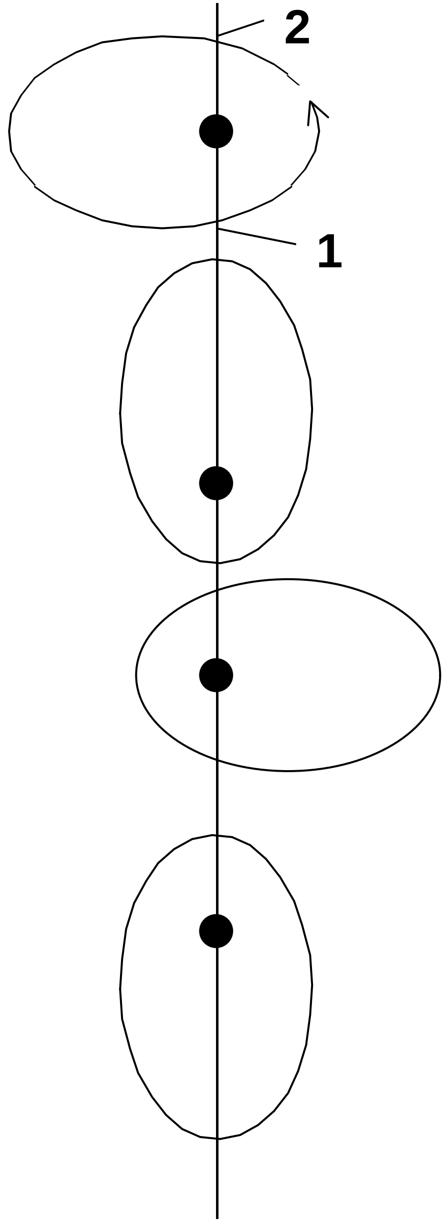
$$P_{sid} \rightarrow P\left(1 - \frac{\dot{w}}{2p}\right)$$

eclipse phases :

$$\frac{t_n - t_0}{P_{sid}} = n - \frac{e}{p}\left(\cos w - \cos w_0 \right) + ..$$

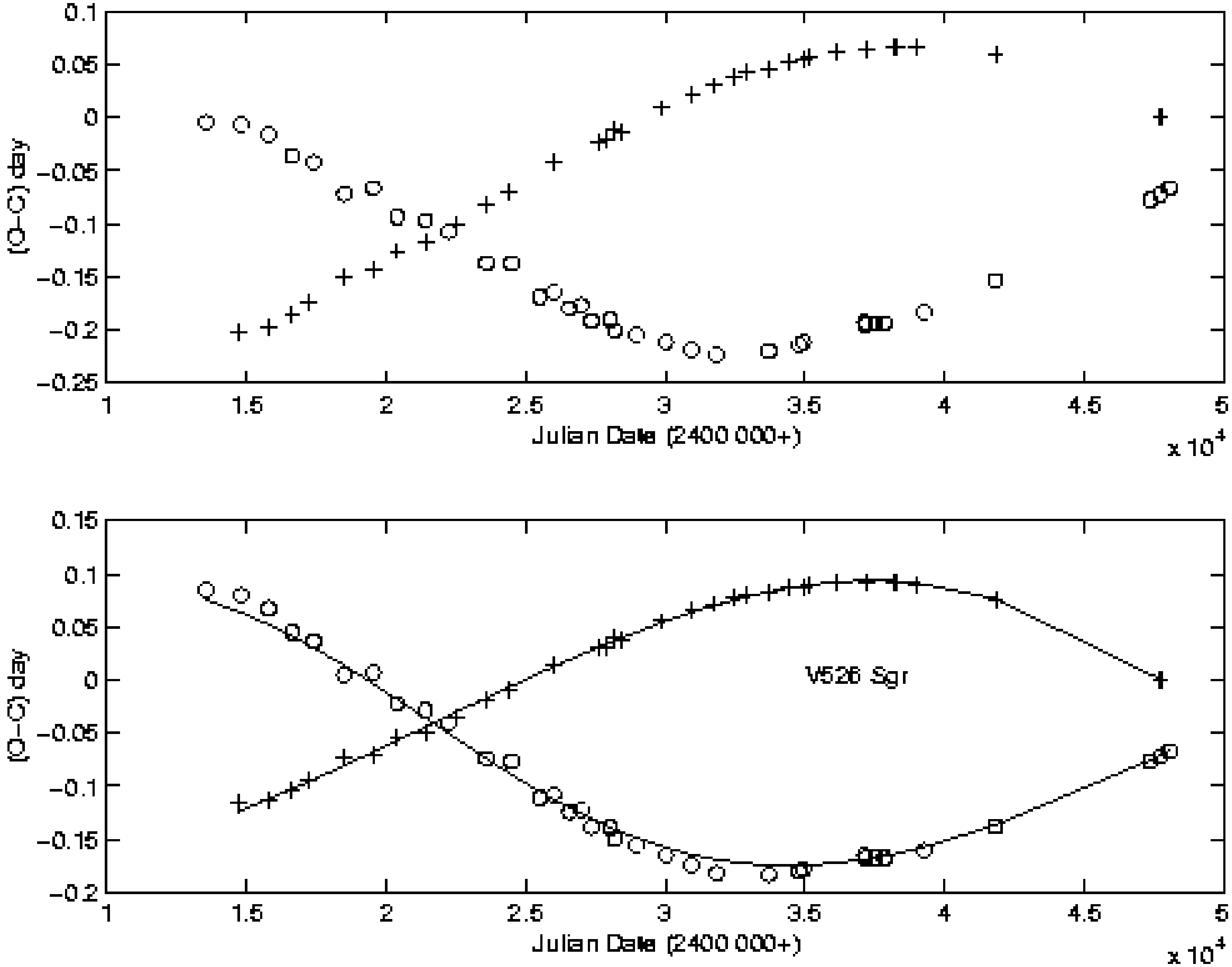
nodding eclipse times





$$\text{eclipse phases}: \quad \frac{t_n - t_0}{P_{sid}} = n - \frac{e}{p}\left(\cos w - \cos w_0 \right) + \ldots.$$

V526 Sgr - eclipse times



Internal Structure of Stars

$$U = \frac{2p}{\dot{w}} = \text{apsidal period}$$

$$\frac{P}{U} = k_1 \left(\frac{R_1}{a} \right)^5 \left(\frac{m_2}{m_1} \left\{ 15 \; f(e) + g(e) \right\} + g(e) \right)$$

$$+ k_2 \left(\frac{R_2}{a} \right)^5 \left(\frac{m_1}{m_2} \left\{ 15 \; f(e) + g(e) \right\} + g(e) \right)$$

$$f(e) = 1 + \frac{13}{2} e^2 + \frac{181}{8} e^4 + \ldots$$

$$g(e) = \left(1 - e^2 \right)^{-2}$$

$$R_1, R_2 = \text{radii of stars}$$

$$k_1, k_2 = \text{apsidal constants -- internal structure -- of stars}$$