Perturbed Two-Body motion

- **"close" to 2-body motion**
  - planetary systems
  - multi-star systems (triples, quadruples, ...)
  - close binaries (non-spherical stars)
  - general relativity corrections (Mercury, binary pulsars)
- **Perturbed potential**

$$Ö = Ö_0 + S$$

point mass potential — small corrections

the ***disturbing function***

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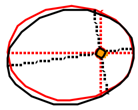
Lagrange's Planetary Equations

- **Developed for Solar System Planets**
  - main potential due to the Sun
  - secondary potentials from other planets
  - slow <u>secular</u> and/or <u>periodic</u> changes to orbital elements due to orbit-averaged effect of *S*
- **Milankovitch Cycles**
  - effects on Earth's climate
- **Apsidal Motion**
  - precession of the orbit in its own plane
  - e.g. Mercury (43 arcsec / century)
  - tests General Relativity
  - observable in binaries (few deg / year)
  - tests stellar structure theory



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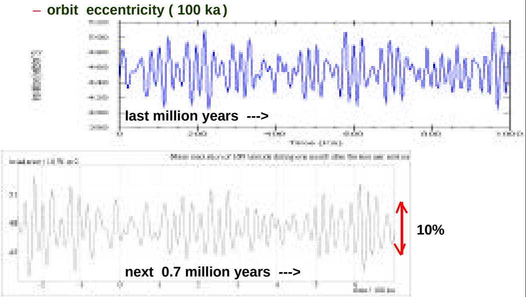
Milankovitch Cycles

- **Northern hemisphere insolation, including**
  - rotation axis precession (23 ka), tilt (41 ka)
  - orbit eccentricity (100 ka)



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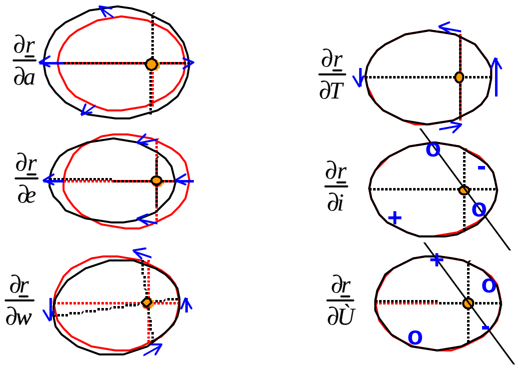
changes in 6 orbital elements



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Lagrange's Equations

Orbit depends on time and 6 orbital elements

$$\underline{r}(\ t, a, e, i, ù\ , Ù, T\ ) \qquad 4p^2 G\ M\ \ P^2 = a^3$$

$$\dot{\underline{r}} = \frac{\partial \underline{r}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i} \dot{e}_i \quad \text{where} \quad e_i = (\ a, e, i, ù\ , Ù, T\ )$$

$$\frac{\partial^2 \underline{r}}{\partial t^2} + \frac{d}{dt}\left[\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i} \dot{e}_i\right] = -\underline{\nabla}\Phi_0 - \underline{\nabla} S$$

2 - body orbit — slow changes in orbital elements

$$\frac{d}{dt}\left[\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i} \dot{e}_i\right] \cdot \frac{\partial \underline{r}}{\partial e_k} = -\underline{\nabla} S \cdot \frac{\partial \underline{r}}{\partial e_k}$$

$$\sum_i M_{ik} \dot{e}_i = -\int \frac{\partial S}{\partial e_k} dt \quad \rightarrow \quad \boxed{\dot{e}_i = -\sum_k M_{ik}^{-1} \int \frac{\partial S}{\partial e_k} dt}$$

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Lagrange's planetary equations

$$\dot{a} = \frac{2}{n\,a} \frac{\partial S}{\partial \mathbf{c}}; \qquad \dot{e} = \frac{1}{n\,a^2 e}\left[(1-e^2) \frac{\partial S}{\partial \mathbf{c}} - \sqrt{1-e^2} \frac{\partial S}{\partial \mathbf{w}} \right]$$

$$\dot{\mathbf{c}} = -\frac{(1-e^2)}{n\,a^2 e} \frac{\partial S}{\partial e} - \frac{2}{n\,a} \frac{\partial S}{\partial a}; \qquad \dot{Ù} = \frac{1}{n\,a^2 \sqrt{1-e^2}\,\sin i} \frac{\partial S}{\partial i}$$

$$\dot{\mathbf{w}} = \frac{\sqrt{1-e^2}}{n\,a^2\,e} \frac{\partial S}{\partial e} - \frac{\cot i}{n\,a^2 \sqrt{1-e^2}} \frac{\partial S}{\partial i}$$

$$\dot{i} = \frac{1}{n\,a^2 \sqrt{1-e^2}}\left[\cot i \frac{\partial S}{\partial \mathbf{w}} - \mathrm{cosec}\, i \frac{\partial S}{\partial Ù} \right]$$

where $n^2 a^3 = G\,M$ and $\mathbf{c} = -n\,T$,

$n = 2\mathbf{p}/P$ is the mean daily motion

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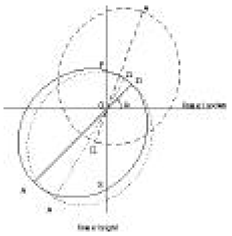
apsidal motion

– longitude of periastron, ϖ, changes as the orbit precesses



$$\dot{\varpi} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial S}{\partial e} - \frac{\cot\, i}{n\,a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

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relativistic apsidal motion

- **Mercury perihelion precession**
- **binary neutron stars (pulsars)**

gravitational potential (Newton + small GR correction)

$$\Phi = -\frac{G\,M}{r}\left[1+\left(\frac{V}{c}\right)^2+\ldots\right] \qquad V = L/r \qquad L^2 = G\,M\,\ell = G\,M\,a\,(1-e^2)$$

disturbing function:

$$S = -\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^3}$$

small inward acceleration

$$-\frac{\partial S}{\partial r} = -\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^4}$$

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relativistic apsidal motion

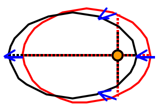
$$S = -\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^3}$$

$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial S}{\partial e} - \frac{\cot\, i}{n\,a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

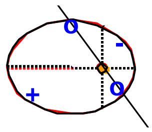
(second term crossed out → 0)

$$\frac{\partial S}{\partial e} \equiv \left\langle \nabla S \cdot \frac{\partial r}{\partial e}\right\rangle$$

$\dfrac{\partial r}{\partial e}$



$\dfrac{\partial r}{\partial i}$



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orbit average of disturbing function:

$$\langle S\rangle = -\left\langle\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^3}\right\rangle = -\left(\frac{G\,M}{c\,a}\right)^2\frac{1}{(1-e^2)^{1/2}}$$

$$\left\langle\frac{1}{r^3}\right\rangle = \frac{1}{P}\int_0^P\frac{dt}{r^3} = \frac{1}{P}\int_0^{2\pi}\frac{1}{r}\frac{d\theta}{r^2\dot{\theta}}$$

$$= \frac{1}{P}\int_0^{2\pi}\frac{(1+e\cos\theta)}{\ell}\frac{d\theta}{L}$$

$$= \frac{2\pi}{P\,\ell\,L}$$

$$= \frac{1}{a^3(1-e^2)^{3/2}}$$

$$r^2\dot{\theta} = L = \sqrt{G\,M\,\ell}$$

$$r = \frac{\ell}{1+e\cos\theta}$$

$$\ell = a\,(1-e^2)$$

$$\frac{2\pi}{P\,L} = \sqrt{\frac{G\,M}{a^3}}\frac{1}{\sqrt{G\,M\,\ell}}$$

$$= \frac{1}{a^2\sqrt{1-e^2}}$$

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An incorrect calculation

apsidal motion:

$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial S}{\partial e}$$

$$= \frac{G^2\,M^2}{n\,c^2\,a^4\,(1-e^2)}$$

$$= \frac{2\pi\,G\,M}{P\,c^2\,a\,(1-e^2)}$$

Factor of 3 too small :{

disturbing function:

$$\langle S\rangle = -\left\langle\frac{G^2\,M^2\,a\,(1-e^2)}{c^2\,r^3}\right\rangle$$

$$= -\left(\frac{G\,M}{c\,a}\right)^2\frac{1}{(1-e^2)^{1/2}}$$

$$\frac{\partial}{\partial e}\langle S\rangle = \left(\frac{G\,M}{c\,a}\right)^2\frac{e}{(1-e^2)^{3/2}}$$

wrong because

$$\frac{\partial}{\partial e}\langle S\rangle \neq \left\langle\nabla S\cdot\frac{\partial r}{\partial e}\right\rangle$$

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Ron's derivation **Why this power ?**

$$\frac{\partial r}{\partial e} = \frac{\frac{\partial}{\partial e}\langle r^{-3}\rangle}{\frac{\partial}{\partial r}\langle r^{-3}\rangle} = \frac{3a^3\,e\,(1-e^2)^{-5/2}}{a^3\,(-3r^{-4})}$$

$$\frac{\partial S}{\partial e} = \nabla S\cdot\frac{\partial r}{\partial e} = \frac{\partial S}{\partial r}\frac{\partial r}{\partial e}$$

$$= \frac{-3G^2\,M^2\,a\,(1-e^2)}{c^2\,r^4}\times\frac{-e\,r^4}{a^3\,(1-e^2)^{5/2}}$$

$$= \left(\frac{G\,M}{c\,a}\right)^2\frac{3e}{(1-e^2)^{3/2}}$$

- **put into equation for apsidal motion**

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Relativistic apsidal motion

$$\dot{w} = \frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial S}{\partial e}$$

with $$\frac{\partial S}{\partial e} = \left(\frac{G\,M}{c\,a}\right)^2 \frac{3\,e}{(1-e^2)^{3/2}}$$

and $n^2 a^3 = G\,M$

gives $$\boxed{\dot{w} = \frac{6p\;G\,M}{P\,c^2 a\,(1-e^2)}}$$

Mass determinations

- **combine apsidal motion and mass function to determine the total mass of the system**
- **measure dw/dt over many years**
- **measure $f(m_2)$, and $a_1 \sin(i)$ from pulsar timing**

$$\dot{w} = \frac{6p\;G\;M}{P\,c^2 a\,(1-e^2)}$$

- since $a / M = a_1 / m_2$

$$\dot{w} = \frac{6p\;G\,(m_2 \sin i)}{P\,c^2\,(1-e^2)(a_1 \sin i)}$$

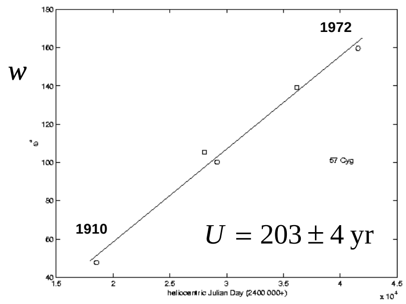
- solve for $m_2 \sin(i)$, then use $f(m_2) = m_2^3 \sin^3(i) / M^2$

$$\therefore\; M = \frac{(m_2 \sin i)^{3/2}}{(f(m_2))^{1/2}}$$

Aspherical stars and apsidal motion

- **For non-spherical stars**
  - potential not that of 2 point masses
- **observe $w(t)$ from radial velocity curves**
- **or from eclipses**
  - changing time of eclipse minima
- **Relate apsidal period to orbital period, *U/P***
  - see handout
  - measureable for *U* ~100s of years
- **tell us about internal structure of stars**
  - need gravitational potential of non-spherical bodies

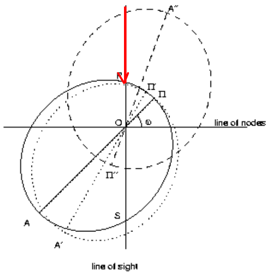
57 Cyg - radial velocity orbit



eclipse times for eccentric orbit

eclipses (for $i = 90^\circ$) at

$q + w = 90^\circ, 270^\circ$



eclipses at minima of

$$d(q) = \frac{1-e^2}{1+e\cos q}\sqrt{1-\sin^2 i\,\sin^2(q+w)}$$

minimum projected separation between stars

eclipse times with apsidal motion

advance of periastron :

$w(t) = w_0 + \dot{w}\,(t - t_0)$

eclipse times :

$$t_n - t_0 = n\,P + \frac{P}{2p}(w_0 - w_n) + \frac{e\,P}{p}(\cos w_0 - \cos w_n) + ...$$

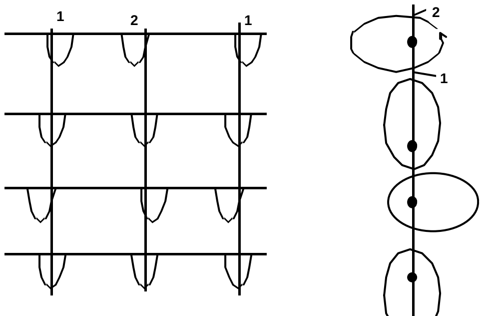
apparent period :

$$P_{sid} \rightarrow P\left(1 - \frac{\dot{w}}{2p}\right)$$

eclipse phases :

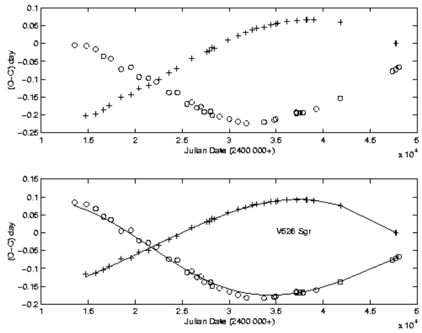
$$\frac{t_n - t_0}{P_{sid}} = n - \frac{e}{p}(\cos w - \cos w_0) + ..$$

nodding eclipse times



eclipse phases: $\frac{t_n - t_0}{P_{sid}} = n - \frac{e}{p}\left(\cos w - \cos w_0 \right) + \ldots$

V526 Sgr - eclipse times



Internal Structure of Stars

$$U = \frac{2p}{\dot{w}} = \text{apsidal period}$$

$$\frac{P}{U} = k_1\left(\frac{R_1}{a}\right)^5 \left(\frac{m_2}{m_1}\{15 f(e) + g(e)\} + g(e) \right)$$

$$+ k_2\left(\frac{R_2}{a}\right)^5 \left(\frac{m_1}{m_2}\{15 f(e) + g(e)\} + g(e) \right)$$

$$f(e) = 1 + \frac{13}{2}e^2 + \frac{181}{8}e^4 + \ldots$$

$$g(e) = \left(1 - e^2\right)^{-2}$$

R_1, R_2 = radii of stars

k_1, k_2 = apsidal constants - internal structure - of stars