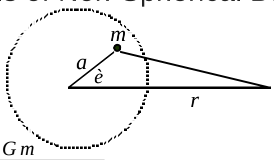
Potentials of Non-Spherical Bodies



$$\Phi = -\frac{G\,m}{\sqrt{r^2 - 2r\,a\cos\theta + a^2}}$$
$$= -\frac{G\,m}{r}\left(1 - \left[2\left(\frac{a}{r}\right)\cos\theta - \left(\frac{a}{r}\right)^2\right]\right)^{-1/2}$$
$$= -\frac{G\,m}{r}\left(1 + \left(\frac{a}{r}\right)P_1(\cos\theta) + \left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \ldots\right)$$

Legendre Polynomials

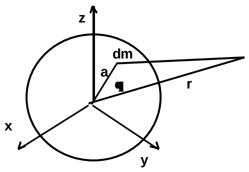
– P_n is the coefficient of $(a / r)^n = x^n$ in the expansion

let $u = \cos\theta \qquad x = a / r$

$$\left(1 - \left[2\cos\theta\left(\frac{a}{r}\right) - \left(\frac{a}{r}\right)^2\right]\right)^{-1/2} = \left(1 - \left[2\,u\,x - x^2\right]\right)^{-1/2}$$
$$= 1 + \frac{1}{2}\left[2\,u\,x - x^2\right] + \frac{1\cdot 3}{2\cdot 4}\left[2\,u\,x - x^2\right]^2 + \ldots$$
$$= 1 + \frac{1}{2}(2\,u)\,x + \left(-\frac{1}{2} + \frac{3}{2}u^2\right)x^2 + \ldots$$

Legendre polynomials
$$\begin{cases} P_0(u) = 1 \\ P_1(u) = u \\ P_2(u) = \frac{1}{2}\left(3\,u^2 - 1\right) \\ P_3(u) = \frac{1}{2}\left(5\,u^3 - 3\,u\right) \end{cases}$$

Potential of the Earth



mass dm inside the Earth.

Potential due to entire Earth:

$$\Phi = -\frac{G}{r}\left[\int P_0\,dm + \int P_1(\cos\theta)\left(\frac{a}{r}\right)dm + \int P_2(\cos\theta)\left(\frac{a}{r}\right)^2 dm + \ldots\right]$$
$$= \Phi_0 + \Phi_1 + \Phi_2 + \ldots$$

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \ldots$$
$$\Phi_0 = -\frac{G}{r}\int dm = -\frac{G\,m}{r} = \text{point mass potential}$$
$$\Phi_1 = -\frac{G}{r}\int\left(\frac{a}{r}\right)\cos\theta\;dm = 0$$
$$\Phi_2 = -\frac{G}{2r}\int\left(\frac{a}{r}\right)^2\left(3\cos^2\theta - 1\right)dm$$

- **Centre of mass** $\rightarrow \Phi_1 = 0$
- **Spherical symmetry** $\rightarrow \Phi_n = 0 \quad n > 0$
- **Ellipsoidal symmetry** $\rightarrow \Phi_n = 0 \quad n$ odd
  – e.g. 3 (possibly unequal) axes

Potential of Spheroids

- **Spheroid = rotationally symmetric body**
  – symmetric in longitude
- **External potential:**

$$\Phi(r,\theta) = -\frac{G\,m}{r}\left(1 + \sum_{n=2}^{\infty} J_n\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right)$$

Jn = dimensionless distortion coefficients
(0 for spherical symmetry)
e.g. due to rotation
measured by observation / experiment
e.g. satellite tracking / laser ranging

– a = mean equatorial radius
– θ = latitude

Acceleration in Rotating Frames e.g. on surface of rotating body

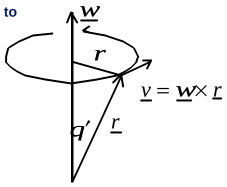
- **The inertial-frame acceleration has 3 parts,**

$$\frac{d^2\underline{r}}{d\,t^2} = \ddot{\underline{r}} + 2\,\underline{\omega}\times\dot{\underline{r}} + \underline{\omega}\times(\underline{\omega}\times\underline{r})$$

– rotating frame acc. + coriolis acc. + centripetal acc.

- $\underline{\omega}\times(\underline{\omega}\times\underline{r})$ inwards perpendicular to the rotation axis

$$\left|\underline{\omega}\times(\underline{\omega}\times\underline{r})\right| = \omega^2\;r\sin\theta'$$
$$\left(\left|\underline{v}\right| = \omega\,r, \quad r = r\sin\theta'\right);$$
$$\theta' = 90^\circ - \theta = \text{co-latitude}$$



Rotational Potential shapes of rotating bodies

- The centripetal acceleration $\omega^2 r$ corresponds to a rotational potential

$$\Phi_{rot} = -\tfrac{1}{2}\omega^2 r^2 = -\tfrac{1}{2}\omega^2 r^2 \sin^2\theta$$

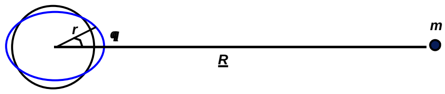
  - the total potential is

$$\Phi_{total} = \Phi_{grav} + \Phi_{rot}$$

  - A rotating fluid body's equilibrium shape is an equi-potential surface
    - called the GEOID for the Earth
    - denotes true sea-level
  - Earth $J_2 = 1.082 \times 10^{-3}$,
  - higher terms $J_n \sim J_2^2$

Tides

- gravitational effect of Moon (and Sun)
  - non-uniform over surface of the Earth



  - Gravitational potential due to moon at point (r, θ) on Earth

$$\Phi_M(r,\theta) = -\frac{G\,m}{|\underline{r}-\underline{R}|} \qquad m = \text{mass of Moon}$$

  - since $r \ll R$, expand potential in powers of (r / R)

Tidal Potential

$$\Phi_M = -\frac{G\,m}{R}\left[1 + \frac{r\cos\theta}{R} + \frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \dots\right]$$

$$= -\frac{G\,m}{R}\left[1 + \frac{x}{R} + \frac{3x^2 - r^2}{2R^2} + \dots\right] \quad (x = r\cos\theta)$$

$$= \Phi_0 + \Phi_1 + \Phi_2 + \dots$$

$$\frac{d}{dx}(\Phi_0 + \Phi_1) = -\frac{G\,m}{R^2} = \text{uniform acceleration in } x$$

moves Earth around Moon-Earth centre of mass.

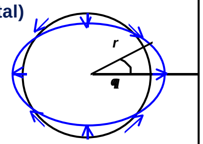
$$\text{Tidal potential: } \Phi_2 = -\frac{G\,m\,r^2}{2R^3}\left(3\cos^2\theta - 1\right)$$

Tidal Accelerations

- Tidal potential

$$\Phi_2 = -\frac{G\,m\,r^2}{2R^3}\left(3\cos^2\theta - 1\right)$$

- acceleration (vertical and horizontal)



$$g_r = -\frac{\partial\Phi}{\partial r} = \frac{G\,m\,r}{R^3}\left(3\cos^2\theta - 1\right)$$

$$g_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{3\,G\,m\,r}{R^3}\cos\theta\,\sin\theta$$

  - g_r = +2 at 0 ,180 deg , -1 at 90 deg
  - (toward and away from moon) 2 tides per day

- $g_{tidal} \ll g_{grav}$ $\left(\frac{G\,m\,r}{R^3}\right)\Big/\left(\frac{G\,M}{r^2}\right) = \frac{m\,r^3}{M\,R^3} \approx \frac{5.7}{2.6}\times 10^{-8}$ $\frac{\text{Moon}}{\text{Sun}}$

Tides in Close Binaries

- In close binary systems, $R \le 10\,r$
  - tides much stronger than Earth-Moon-Sun

- Potential on surface of star 1

$$\Phi = \Phi_{grav} + \Phi_{rot} + \Phi_{tide}$$

$$\Phi_{grav} = -\frac{G\,m_1}{r} - \frac{J_2\,P_2(\cos\theta)}{r^3}$$ — internal structure (drives apsidal motion)

$$\Phi_{rot} = -\tfrac{1}{2}\omega_1^2 r^2 \sin^2\theta'$$ — star 1 rotates

$$\Phi_{tide} = -\frac{G\,m_2}{R^3} r^2\,P_2(\cos\theta)$$ — tide from star 2 at distance R

- Symmetric expression for star 2

Tidal Synchronisation and Circularisation of Orbits

- Theory by Zahn 1975 - 92
- The tidal potentials create tidal torques
  - transfer energy and angular momentum
    - from the stars (rotation) to the orbit
- until rotation and orbital periods match
  - synchronous rotation
- also circularises the orbit

Two parts of tide

- **equilibrium tides** -
  - late-type stars (convective envelopes)
  - hydrostatic adjustment of star's structure
  - friction (turbulence, convection) dissipates energy
  - friction timescale ~1 yr

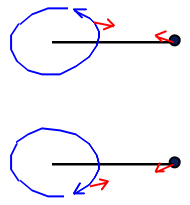
$$t_f = \left(\frac{m\,R^2}{L} \right)^{1/3}$$

- **dynamical tides**
  - early-type stars (radiative envelopes)
  - free modes of oscillation- called *dynamical tides*
  - weaker than equilibrium tides
  - dissipation by radiative damping
  - dynamical timescale

$$t_d = \left(\frac{G\,m}{R^3} \right)^{1/2}$$

Tidal Synchronisation

Friction causes tidal bulge to lag behind



Earth spins faster

bulge pulls moon forward

boosts moon into larger, longer period orbit

moon pulls bulge back, slowing Earth's spin

If Earth spins slower, vice-versa

equilibrium tides

Zahn's equations

$$\frac{\dot a}{a} = \frac{k_2}{t_f}\, q\,(1+q)\left(\frac{R}{a}\right)^8 \left[\, 12\left(\frac{w}{w_K} - 1 \right) + O(e^2) \,\right] \qquad w = \frac{2p}{P_{rot}}$$

$$\frac{\dot e}{e} = \frac{k_2}{t_f}\, q\,(1+q)\left(\frac{R}{a}\right)^8 \left[\, \frac{33}{2}\left(\frac{w}{w_K} - \frac{18}{11} \right) + O(e^2) \,\right] \qquad w_K = \frac{2p}{P_{orb}}$$

$$\frac{d}{dt}(I\,w) = \frac{k_2}{t_f}\, q^2 m\,R^2 \left(\frac{R}{a}\right)^6 \left[\, 6\,(w_K - w) + O(e^2) \,\right] \qquad t_f = \left(\frac{m\,R^2}{L} \right)^{1/3}$$

timescales

$$t_{synch} = \frac{I\,w}{\frac{d}{dt}(I\,w)} \approx 10^4\ \mathrm{yr} \left(\frac{1+q}{2\,q} \right)^2 \left(\frac{P}{\mathrm{day}} \right)^4$$

$$t_{circ} \equiv \frac{e}{\dot e} \approx 10^6\,\mathrm{yr} \left(\frac{1+q}{2} \right)^{5/3} \left(\frac{P}{\mathrm{day}} \right)^{16/3}$$

P	t_{sync}	t_{circ}
day	yr	yr
1	10^4	10^6
10	10^8	10^{11}

dynamical tides

timescales

$$t_{synch} = \frac{I\,w}{\frac{d}{dt}(I\,w)} = \frac{1}{5 \times 2^{5/3}} \frac{1}{q^2 (1+q)^{5/6}} \left(\frac{R^3}{G\,m} \right)^{1/2} \left(\frac{I}{m\,R^2} \right) \frac{1}{E_2} \left(\frac{a}{R} \right)^{17/2}$$

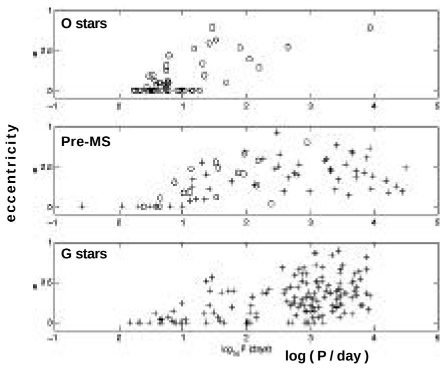
$$t_{circ} \equiv \frac{e}{\dot e} \approx \frac{2}{21} \frac{1}{q\,(1+q)^{11/6}} \left(\frac{R^3}{G\,m} \right)^{1/2} \frac{1}{E_2} \left(\frac{a}{R} \right)^{21/2}$$

example

$M = 10\,M_{sun} \quad R = 5\,R_{sun} \quad E_2 \sim 10^{-6} \quad I \sim 0.1\,m\,R^2$

$P \approx 5$ days $\quad a \approx 33\,R_{sun} \quad t_{sync} \sim 10^{6.8}$ yr $\quad t_{circ} \sim 10^{9.3}$ yr

Circularisation in Close Binaries



Circularisation of Exoplanet Orbits