Lecture 3

Metrics for Curved Geometry

Cosmological Observations in a Curved and Evolving Universe

Non-Euclidian geometries:

(positive / negative curvature)

Evolving geometries:

(expanding / accelerating / decelerating)

Time-Redshift-Distance relations

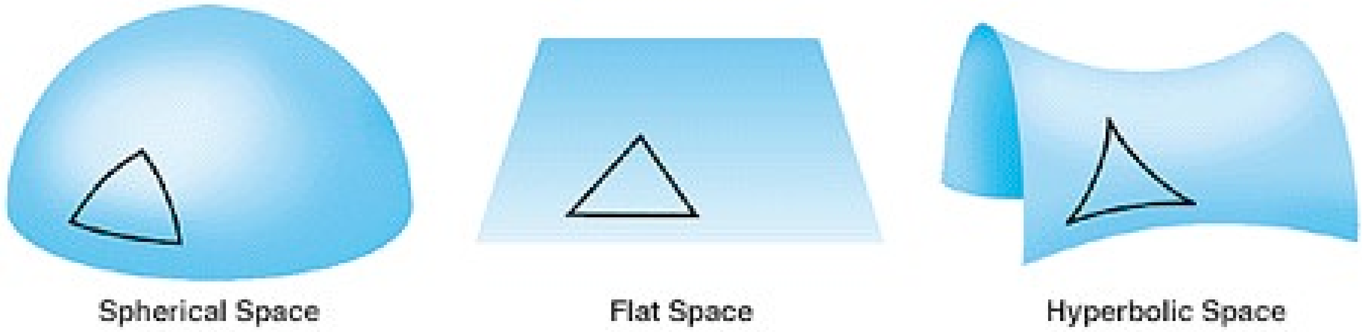
Non-Euclidean Geometry

Curved 3-D Spaces

How Does Curvature affect
Distance Measurements ?

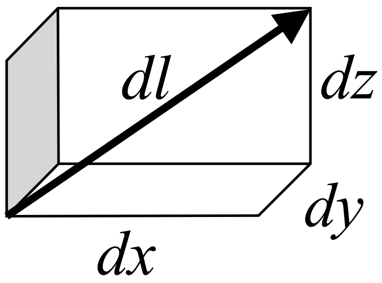
Is our Universe Curved?

	Closed	**Flat**	**Open**



	Closed	Flat	Open
Curvature:	+	0	--
Sum of angles of triangle:	$> 180^o$	$= 180^o$	$< 180^o$
Circumference of circle:	$< 2\pi r$	$= 2\pi r$	$> 2\pi r$
Parallel lines:	converge	remain parallel	diverge
Size:	finite	infinite	infinite
Edge:	no	no	no

Flat Space: Euclidean Geometry



Cartesian coordinates:

1 D: $dl^2 = dx^2$

2 D: $dl^2 = dx^2 + dy^2$

3 D: $dl^2 = dx^2 + dy^2 + dz^2$

4 D: $dl^2 = dw^2 + dx^2 + dy^2 + dz^2$

Metric tensor : coordinates -> distance

$$dl^2 = (\, dx \quad dy \quad dz \,) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Summation convention:

$$dl^2 = g_{ij}\, dx^i\, dx^j \equiv \sum_i \sum_j g_{ij}\, dx^i\, dx^j$$

Orthogonal coordinates
<--> diagonal metric

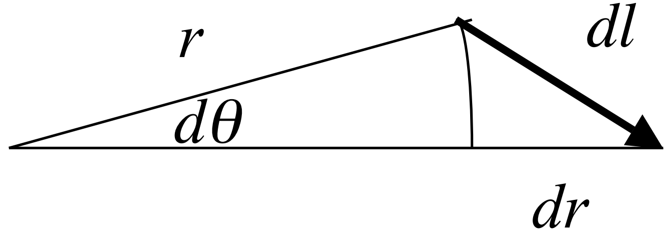
$g_{xx} = g_{yy} = g_{zz} = 1$

$g_{xy} = g_{xz} = g_{yz} = 0$

symmetric: $g_{ij} = g_{ji}$

Polar Coordinates

Radial coordinate r, angles $\phi, \theta, \alpha, ...$



1 D: $dl^2 = dr^2$

2 D: $dl^2 = dr^2 + r^2\ d\theta^2$

3 D: $dl^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\ d\phi^2 \right)$

4 D: $dl^2 = dr^2 + r^2 \left[d\theta^2 + \sin^2\theta \left(d\phi^2 + \sin^2\phi\ d\alpha^2 \right) \right]$

$dl^2 = dr^2 + r^2\ d\psi^2$ generic angle: $d\psi^2 = d\theta^2 + \sin^2\theta\ d\phi^2 + ...$

$$dl^2 = (\ dr \quad d\theta \quad d\phi\ ) \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

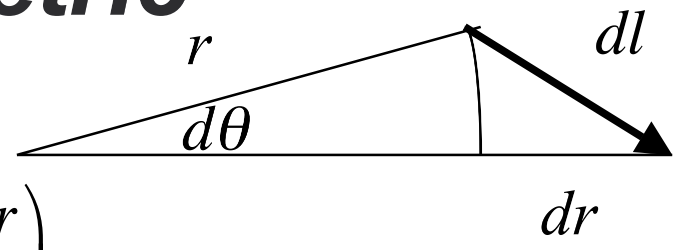
$g_{rr} = ?$ $g_{r\theta} = ?$

$g_{\theta\theta} = ?$

$g_{\phi\phi} = ?$

$g_{\alpha\alpha} = ?$

Using the Metric



$$dl^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

$$dl^2 = \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

$$dl_r = \sqrt{g_{rr}}\, dr = dr, \qquad dl_\theta = \sqrt{g_{\theta\theta}}\, d\theta = r\, d\theta, \qquad dl_\phi = ?$$

$$\text{Radial Distance}: \; D \equiv \int dl_r = \int_0^r \sqrt{g_{rr}}\, dr = \int_0^r dr = r$$

$$\text{Circumference}: C \equiv \oint dl_\theta = \int_0^{2\pi} \sqrt{g_{\theta\theta}}\, d\theta = \int_0^{2\pi} r\, d\theta = 2\pi r$$

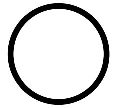
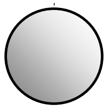
$$\text{Area}: \; A = \int dA_{r\theta} = \int dl_r \, dl_\theta = \int_0^r \int_0^{2\pi} \sqrt{g_{rr}}\, dr \sqrt{g_{\theta\theta}}\, d\theta = \int_0^r dr \int_0^{2\pi} r\, d\theta = \pi r^2$$

$$Note: \int dx\, dy = \int r\, dr\, d\theta$$

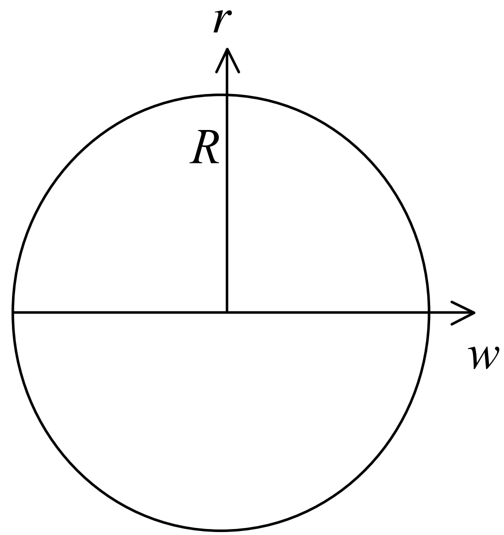
Same result using metric for any choice of coordinates.

Embedded Spheres

R = radius of curvature

1 – D:	$R^2 = x^2$	0 - D 2 points	
2 – D:	$R^2 = x^2 + y^2$	1 - D circle	
3 – D:	$R^2 = x^2 + y^2 + z^2$	2 - D surface of 3 - sphere	
4 – D:	$R^2 = x^2 + y^2 + z^2 + w^2$	3 - D surface of 4 - sphere	?

Metric for 3-D surface of 4-D sphere



$$4-\text{sphere}: \quad R^2 = x^2 + y^2 + z^2 + w^2$$

$$\text{i.e.} \quad R^2 = r^2 + w^2 \quad \text{with} \quad r^2 \equiv x^2 + y^2 + z^2 \; .$$

$$0 = 2\,r\,dr + 2\,w\,dw \quad \rightarrow \quad dw^2 = \left(\frac{r\,dr}{w}\right)^2 = \frac{r^2\,dr^2}{R^2 - r^2}$$

$$dl^2 = dw^2 + dr^2 + r^2 d\psi^2 \qquad \text{4 - space metric}$$

$$= \frac{r^2\,dr^2}{R^2 - r^2} + dr^2 + r^2 d\psi^2 \qquad \text{confined to } R^2 = r^2 + w^2$$

$$dl^2 = \frac{dr^2}{1-\left(r/R\right)^2} + r^2 d\psi^2 \qquad d\psi^2 = d\theta^2 + \sin^2\theta\; d\phi$$

Metric for a 3 - D space with constant curvature radius R

Non-Euclidean Metrics

$$k = -1, 0, +1 \quad (\text{ open, flat, closed })$$

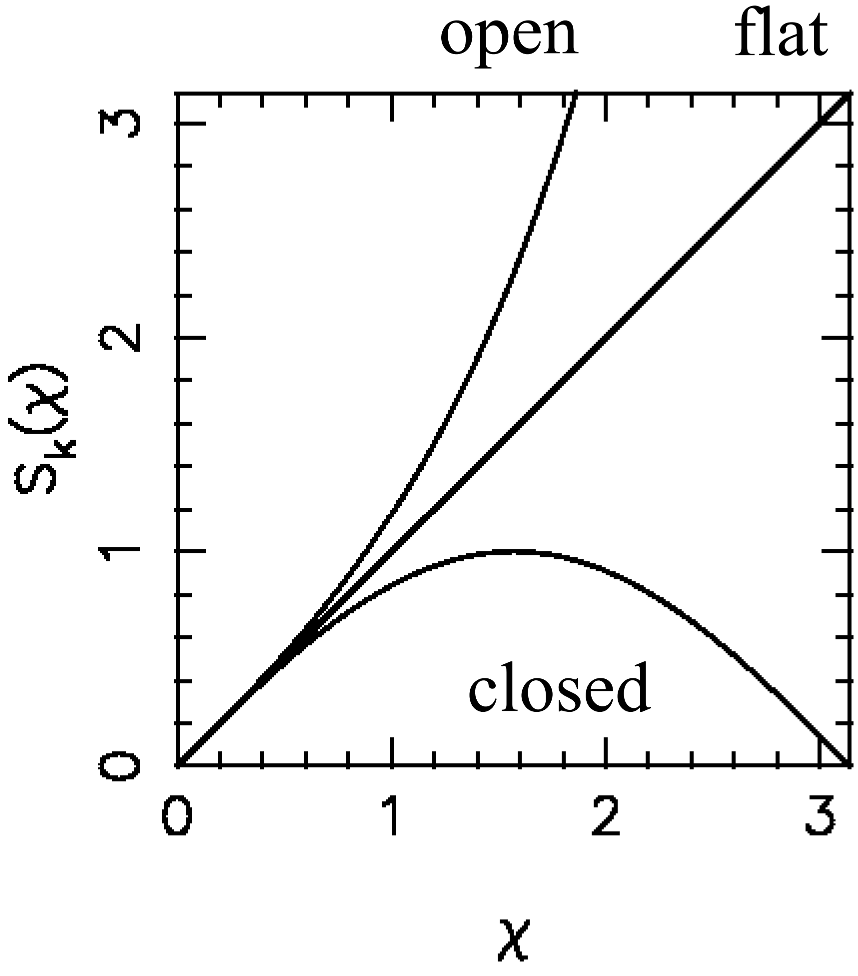
$$dl^2 = \frac{dr^2}{1 - k\left(r/R\right)^2} + r^2 d\psi^2$$

dimensionless radial coordinates :

$$u = r/R = S_k(\chi)$$

$$dl^2 = R^2 \left(\frac{du^2}{1 - k\,u^2} + u^2 d\psi^2 \right)$$

$$= R^2 \left(d\chi^2 + S_k^{\,2}(\chi)\, d\psi^2 \right)$$

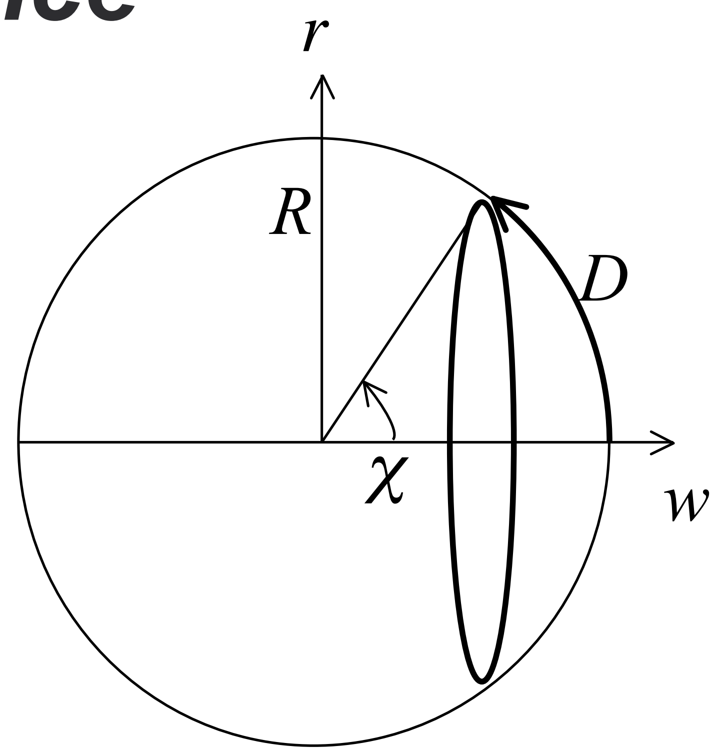


$$S_{-1}(\chi) \equiv \sinh(\chi)\,, \quad S_0(\chi) \equiv \chi\,, \quad S_{+1}(\chi) \equiv \sin(\chi)$$

Circumference

metric :

$$dl^2 = \frac{dr^2}{1 - k\left(r/R\right)^2} + r^2\ d\theta^2$$



radial distance (for $k = +1$) :

$$D = \int_0^r \frac{dr}{\sqrt{1 - k\left(r/R\right)^2}} = R \sin^{-1}\left(r/R\right)$$

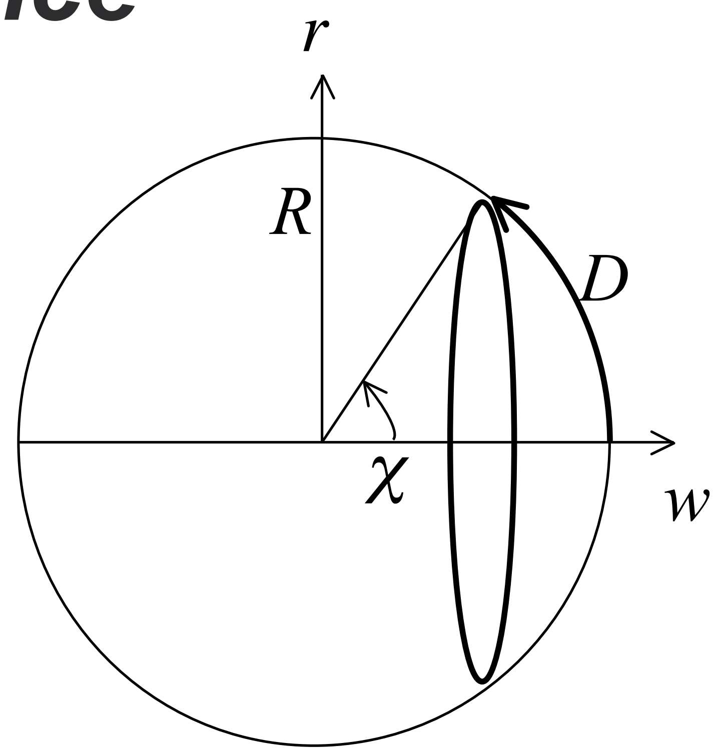
circumference :

$$C = \int_0^{2\pi} r\ d\theta = 2\pi\ r$$

"circumferencial" distance : $r \equiv \dfrac{C}{2\pi} = R\ S_k(D/R) = R\ S_k(\chi)$

If $k = +1$, coordinate r breaks down for $r > R$

Circumference



metric:

$$dl^2 = R^2 \left(d\chi^2 + S_k^2(\chi)\, d\theta^2 \right)$$

radial distance :

$$D = \int \sqrt{g_{\chi\chi}}\, d\chi = \int_0^{\chi} R\, d\chi = R\,\chi$$

circumference :

$$C = \oint \sqrt{g_{\theta\theta}}\, d\theta = \int_0^{2\pi} R\; S_k(\chi)\, d\theta = 2\pi\; R\; S_k(\chi)$$

$$= 2\pi\; D\; \frac{S_k(\chi)}{\chi}$$

Same result for any choice of coordinates.

Angular Diameter

metric :

$$dl^2 = R^2 \left(d\chi^2 + S_k^2(\chi)\, d\theta^2 \right)$$

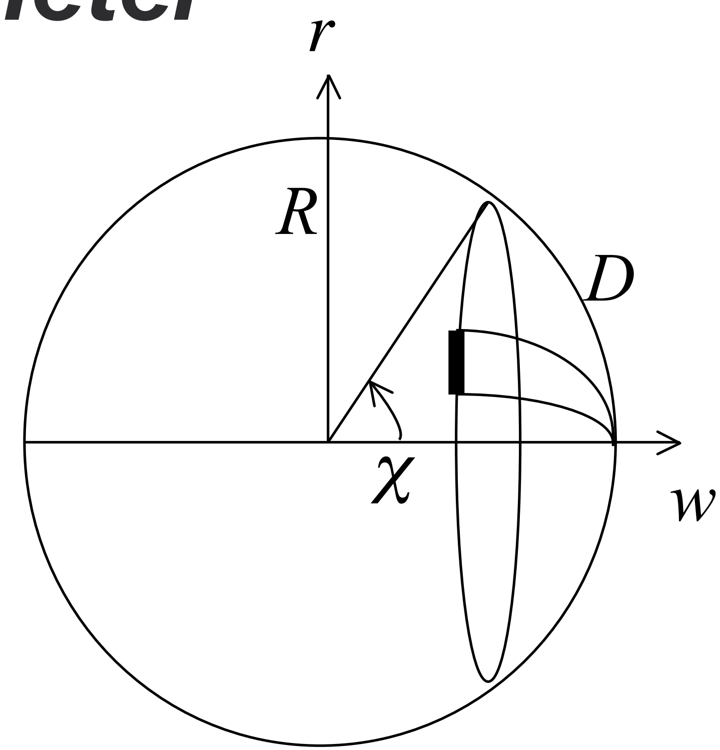
radial distance :

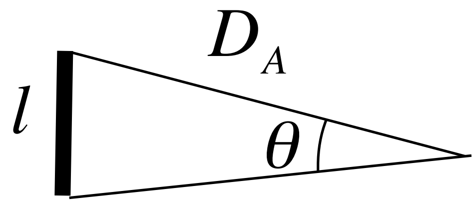
$$D = \int \sqrt{g_{\chi\chi}}\, d\chi = \int_0^{\chi} R\, d\chi = R\, \chi$$

linear size : $(l << D)$

$$l = \int \sqrt{g_{\theta\theta}}\; d\theta = R\, S_k(\chi)\, \theta$$

angular size :



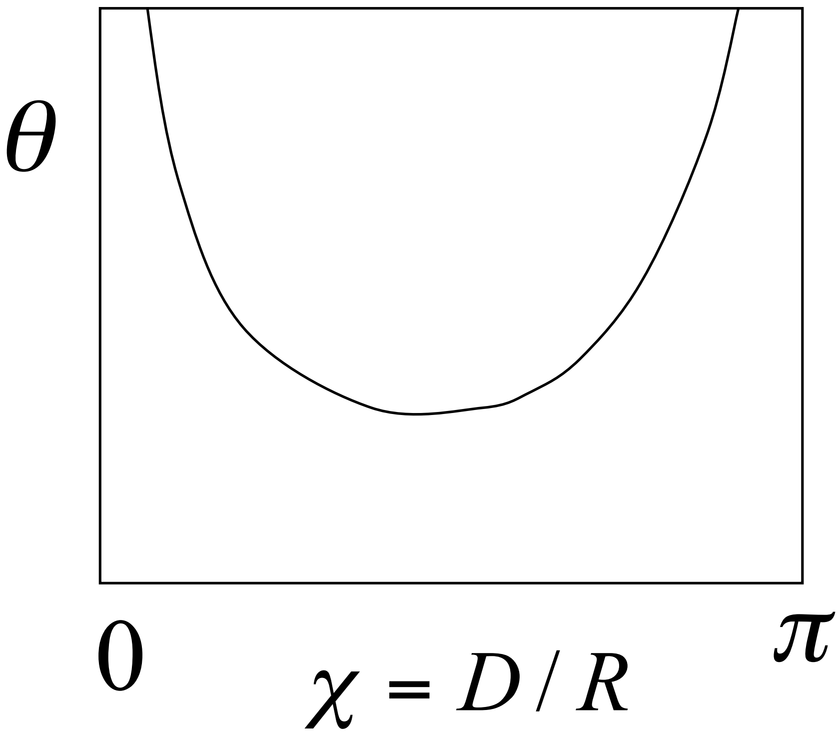


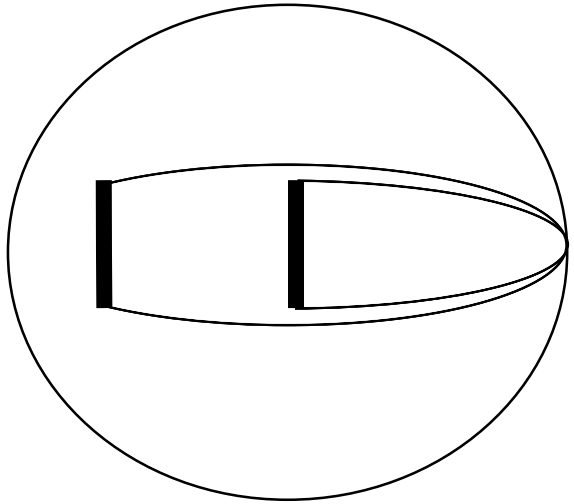
$$\theta = \frac{l}{D_A}$$

$D = R\ \chi =$ Radial Distance

$D_A = R\, S_k(\chi) =$ Angular Diameter Distance

Angular Diameter





$$\theta = \frac{l}{D_A} \qquad D_A = R\, S_k(\chi) = \text{ Angular Diameter Distance}$$

Positive curvature makes objects look larger, hence closer.

Area of Spherical Shell

radial coordinate χ, angles θ, ϕ :

$$dl^2 = R^2 \left[d\chi^2 + S_k^2(\chi) \left(d\theta^2 + \sin^2\theta \;\; d\phi^2 \right) \right]$$

area of shell :

$$A = \int \sqrt{g_{\theta\theta}}\, d\theta \sqrt{g_{\phi\phi}}\, d\phi$$

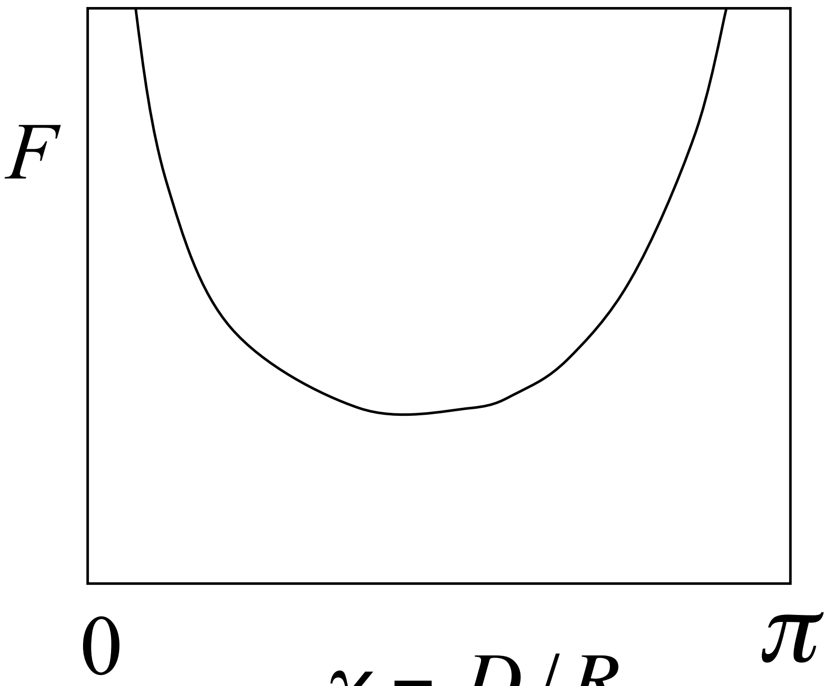
$$= R^2 \;\; S_k^2(\chi) \int_0^{\pi} d\theta \;\; \sin\theta \int_0^{2\pi} d\phi$$

$$= 4\pi \;\; R^2 \;\; S_k^2(\chi)$$

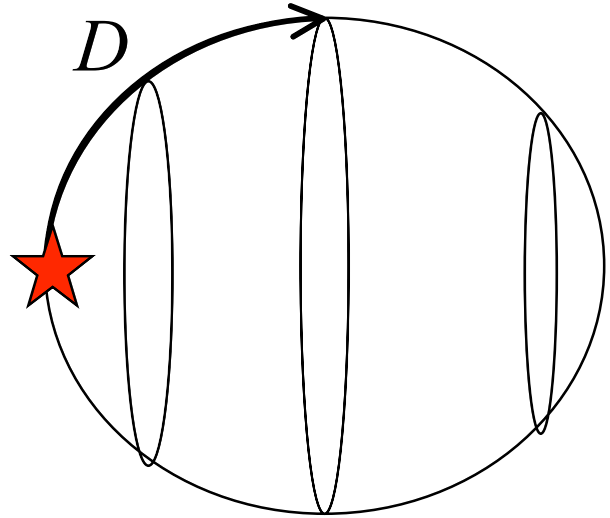
flux :

$$F = \frac{L}{A} = \frac{L}{4\pi \;\; D_L^2} \qquad D_L = R \; S_k(\chi) = \text{ Luminosity Distance}$$

Fluxes



$$\chi = D / R$$



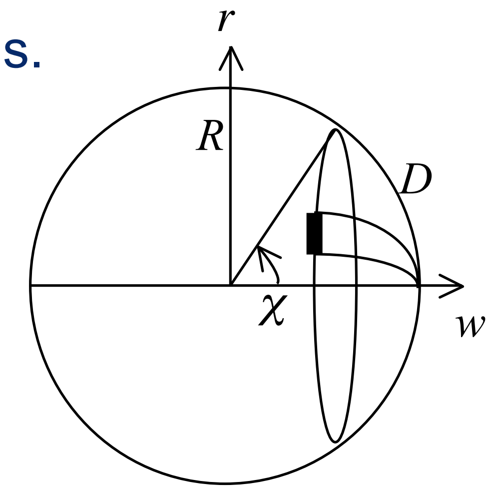
$$F = \frac{L}{A} = \frac{L}{4\pi\ D_L^2} \qquad D_L = R\ S_k(\chi) = \text{ Luminosity Distance}$$

Positive curvature makes sources look brighter, hence closer.

$$\text{Note}:\ D_L = D_A \quad \text{if} \quad R = \text{const.}$$

Summary

- The **metric** converts coordinate steps to physical lengths.

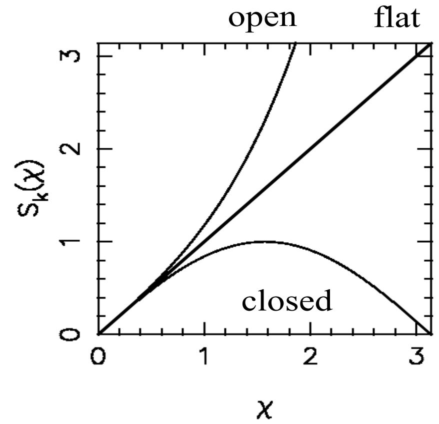


- Use the metric to compute lengths, areas, volumes, …
- Radial distance: $$D \equiv \int \sqrt{g_{rr}}\, dr = R\,\chi$$
- "Circumferencial" distance

$$r \equiv \frac{C}{2\pi} = \left(\frac{A}{4\pi}\right)^{1/2} = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi = R\,S_k(\chi) = R\,S_k(D/R)$$

- "Observable" distances, defined in terms of local observables (angles, fluxes), give r, not D.

$$D_A \equiv \frac{l}{\theta} = r \qquad D_L \equiv \left(\frac{L}{4\pi F}\right)^{1/2} = r$$



- r can be smaller than D (positive curvature) or larger (negative curvature) or the same (flat).