Lecture 4

Space-Time Metric

Olber's Paradox

Why is the sky dark at night ?

Flux from all stars in the sky :

$$F = \int n_* \, F_* \, \mathrm{d}(\mathrm{Vol}) = \int_0^{\chi_{\max}} n_* \left(\frac{L_*}{A(\chi)} \right) \left(A(\chi) \, R \, d\chi \right)$$

$$= n_* \, L_* \, R \, \chi_{\max}$$

$$\Rightarrow \infty \quad \text{for flat space, } R \rightarrow \infty.$$

A dark sky may imply :

(1) an edge (we don't observe one)

(2) a curved space (finite size)

(3) expansion ($R(t)$ => finite age, redshift)

Minkowski Spacetime Metric

$$ds^2 = -c^2 dt^2 + dl^2$$

$$d\tau^2 = dt^2 - \frac{dl^2}{c^2} = dt^2 \left(1 - \frac{1}{c^2} \left(\frac{dl}{dt} \right)^2 \right)$$

Time-like intervals:
$ds^2 < 0, \quad d\tau^2 > 0$
Inside light cone.
Causally connected.

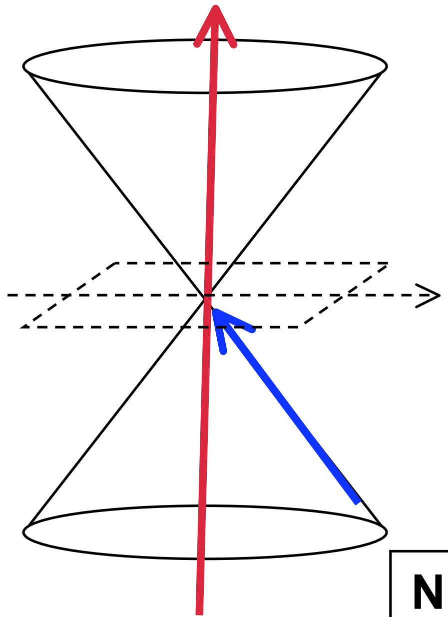
***Proper time** (moving clock):*

$$d\tau = \sqrt{-ds^2 / c^2}$$

$$= dt \sqrt{1 - \frac{v^2}{c^2}} > 0$$

Space-like intervals:
$ds^2 > 0 \,, \quad d\tau^2 < 0$

Outside light cone.
Causally disconnected.



World line of massive particle at rest.

Null intervals light cone:
v = c, $ds^2 = 0$

Photons arrive from our past light cone.

Minkowski Spacetime Metric

$$ds^2 = -c^2 dt^2 + dl^2$$

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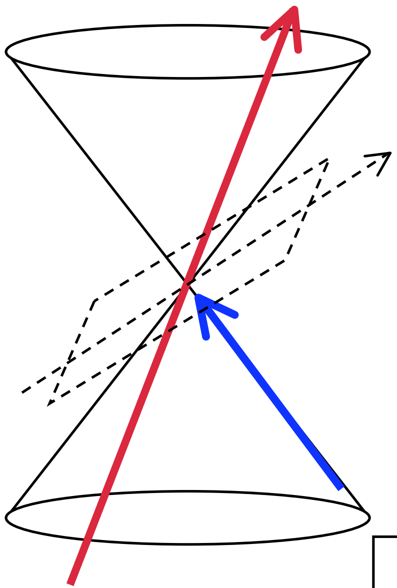
Time-like intervals:
$ds^2 < 0, \quad d\tau^2 > 0$
Inside light cone.
Causally connected.

Space-like intervals:
$ds^2 > 0\ , \quad d\tau^2 < 0$

Outside light cone.
Causally disconnected.



World line of massive particle in motion.

Null intervals light cone:
$v = c, \quad ds^2 = 0$

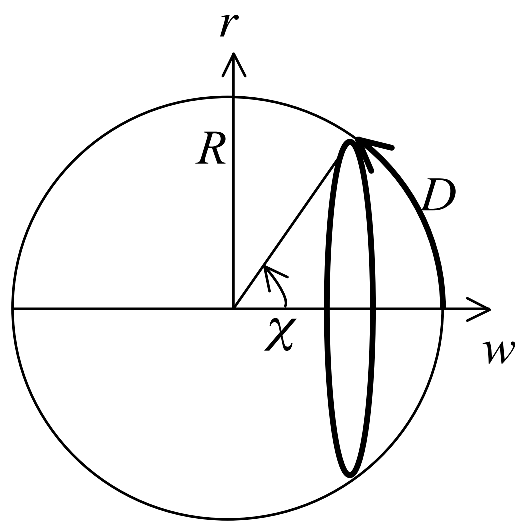
Photons arrive from our past light cone.

Robertson-Walker metric
uniformly curved, evolving spacetime

$$ds^2 = -c^2 dt^2 + R^2(t)\left(d\chi^2 + S_k^2(\chi)\, d\psi^2\right)$$

$$= -c^2 dt^2 + R^2(t)\left(\frac{du^2}{1-k\,u^2} + u^2\, d\psi^2\right)$$

$$= -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-k\left(r/R_0\right)^2} + r^2\, d\psi^2\right)$$



$$S_k(\chi) = \begin{cases} \sin\chi & (k=+1) & \text{closed} \\ \chi & (k=0) & \text{flat} \\ \sinh\chi & (k=-1) & \text{open} \end{cases}$$

$$d\psi^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$$

$$a(t) \equiv R(t)/R_0$$

$$R_0 \equiv R(t_0)$$

radial distance $= D(t) = R(t)\,\chi$

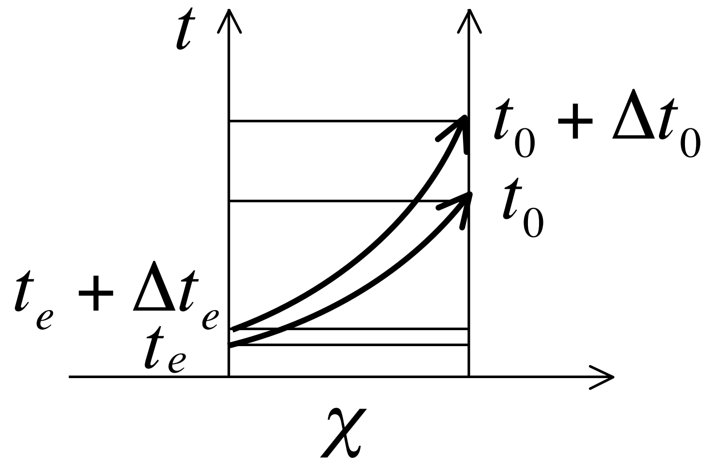
circumference $= 2\pi\, r(t)$ $\qquad r(t) = a(t)\, r = R(t)\, u = R(t)\, S_k(\chi)$

Redshift and Time Dilation

Light rays are null geodessics:

$$ds^2 = R^2(t)\, d\chi^2 - c^2\, dt^2 = 0$$

$$d\chi = \frac{c\, dt}{R(t)}$$



$$\chi = \int_{t_e}^{t_e+\Delta t_e} \frac{c\, dt}{R(t)} + \int_{t_e+\Delta t_e}^{t_o} \frac{c\, dt}{R(t)} = \int_{t_e+\Delta t_e}^{t_o} \frac{c\, dt}{R(t)} + \int_{t_o}^{t_o+\Delta t_o} \frac{c\, dt}{R(t)}$$

$$\frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_o}{R(t_o)} \qquad \rightarrow \qquad \frac{R(t_o)}{R(t_e)} = \frac{\Delta t_o}{\Delta t_e} = \frac{\lambda_o}{\lambda_e} = 1 + z$$

$\therefore$ Observed wavelengths and time intervals appear "stretched" by a factor $x = 1 + z = R_0 / R(t)$.

Fidos and co-moving coordinates

Distance varies in time:

$$D(t)$$

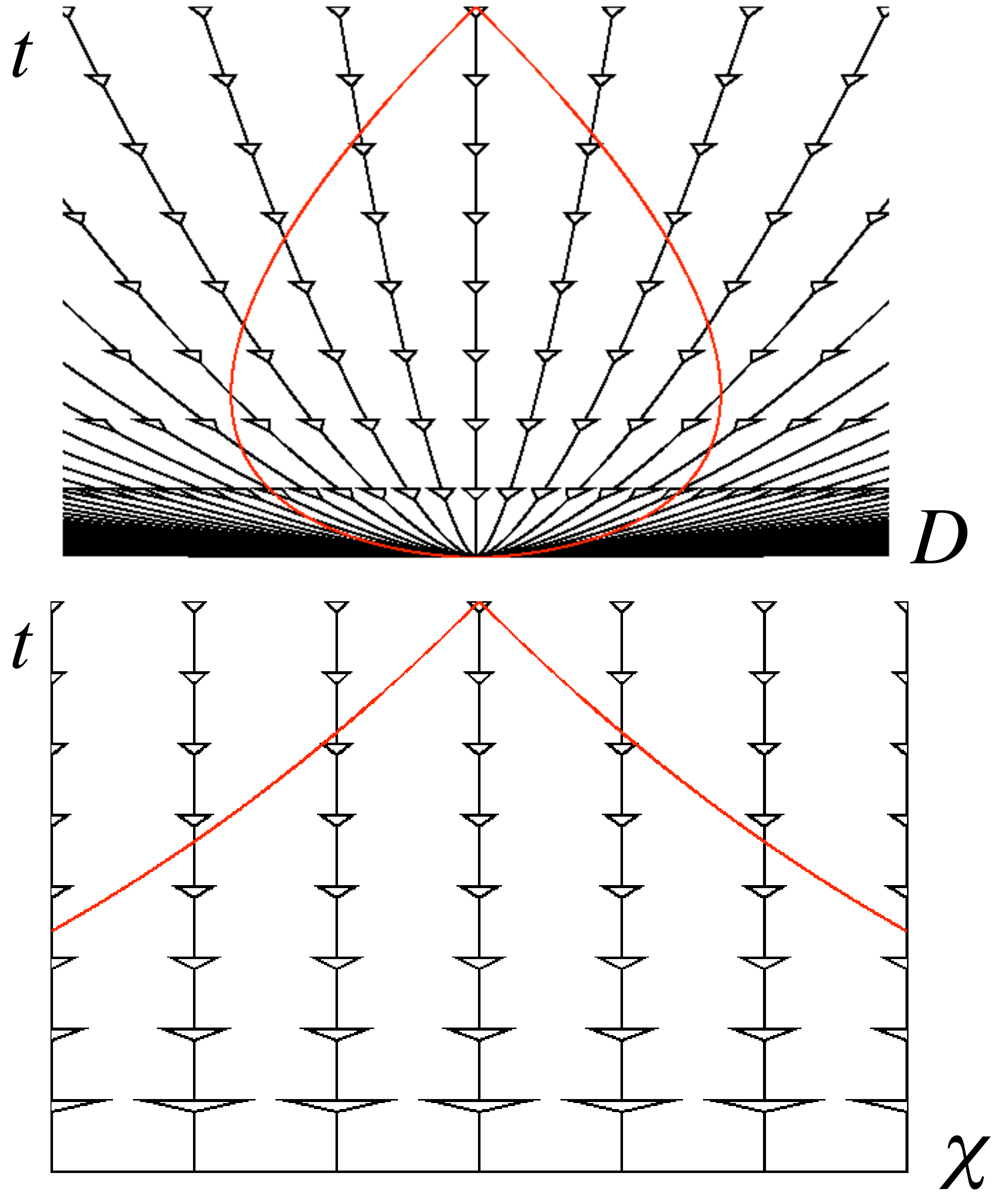
"Fiducial observers" (Fidos)

$$D(t) = R(t)\,\chi$$

"Co-moving" coordinates

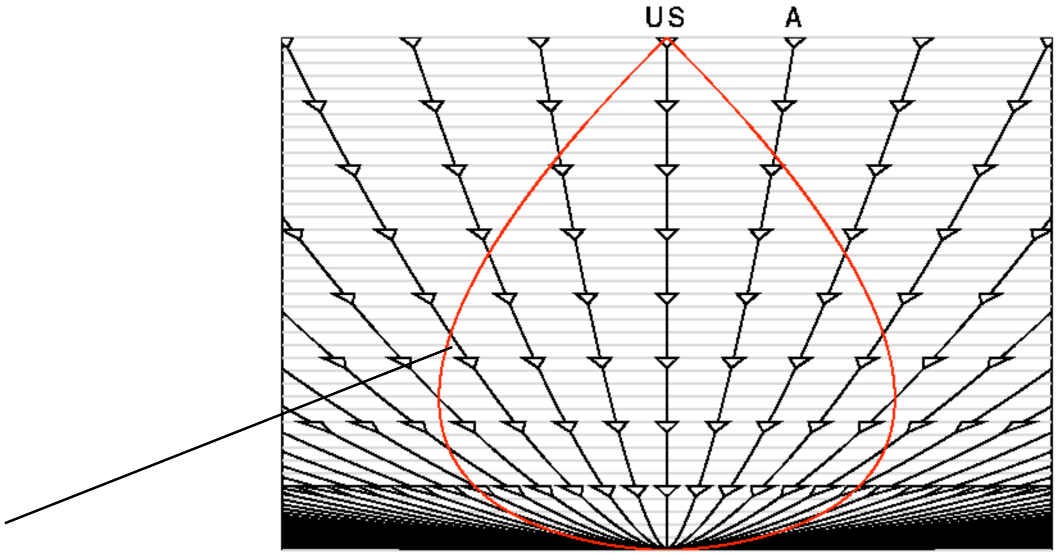
$$\chi \quad \text{or} \quad D_0 \equiv R_0\,\chi$$

Labels the Fidos



Cosmological Principle

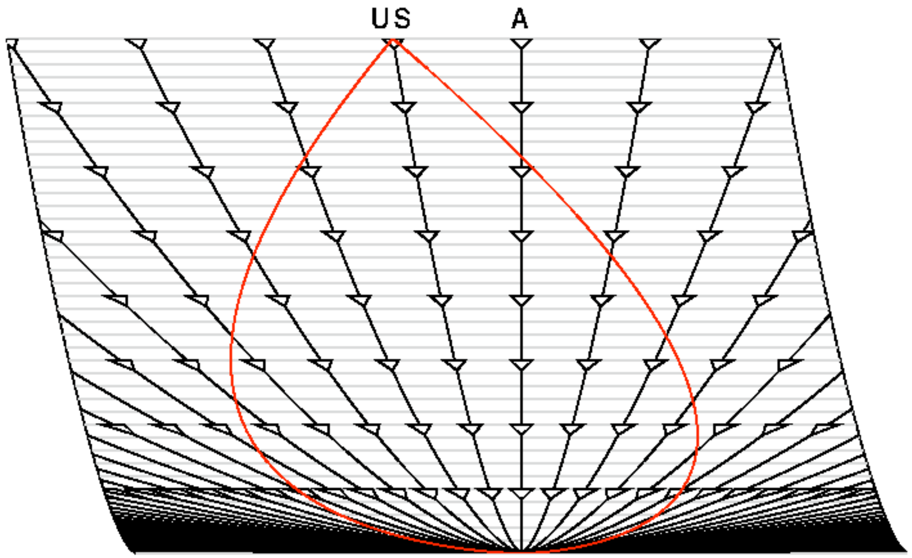
All "co-moving" observers see an equivalent view.



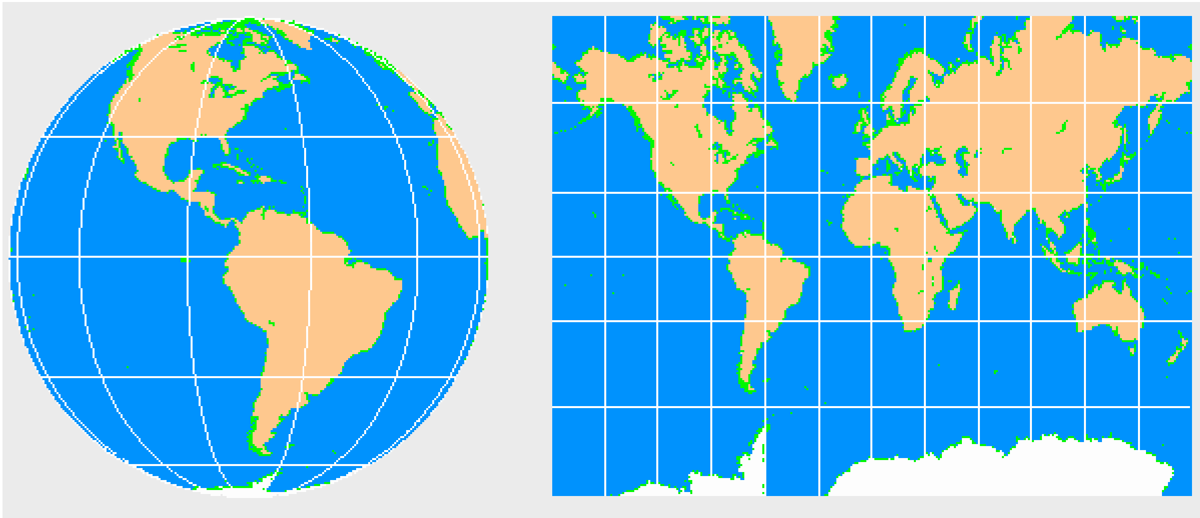
Light ray

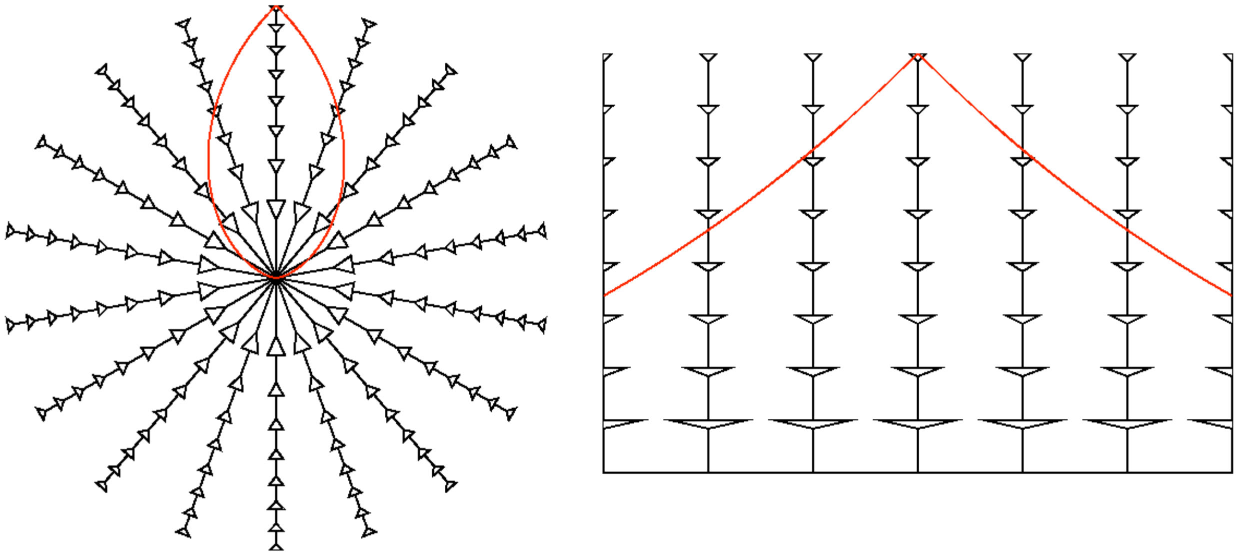
Past light cone:

- looking out = back in time

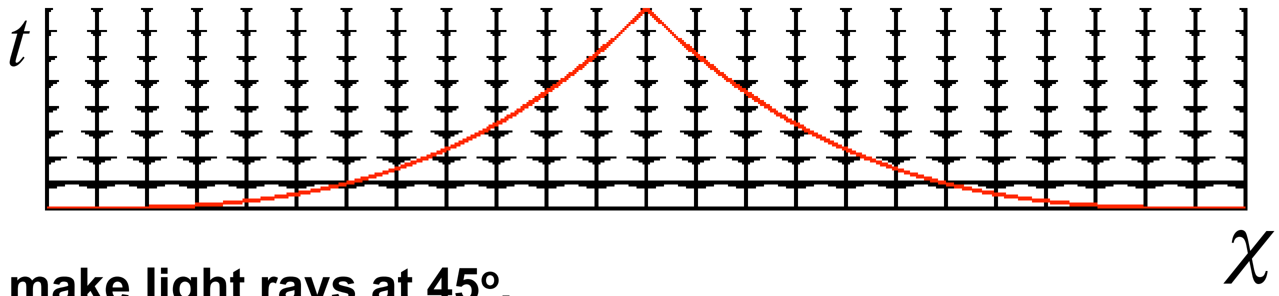


Coordinate Systems



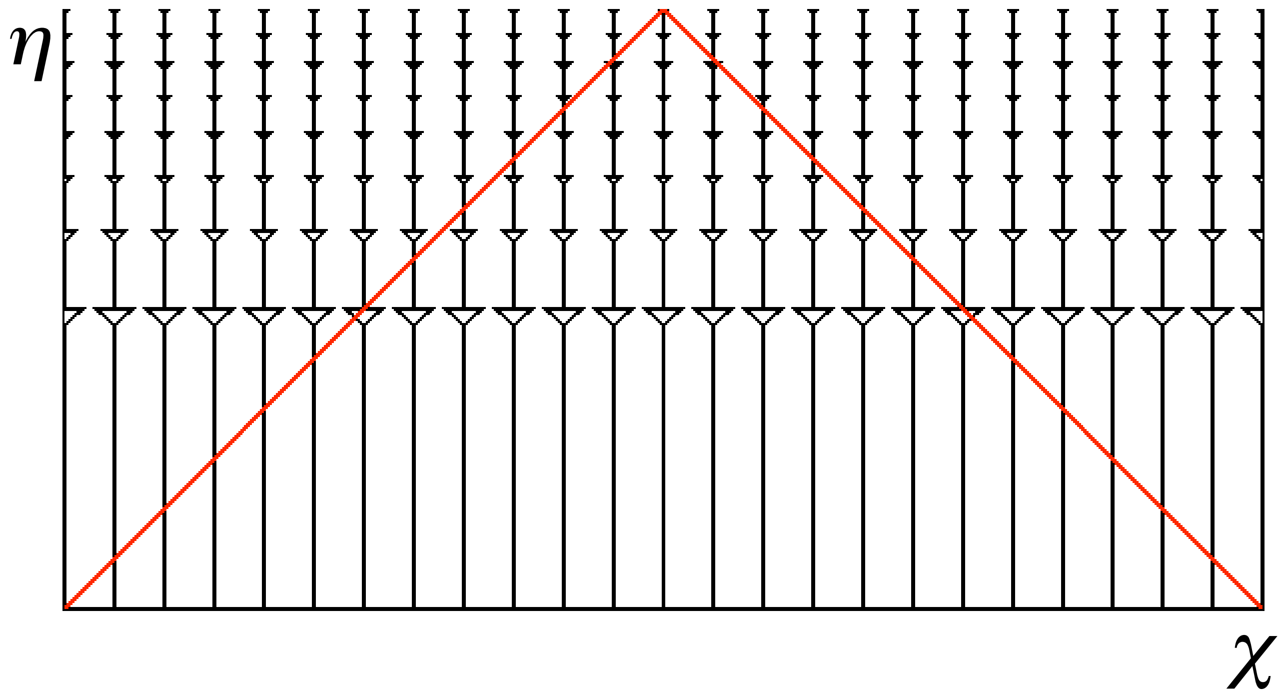


Conformal time



Stretch time axis to make light rays at 45°.

$$\eta \equiv \int \frac{c\ dt}{R(t)}$$



$$ds^2 = R^2(t)\left(-d\eta^2 + d\chi^2 + S_k{}^2(\chi)\, d\psi^2\right)$$

The Horizon: How far can we see?

- **We see only a finite patch of the Universe.**
- **The Horizon may grow with time (if expansion decelerates) or shrink in time (e.g. inflation).**

suppose $\quad R(t) = R_0 \left(t/t_0 \right)^{\alpha}$

$$\chi_H = \eta_0 = \int_0^{t_0} \frac{c\,dt}{R(t)} = \frac{c\,t_0}{\left(1-\alpha\right) R_0} \quad \text{if} \quad \alpha < 1$$

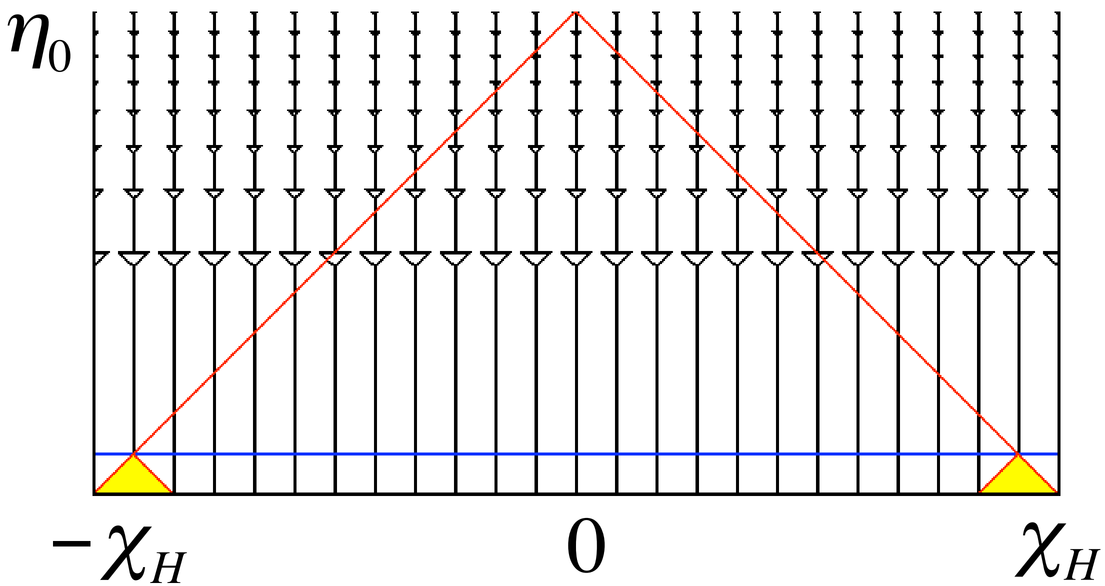
$$\Rightarrow \quad \infty \qquad \text{as} \quad \alpha \Rightarrow 1$$

matter - dominated :

$$\alpha = 2/3$$

radiation - dominated :

$$\alpha = 1/2$$



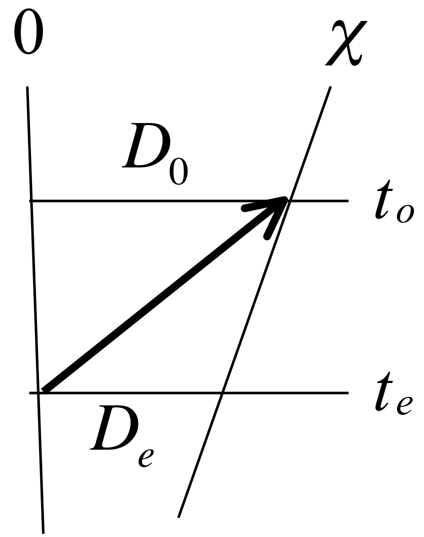
Angular Diameter Distance

- **radial distance**
  - now (when photon received):

$$D_0 = R(t_0)\,\chi = R_0\,\chi$$

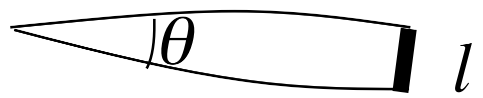
  - when photon emitted:

$$D_e = R(t_e)\,\chi = \frac{R(t_e)\,R_0\,\chi}{R_0} = \frac{D_0}{1+z}$$



- **angular size**
  - Fraction of circumference when photon was emitted:

$$\frac{\theta}{2\pi} = \frac{l}{2\pi\,r(t_e)}$$



- **angular diameter distance**

$$D_A \equiv \frac{l}{\theta} = r(t_e) = R(t_e)\,S_k(\chi) = \frac{R(t_e)}{R_0}\,R_0\,S_k(\chi)$$

$$= \frac{R_0\,S_k(\chi)}{1+z} = \frac{r(t_0)}{1+z} \equiv \frac{r_0}{1+z}$$

Circumference was smaller by factor x=1+z.

Souces look larger/closer.

Luminosity Distance

– Luminosity (erg s^{-1}) $L = \dfrac{N\,h\,\nu_e}{\Delta t_e}$

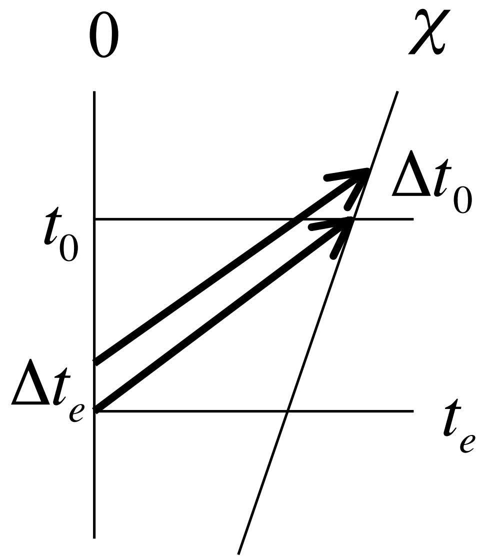
– area of photon sphere (when photons observed):

$$A_0 = 4\pi\, r_0^{\,2} = 4\pi\, R_0^{\,2} S_k^{\,2}(\chi)$$

– redshift:

$$\lambda_0 = \lambda_e\,(1+z)$$

$$\nu_0 = \nu_e \big/ (1+z)$$



– time dilation: lower photon arrival rate

$$\Delta t_0 = \Delta t_e\,(1+z)$$

– observed flux (erg cm^{-2} s^{-1})

$$F = \frac{N\,h\,\nu_0}{A_0\,\Delta t_0} = \frac{L}{4\pi\, r_0^{\,2}\,(1+z)^2} = \frac{L}{4\pi\, D_L^2}$$

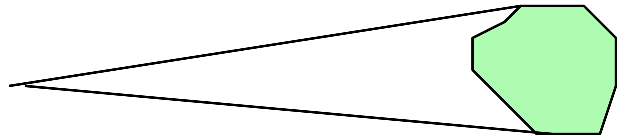
- **Luminosity distance**

$$D_L = (1+z)\,r_0 = (1+z)\,R_0\,S_k(\chi)$$

Sources look fainter/farther.

Surface Brightness

- **Solid angle** $\Omega = A / D_A^2$



- **Surface brightness**
  - Flux per solid angle (erg s^{-1} cm^{-2} arcsec^{-2})

$$\Sigma \equiv \frac{F}{\Omega} = \frac{L}{4\pi\, D_L^{\,2}} \, \frac{D_A^{\,2}}{A} = \frac{L}{4\pi\, A\,(1+z)^4}$$

  - decreases very rapidly with z because:
  - expansion spreads out the photons
  - decreases their energy
  - decreases their arrival rate

$$D_L = (1+z)\, r_0$$

$$D_A = \frac{r_0}{(1+z)}$$

$$r_0 = R_0\, S_k(\chi)$$

Flux Density Spectra

- **emitted photons**
  - (erg s^{-1} Hz^{-1})

$$L_\nu(\lambda_e) = \frac{N(\lambda_e)\, h\, \nu_e}{\Delta\nu_e\, \Delta t_e}$$

- **redshift**

$$\lambda_o = \lambda_e(1+z)$$

photon numbers

$$N(\lambda_o) = N(\lambda_e)$$

- **observed flux density spectra**
  - (erg cm^{-2} s^{-1} Hz^{-1})

$$F_\nu(\lambda_o) = \frac{N(\lambda_o)\, h\, \nu_0}{\Delta\nu_0\, \Delta t_0\, A} = \frac{L_\nu(\lambda_e)}{4\pi\, D_A^2} \frac{1}{(1+z)}$$

  - (erg cm^{-2} s^{-1} A^{-1})

$$F_\lambda(\lambda_o) = \frac{N(\lambda_o)\, h\, \nu_0}{\Delta\lambda_0\, \Delta t_0\, A} = \frac{L_\lambda(\lambda_e)}{4\pi\, D_A^2} \frac{1}{(1+z)^3}$$