Lecture 5

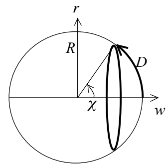
Dynamics of the Universe

R(t) = ?

Robertson-Walker metric

uniformly curved, evolving spacetime

$$ds^2 = -c^2dt^2 + R^2(t)\left(d\chi^2 + S_k^2(\chi)\,d\psi^2\right)$$
$$= -c^2dt^2 + R^2(t)\left(\frac{du^2}{1-k\,u^2} + u^2\,d\psi^2\right)$$
$$= -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-k\left(r/R_0\right)^2} + r^2\,d\psi^2\right)$$



$$S_k(\chi) = \begin{cases} \sin\chi & (k=+1) \quad \text{closed} \\ \chi & (k=0) \quad \text{flat} \\ \sinh\chi & (k=-1) \quad \text{open} \end{cases}$$

$$d\psi^2 \equiv d\theta^2 + \sin^2\theta\,d\phi^2$$
$$a(t) \equiv R(t)/R_0$$
$$R_0 \equiv R(t_0)$$

radial distance $= D(t) = R(t)\,\chi$

circumference $= 2\pi\,r(t) \qquad r(t) = a(t)\,r = R(t)\,u = R(t)\,S_k(\chi)$

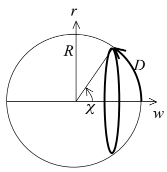
Time and Distance vs Redshift

- We observe the **redshift** : $z \equiv \frac{\lambda-\lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 \quad \lambda = \text{observed},$

  $\lambda_0 = \text{emitted (rest)}$
- Hence we know the **expansion factor:**

$$x \equiv 1+z = \frac{\lambda}{\lambda_0} = \frac{\lambda(t_0)}{\lambda(t)} = \frac{R(t_0)}{R(t)} = \frac{R_0}{R(t)}$$

- When was the light emitted? $t(z) = ?$
- How far away was the source? $\chi(z) = ?$



$$D(t,\chi) = R(t)\,\chi \qquad D_A = r_0(\chi)/\left(1+z\right)$$
$$r(t,\chi) = R(t)\,S_k(\chi) \qquad D_L = r_0(\chi)\left(1+z\right)$$

- How do these depend on cosmological parameters?

$$H_0 \quad \Omega_M \quad \Omega_\Lambda$$

Time -- Redshift relation

$$x = 1+z = \frac{R_0}{R}$$
$$\frac{dx}{dt} = -\frac{R_0}{R^2}\frac{dR}{dt}$$
$$= -\frac{R_0}{R}\frac{\dot R}{R}$$
$$= -x\,H(x)$$

Memorise this derivation!

Hubble parameter : $H \equiv \frac{\dot R}{R}$

$$\therefore\ dt = \frac{-dx}{x\,H(x)} = \frac{-dz}{\left(1+z\right)H(z)}$$

Time and Distance vs Redshift

$$\frac{d}{dt}\left(x = 1+z = \frac{R_0}{R}\right) \rightarrow dt = \frac{-dx}{x\,H(x)}$$

Look - back time :

$$t(z) = \int_t^{t_0} dt = \int_{1+z}^{1} \frac{-dx}{x\,H(x)} = \int_1^{1+z} \frac{dx}{x\,H(x)}$$

Age : $t_0 = t(z \rightarrow \infty)$

Distance : $D = R\,\chi \qquad r = R\,S_k(\chi)$

$$\chi(z) = \int d\chi = \int_t^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{R_0}\int_1^{1+z} \frac{R_0}{R(t)}\frac{dx}{x\,H(x)} = \frac{c}{R_0}\int_1^{1+z}\frac{dx}{H(x)}$$

Horizon : $\chi_H = \chi(z \rightarrow \infty)$

Need to know *R(t)*, or R_0 and *H(x)*.

Einstein's General Relativity

- ***1. Spacetime geometry tells matter how to move***
  - gravity = effect of curved spacetime
  - free particles follow geodesic trajectories
    - *ds²* < 0 v < c time-like massive particles
    - *ds²* = 0 v = c null massless particles (photons)
    - *ds²* > 0 v > c space-like tachyons (not observed)
- ***2. Matter (+energy) tells spacetime how to curve***
  - Einstein field equations
    - nonlinear
    - second-order derivatives of metric with respect to space/time coordinates

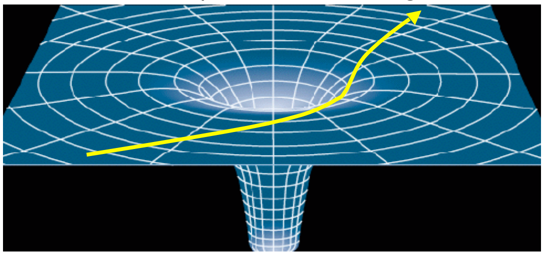
Geodesics

Gravity = curvature of space-time by matter/energy.

Freely-falling bodies follow **geodesic trajectories**.

Shortest possible path in curved space-time.

Local curvature replaces forces acting at distance.



Space-Time Geodesics

Schwarzschild Metric:

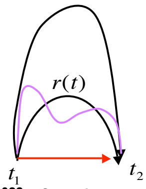
(curved space-time outside a compact mass *M*)

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{r_S}{r}\right)c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r_S}{r}\right)} + r^2 d\psi^2$$

Schwarzschild Radius: $r_S = \frac{2GM}{c^2}$

Freely-falling test particles follow a ***geodessic path.***

Free fall = maximum τ proper time.



$$\tau = \int d\tau = \int_{t_1}^{t_2} \frac{d\tau}{dt}\, dt$$

$$d\tau^2 = \frac{-ds^2}{c^2} = \left(1 - \frac{r_S}{r}\right)dt^2 - \frac{1}{c^2}\left[\frac{dr^2}{\left(1 - \frac{r_S}{r}\right)} + r^2 d\varphi^2\right]$$

Space-Time Geodesics

Schwarzschild Metric:

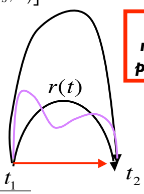
$$ds^2 = -c^2 d\tau^2 = -[1 - (r_S/r)]c^2 dt^2 + \frac{dr^2}{[1 - (r_S/r)]} \qquad r_S = \frac{2GM}{c^2}$$

Proper time:

High up clock runs faster **:)**

$$\frac{d\tau}{dt} = \sqrt{\frac{-ds^2}{c^2 dt^2}} = \frac{1}{c}\sqrt{-g_{tt}} = \sqrt{1 - \frac{2GM}{c^2 r}}$$

Free fall = maximum τ proper time.



Moving clock runs slower **: (**

$$\frac{d\tau}{dt} = \frac{1}{c}\sqrt{-g_{tt} - g_{rr}\left(\frac{dr}{dt}\right)^2} = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2} + \ldots}$$

$$mc^2\tau = \int_{t_1}^{t_2}\left(mc^2 - \frac{GMm}{r} - \frac{mv^2}{2} + \ldots\right)dt \approx (t_2 - t_1)mc^2 - \int L(t)\, dt$$

$$\max[\tau] = \min[\int L\, dt] \qquad L \equiv E_{kin} - E_{pot} = \text{Lagrangian}$$

Tests of General Relativity

- **Matter bends light rays (e.g. solar eclipses)**
  - 1.75 arcsec for Sun grazing ray
- **Perihelion shift of planetary orbits**

	GR	observed
Mercury	43.03	43.11 +/- 0.45 arcsec/century
Venus	8.6	8.4 +/- 4.8
Earth	3.8	5.0 +/- 1.2
Icarus	10.3	9.8 +/- 0.8

- **Time delay of Venus radar reflections**
  - ~200 microsec
- **Decay of orbit of binary pulsars**
  - due to emission of gravitational waves
- **All tests on small size scales.**
- **For cosmology, assume GR valid on larger scales.**