Lecture 6

General Relativity:
Field Equations

Dynamics of the Universe:
$R(t) = ?$
$H(x) = ?$

Friedmann Equation

Time and Distance vs Redshift

$$\frac{d}{dt}\left(x = 1 + z = \frac{R_0}{R} \right) \;\rightarrow\; dt = \frac{-dx}{x\,H(x)}$$
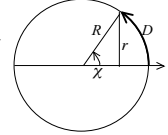
Look - back time :

$$t(z) = \int_t^{t_0} dt = \int_{1+z}^{1} \frac{-dx}{x\,H(x)} = \int_1^{1+z} \frac{dx}{x\,H(x)}$$

Age : $t_0 = t(z \rightarrow \infty)$

Distance : $D = R\chi$ $r = R\,S_k(\chi)$

$$\chi(z) = \int d\chi = \int_t^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{R_0}\int_1^{1+z} \frac{R_0}{R(t)}\frac{dx}{x\,H(x)} = \frac{c}{R_0}\int_1^{1+z} \frac{dx}{H(x)}$$

Horizon : $\chi_H = \chi(z \rightarrow \infty)$

Need to know $R(t)$, or R_0 and $H(x)$.

Einstein's General Relativity

- **1. *Spacetime geometry tells matter how to move***

  - gravity = effect of <u>curved spacetime</u>
  - free particles follow <u>geodesic</u> trajectories

- **2. *Matter (energy) tells spacetime how to curve***

  - Einstein field equations
    - nonlinear
    - second-order derivatives of metric
      - with respect to space/time coordinates

Einstein Field Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R^{\alpha}{}_{\alpha}\, g_{\mu\nu} = \frac{8\pi\,G}{c^2} T_{\mu\nu} - \Lambda\, g_{\mu\nu}$$

$g_{\mu\nu}$ = spacetime metric ($ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$)

$G_{\mu\nu}$ = Einstein tensor (spacetime curvature)

$R_{\mu\nu}$ = Ricci curvature tensor

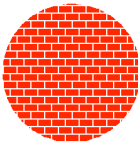$R^{\alpha}{}_{\alpha}$ = Ricci curvature scalar

G = Netwon's gravitational constant
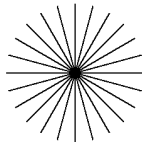
$T_{\mu\nu}$ = energy - momentum tensor

Λ = cosmological constant

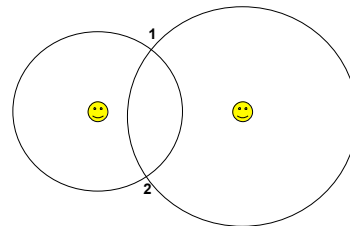## Homogeneity and Isotropy

homogeneous
not isotropic

isotropic
not homogeneous

For cosmology, assume Universe is Homogeneous.
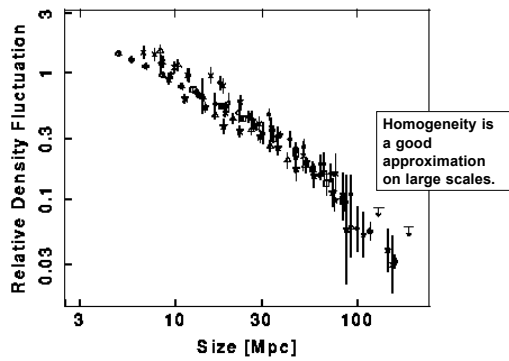*Simplifies the equations. :)*

Cosmological Principle (assumed)
+ Isotropy (observed)
=> Homogeneity

$\rho_1 = \rho_2$ otherise not isotropic

for equidistant fidos

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Density Fluctuations vs Scale

Homogeneity is a good approximation on large scales.

Homogeneous perfect fluid

density ρ pressure p

Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi\,G}{c^2}\begin{pmatrix} \rho\,c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} - \Lambda\begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

---> Friedmann equations :

$$\dot{R}^2 = \left(\frac{8\pi\,G\,\rho + \Lambda}{3}\right)R^2 - k\,c^2 \qquad \text{energy}$$

$$\ddot{R} = -\frac{4\pi\,G}{3c^2}\left(\rho c^2 + 3p\right)R + \frac{\Lambda}{3}R \qquad \text{momentum}$$

Note: energy density and pressure decelerate, Λ accelerates.

Local Conservation of Energy

$$d[energy] = work$$

$$d[\rho\,c^2 R^3] = -p\,d[R^3]$$

$$\dot{\rho}\,c^2 R^3 + \rho\,c^2\,(3R^2\dot{R}) = -p\,(3R^2\dot{R})$$
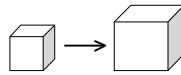
$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} \qquad p = p(\rho) = \text{equation of state}$$

$$\text{Friedmann 1}: \dot{R}^2 = \frac{8\pi\,G}{3}\rho\,R^2 + \frac{\Lambda}{3}R^2 - k\,c^2$$

$$\left(2\,\dot{R}\,\ddot{R}\right) = \frac{8\pi\,G}{3}\left(\dot{\rho}\,R^2 + 2\,R\,\dot{R}\,\rho\right) + \frac{\Lambda}{3}\left(2\,R\,\dot{R}\right)$$

$$\ddot{R} = \frac{8\pi\,G}{3}\left(\frac{\dot{\rho}\,R^2}{2\,\dot{R}} + R\,\rho\right) + \frac{\Lambda}{3}R$$

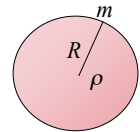$$\ddot{R} = -\frac{4\pi\,G}{3}\left(\rho + \frac{3\,p}{c^2}\right)R + \frac{\Lambda}{3}R = \text{Friedmann 2}$$

Newtonian Analogy

$$E = \frac{m}{2}\dot{R}^2 - \frac{G\,M\,m}{R} \qquad M = \frac{4\pi}{3}R^3\rho$$

$$\dot{R}^2 = \frac{8\pi\,G}{3}\rho\,R^2 + \frac{2E}{m}$$

Friedmann equation:

$$\dot{R}^2 = \left(\frac{8\pi\,G\,\rho + \Lambda}{3}\right)R^2 - k\,c^2$$

same equation if $\rho \to \rho + \dfrac{\Lambda}{8\pi\,G}$, $\dfrac{2E}{m} \to -k\,c^2$

Newtonian Analogy
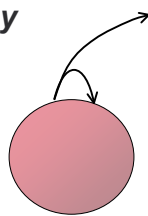
$$E = \frac{m}{2}\dot{R}^2 - \frac{G\,M\,m}{R}$$

$$V_{esc} = \sqrt{\frac{2\,G\,M}{R}}$$

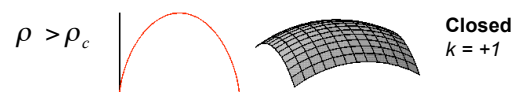$$E > 0 \quad V > V_{esc} \quad R \to \infty \quad V_\infty > 0$$
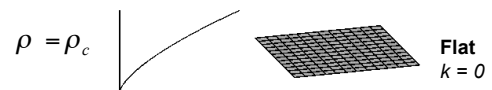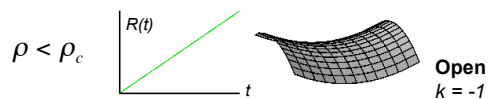$$E = 0 \quad V = V_{esc} \quad R \to \infty \quad V_\infty = 0$$
$$E < 0 \quad V < V_{esc} \quad R \to 0$$

Density - Evolution - Geometry

$$\rho < \rho_c$$ **Open** $k = -1$

$$\rho = \rho_c$$ **Flat** $k = 0$

$$\rho > \rho_c$$ **Closed** $k = +1$

Critical Density

- **Derive using Newtonian analogy:**

escape velocity :

$$V_{esc}^2 = \frac{2\,G\,M}{R} = \frac{2\,G}{R}\left(\frac{4\pi\,R^3\,\rho}{3}\right) = \frac{8\pi\,G\,R^2\,\rho}{3}$$
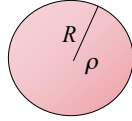
Hubble expansion :

$$V = \dot{R} = H_0\,R$$

critical density :

$$\left(\frac{V_{esc}}{V}\right)^2 = \frac{8\pi\,G\,\rho}{3\,H_0^2} \equiv \frac{\rho}{\rho_c}$$
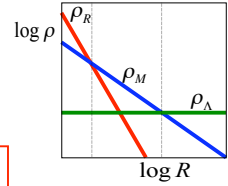
$$\boxed{\rho_c = \frac{3\,H_0^2}{8\pi\,G}}$$

radiation -> matter -> vacuum

radiation : $\rho_R \propto R^{-4}$

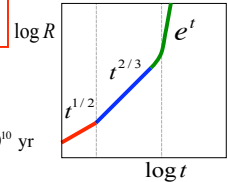matter : $\rho_M \propto R^{-3}$

vacuum : $\rho_\Lambda = \text{const}$

$$\boxed{\begin{aligned}\rho(x) &= \rho_R\,x^4 + \rho_M\,x^3 + \rho_\Lambda \\ x &= 1+z = \frac{R_0}{R} = \frac{1}{a}\end{aligned}}$$

$\rho_R = \rho_M$ at $x \sim \dfrac{\rho_M}{\rho_R} \sim 10^4 \quad t \sim 10^4\ \text{yr}$

$\rho_M = \rho_\Lambda$ at $x \sim \left(\dfrac{\rho_\Lambda}{\rho_M}\right)^{1/3} z \sim 0.3 \quad t \sim 10^{10}\ \text{yr}$
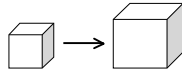
Equation of State ----- w

$$\boxed{\begin{aligned}&\text{Equation of state :}\\ &\rho \propto R^{-n} \quad n = 3(1+w) \\ &w \equiv \frac{\text{pressure}}{\text{energy density}} = \frac{p}{\rho\,c^2} = \frac{n}{3} - 1\end{aligned}}$$

<u>Radiation</u> : $(n = 4,\ w = 1/3)$

$$p_R = \frac{1}{3}\rho_R\,c^2$$

<u>Matter</u> : $(n = 3,\ w = 0)$

$$p_M \sim \rho_M\,c_S^2 << \rho_M\,c^2$$

<u>Vacuum</u> : $(n = 0,\ w = -1)$

$$p_\Lambda = -\rho_\Lambda\,c^2$$

Negative Pressure ! ?

$$\begin{aligned}&d[\text{energy}] = \text{work} \\ &d\left[\rho\,c^2 R^3\right] = -p\,d\left[R^3\right] \\ &\rho\,c^2\left(3\,R^2 dR\right) + R^3\,c^2 d\rho = -p\left(3\,R^2 dR\right) \\ &1 + \frac{R\,d\rho}{3\,\rho\,dR} = -\frac{p}{\rho c^2} = -w \\ &w = -\frac{1}{3}\frac{d[\ln\rho]}{d[\ln R]} - 1 \\ &w = \frac{n}{3} - 1\end{aligned}$$

Density Parameters

critical density :   density parameters (today) :

$$\rho_c \equiv \frac{3\,H_0^2}{8\pi\,G} \qquad \Omega_R \equiv \frac{\rho_R}{\rho_c} \quad \Omega_M \equiv \frac{\rho_M}{\rho_c} \quad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\,H_0^2}$$

total density parameter today :

$$\boxed{\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda}$$

density at a past/future epoch in units of today's critical density :

$$\boxed{\Omega \equiv \frac{\rho}{\rho_c} = \sum_w \Omega_w\,x^{3(1+w)} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda} \qquad x \equiv 1 + z = R_0/R$$

in units of critical density at the past/future epoch :

$$\Omega(x) = \frac{8\pi\,G\rho}{3H^2} = \frac{H_0^2}{H^2}\sum_w \Omega_w\,x^{3(1+w)} = \frac{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda}{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2}$$

Note: radiation dominates at high z, can be neglected at lower z.

Hubble Parameter Evolution -- H(z)

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\,G}{3}\rho + \frac{\Lambda}{3} - \frac{k\,c^2}{R^2}$$

$$\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda - \frac{k\,c^2}{H_0^2 R_0^2}x^2$$

evaluate at $x = 1 \quad \to \quad 1 = \Omega_0 - \dfrac{k\,c^2}{H_0^2 R_0^2}$

Dimensionless Friedmann Equation:

$$\boxed{\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)\,x^2}$$

Curvature Radius today:

$$\boxed{R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}} \to \begin{cases} k = +1 & \Omega_0 > 1 \\ k = 0 & \Omega_0 = 1 \\ k = -1 & \Omega_0 < 1 \end{cases}}$$

Density determines Geometry

$$x = 1 + z = R_0/R$$

$$\rho_c = \frac{3\,H_0^2}{8\pi\,G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c},\ \ \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\,H_0^2}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$
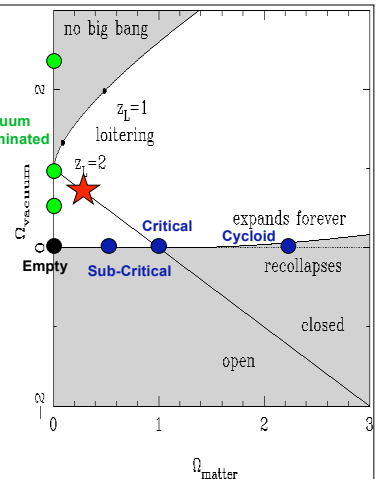
Possible Universes

$$\boxed{\begin{aligned}H_0 &\approx 70\ \frac{\text{km/s}}{\text{Mpc}} \\ \Omega_M &\sim 0.3 \\ \Omega_\Lambda &\sim 0.7 \\ \Omega_R &\sim 8\times 10^{-5} \\ \Omega &= 1.0\end{aligned}}$$