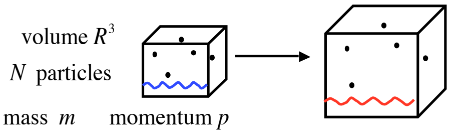
Lecture 15

Early Universe

Thermal History

Energy Density of expanding box



particle mass m momentum p

energy $E = h\nu = \sqrt{m^2c^4 + p^2c^2} = m\,c^2 + \frac{p^2}{2m} + ...$

Assuming that N is conserved:

Cold Matter: ($m > 0,\ p << mc$)

$E \approx m\,c^2 = \text{const}$

$$\varepsilon_M \approx \frac{N\,m\,c^2}{R^3} \propto R^{-3}$$

Radiation: ($m = 0$)
Hot Matter: ($m > 0,\ p >> mc$)

$\lambda \propto R$ (wavelengths stretch):

$E = h\,\nu = \frac{h\,c}{\lambda} \propto R^{-1}$

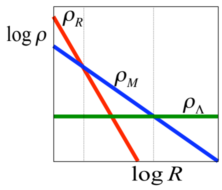
$$\varepsilon_R = \frac{N\,h\,\nu}{R^3} \propto R^{-4}$$

3 Eras: Radiation… Matter…Vacuum

radiation: $\rho_R \propto R^{-4}$

matter: $\rho_M \propto R^{-3}$

vacuum: $\rho_\Lambda = \text{const}$

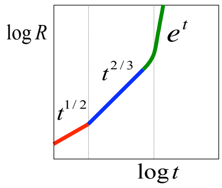


$x \equiv 1 + z = \frac{R_0}{R}$

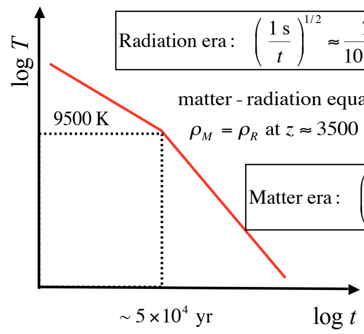
$\rho = \rho_c \left(\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda \right)$

$\rho_R = \rho_M$ at $x = \frac{\Omega_M}{\Omega_R} = \frac{0.3}{8.5\times10^{-5}} \sim 3500$

$\rho_M = \rho_\Lambda$ at $x = \left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{1/3} = \left(\frac{0.7}{0.3}\right)^{1/3} \approx 1.3$



Cooling History: T(t)



Radiation era: $\left(\frac{1\text{ s}}{t}\right)^{1/2} \approx \frac{T}{10^{10}\text{K}} \approx \frac{k\,T}{\text{MeV}}$

matter - radiation equality: $\rho_M = \rho_R$ at $z \approx 3500$

Matter era: $\left(\frac{10^{10}\text{ yr}}{t}\right)^{2/3} = \frac{T}{2.7\text{ K}}$

Relativistic Pairs

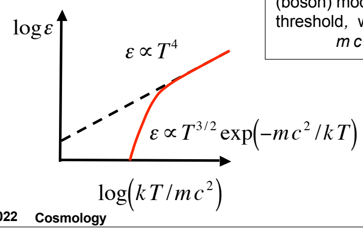
Relativistic particle-antiparticle pairs augment thermal radiation background.

Particle-antiparticle pairs created when E > 2 m c²

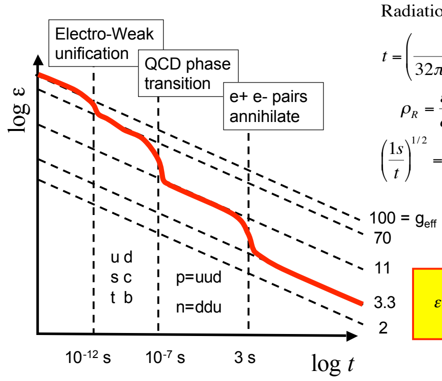
Energy density of pairs "switches on" at the threshold, when kT > m c²

Effective number of relativistic particle species:

g$g_{eff}(T)$ = number of particle (boson) modes above threshold, with $m\,c^2 << k\,T$



Early Cooling History:



$1 + z = \frac{R_0}{R} = \frac{T}{T_0}$

Radiation era:

$t = \left(\frac{3}{32\pi\,G\rho_R}\right)^{1/2}$

$\rho_R = \frac{\varepsilon_R}{c^2} \propto g_{eff}\,T^4$

$\left(\frac{1s}{t}\right)^{1/2} = \left(\frac{T}{10^{10.26}K}\right)\left(g_{eff}\right)^{1/4}$

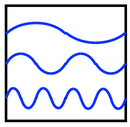
$\varepsilon \propto T^4 \propto R^{-4} \propto t^{-2}$

Significant Events

Event	T	kT	g_{eff}	z	t
Now	2.7 K	0.0002 eV	3.3	0	13 Gyr
First Galaxies	16 K	0.001 eV	3.3	5	1 Gyr
Recombination	3000 K	0.3 eV	3.3	1100	300,000 yr
$\rho_M = \rho_R$	9500 K	0.8 eV	3.3	3500	50,000 yr
$e^+ e^-$ pairs	$10^{9.7}$ K	0.5 MeV	11	$10^{9.5}$	3 s
Nucleosynthesis	10^{10} K	1 MeV	11	10^{10}	1 s
Nucleon pairs	10^{13} K	1 GeV	70	10^{13}	10^{-7} s
E-W unification	$10^{15.5}$ K	250 GeV	100	10^{15}	10^{-12} s
Grand unification	10^{28} K	10^{15} GeV	100(?)	10^{28}	10^{-36} s
Quantum gravity	10^{32} K	10^{19} GeV	100(?)	10^{32}	10^{-43} s

Thermal Equilibrium

Waves in a box.



Density of states in 6-D phase space.

$$\lambda = \frac{L}{n} \qquad k = \frac{2\pi}{\lambda} = n\,\Delta k \qquad \Delta k = \frac{2\pi}{L}$$

$$\frac{dn}{d^3\mathbf{k}\; d^3\mathbf{x}} = \frac{g}{L^3 \Delta k^3} = \frac{g}{(2\pi)^3} \qquad d^3\mathbf{k} = 4\pi k^2 dk \qquad p = \hbar k$$

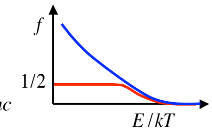
Particle density:

$$\frac{dN}{d^3\mathbf{x}} = \int \frac{g\, f\, d^3\mathbf{k}}{(2\pi)^3} = \frac{g}{(2\pi\hbar)^3} \int f(p)\, 4\pi p^2\, dp$$

Thermal equilibrium occupation number.

$$f = \frac{1}{\exp(E/kT) \pm 1} \qquad \begin{matrix} + & \text{for fermions} \\ - & \text{for bosons} \end{matrix}$$

$$E = \left(p^2c^2 + m^2c^4\right)^{1/2} \Rightarrow \begin{cases} pc & p >> mc \\ mc^2 + \dfrac{p^2}{2m} & p << mc \end{cases}$$



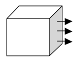
Thermal Equilibrium

Particle density: $$n = \frac{g}{(2\pi\hbar)^3} \int \frac{4\pi p^2 dp}{\exp(E/kT) \pm 1}$$

Energy density: $$\varepsilon = \frac{g}{(2\pi\hbar)^3} \int \frac{E\;\, 4\pi p^2 dp}{\exp(E/kT) \pm 1} \qquad E = \left(p^2c^2 + m^2c^4\right)^{1/2}$$

Pressure : $$P \equiv \frac{dp}{dA\,dt} = \frac{1}{3} n \langle p\, v \rangle = \frac{1}{3} n \left\langle \frac{p^2c^2}{E} \right\rangle \qquad v \equiv \frac{pc^2}{E}$$

Sanity check:
v => c
v => p / m



$$P = \frac{g}{(2\pi\hbar)^3} \int \left(\frac{p^2c^2}{3E}\right) \frac{4\pi p^2 dp}{\exp(E/kT) \pm 1}$$

Entropy :

$$dE = T\,dS - P\,dV$$

$$\frac{E}{V} dV + \frac{\partial E}{\partial T} dT = T\left(\frac{S}{V} dV + \frac{\partial S}{\partial T} dT\right) - P\,dV$$

$$E = TS - PV \qquad S = \frac{E + PV}{T} \qquad s \equiv \frac{S}{V} = \frac{\varepsilon + P}{T}$$

Relativistic Limit

$$kT >> mc^2 \qquad E \Rightarrow pc \qquad y \equiv pc/kT$$

Particle density:

$$n \Rightarrow \frac{g}{(2\pi\hbar)^3} \int \frac{4\pi p^2 dp}{\exp(pc/kT) \pm 1} = \frac{4\pi g}{(2\pi\hbar)^3} \left(\frac{kT}{c}\right)^3 \int \frac{y^2 dy}{e^y \pm 1}$$

Energy density:

$$\varepsilon \Rightarrow \frac{4\pi\, g}{(2\pi\hbar)^3} \frac{(kT)^4}{c^3} \int \frac{y^3 dy}{e^y \pm 1} = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \begin{cases} 7/8 & \text{fermions} \\ 1 & \text{bosons} \end{cases}$$

Pressure : $$P = \frac{1}{3}\varepsilon$$

Entropy : $$\frac{s}{k} = \frac{\varepsilon + P}{kT} = \frac{4}{3} \frac{\varepsilon}{kT} = 3.602\, n \begin{cases} 3/4 & \text{fermions} \\ 1 & \text{bosons} \end{cases}$$

$$\varepsilon \propto g T^4 \qquad w \equiv P/\varepsilon = 1/3$$
$$n \propto g T^3 \qquad s \propto g T^3$$

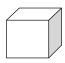
Relativistic fermions behave (almost) like photons.

Photon / Baryon ratio

Photons: $$g = 2 \qquad \varepsilon_\gamma = \frac{\pi^2}{15} \frac{(kT)^4}{(\hbar c)^3} = \frac{0.261 \text{ eV}}{\text{cm}^3} \left(\frac{T}{2.725\text{K}}\right)^4$$

$$\Omega_\gamma = \frac{0.261}{5200} = 5 \times 10^{-5} \qquad x_{M\gamma} = \frac{\Omega_M}{\Omega_\gamma} = \frac{0.3}{5 \times 10^{-5}} = 6000$$

$$n_\gamma = \frac{411}{\text{cm}^3} \left(\frac{T}{2.725\text{K}}\right)^3$$



Baryons: $$\varepsilon_b = \Omega_b \frac{3H_0^2c^2}{8\pi G} = 0.04 \frac{5200 \text{ eV}}{\text{cm}^3} \left(\frac{h}{0.7}\right)^2 = \frac{210 \text{ eV}}{\text{cm}^3} \left(\frac{h}{0.7}\right)^2$$

$$n_b = \frac{\varepsilon_b}{E_b} = \frac{0.22}{\text{m}^3} \qquad E_b \approx m_p c^2 = 939 \text{ MeV}$$

Photons/Baryon : $$\eta \equiv \frac{n_\gamma}{n_b} = \frac{411}{2.2 \times 10^{-7}} = 2 \times 10^9 \left(\frac{\Omega_b}{0.04}\right)^{-1} \left(\frac{h}{0.7}\right)^{-2}$$

How does η scale with redshift ?

Fermions vs Bosons

Relativistic limit: $kT >> mc^2 \qquad E \Rightarrow pc \qquad y \equiv pc/kT$

$$n \Rightarrow \frac{4\pi\, g}{(2\pi\hbar)^3} \left(\frac{kT}{c}\right)^3 \int \frac{y^2 dy}{e^y \pm 1} \qquad \varepsilon \Rightarrow \frac{4\pi\, g}{(2\pi\hbar)^3} \frac{(kT)^4}{c^3} \int \frac{y^3 dy}{e^y \pm 1}$$

$$\frac{1}{e^x + 1} = \frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1} \qquad \frac{n_F(T)}{g_F} = \frac{n_B(T) - 2\,n_B(T/2)}{g_B}$$

Trick:
Fermions at T behave like bosons at T minus twice bosons at T/2.

$$\frac{n_F(T)/g_F}{n_B(T)/g_B} = 1 - 2\left(\frac{T/2}{T}\right)^3 = 1 - \frac{2}{8} = \frac{3}{4}$$

$$\frac{\varepsilon_F(T)/g_F}{\varepsilon_B(T)/g_B} = 1 - 2\left(\frac{T/2}{T}\right)^4 = 1 - \frac{2}{16} = \frac{7}{8}$$

$$g_{eff} \equiv \sum_{bosons} g_i + \frac{7}{8} \sum_{fermions} g_j$$

Relativistic Degrees of Freedom

Relativistic limit: $kT >> mc^2 \quad E \Rightarrow pc$

$$\varepsilon_R = \rho_R c^2 = g_{\rm eff}\,\frac{\pi^2}{30}\frac{(kT)^4}{(c\hbar)^3}$$

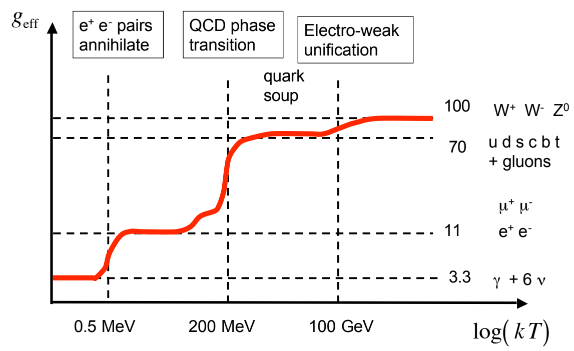
Sum over all **relativistic** fermion and boson degrees of fredom:

$$g_{\rm eff} = \sum_{bosons} g_i + \frac{7}{8}\sum_{fermions} g_j \qquad \frac{n_F}{g_F} = \frac{3}{4}\frac{n_B}{g_B} \qquad \frac{\varepsilon_F}{g_F} = \frac{7}{8}\frac{\varepsilon_B}{g_B}$$

Photons: g = 2 polarizations.
Leptons: g = 2 spins x 3 generations (e, μ, τ)
Neutrinos: g = 1 spin x 3 generations (e, μ, τ)
Quarks: g = 2 spins x 3 colours x 6 flavours (u d s c b t)
Vector bosons: g = 3 spins x 3 (W^+ W^- Z^0)
Gluons: g = 3 colour changes x 8 flavour changes
Higgs g = 1
Particle - antiparticle distinguishable (except photons).

$g_{\rm eff}$ = 2 + 2 x (7/8) x (6 + 3 + 36) + 9 + 24 +1 = 113

Relativistic Degrees of Freedom



Annihilation of e^+ e^- pairs

When does this occur ?

$$kT \sim m_e c^2 = 0.511\ {\rm MeV} \qquad \frac{t}{1{\rm s}} \sim \left(\frac{\rm MeV}{kT}\right)^2 \sim \left(\frac{\rm MeV}{m_e c^2}\right)^2 = \left(\frac{1}{0.511}\right)^2 \sim 4$$

$$g(\gamma) = 2 \qquad g(e^-) = g(e^+) = 2\times\frac{7}{8} \qquad g(\nu) = g(\bar\nu) = 1\times\frac{7}{8}$$

Before : $$g(\gamma + e^+ + e^- + 3(\nu+\bar\nu)) = 2 + \frac{7}{8}(4+6) = \frac{43}{4} = 10.8$$

After : $$g(\gamma + 3(\nu+\bar\nu)) = 2 + 6\times\frac{7}{8}\left(\frac{4}{11}\right)^{4/3} = 3.36$$

Neutrinos cooler than photons after e^+ e^- pairs annihilate:

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} = \frac{1.945\ K}{2.725\ K}$$

Homework problem:

1. Calculate *n(γ)* , the number of CMB photons per unit volume (per cm^3), for the presently observed photon temperature *T(γ)* = 2.725 K
2. Calculate Ω_R for the CMB photons.
3. Calculate x_{RM} at which $\Omega_M x^3 = \Omega_R x^4$ assuming CMB photons only.
4. Calculate the temperature *T(υ)*, and the number density *n(υ)*, of relic neutrinos. By how much do Ω_R and x_{RM} change when neutrinos are included. Assume 3 types of neutrino, and their anti-neutrinos, and note that

$$\frac{T(\nu)}{T(\gamma)} = \left(\frac{4}{11}\right)^{1/3} \qquad \frac{n(\nu)}{n(\gamma)} = \frac{3}{4}\frac{g(\nu)}{g(\gamma)}\left(\frac{T(\nu)}{T(\gamma)}\right)^3 \qquad \frac{\varepsilon(\nu)}{\varepsilon(\gamma)} = \frac{7}{8}\frac{g(\nu)}{g(\gamma)}\left(\frac{T(\nu)}{T(\gamma)}\right)^4$$

$$g(\nu) = 1 \quad g(\gamma) = 2$$