GRAVITY IN A NUTSHELL

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Pedagogical introduction to gravity and general relativity for nonexperts and graduate students in five lectures

P. D. Mannheim, *Alternatives to dark matter and dark energy*, Progress in Particle and Nuclear Physics **56**, 340 (2006). (astro-ph/0505266)

P. D. Mannheim, *Mass generation, the cosmological constant problem, conformal symmetry, and the Higgs boson*, Progress in Particle and Nuclear Physics **94**, 125 (2017). (arXiv:1610.08907 [hep-ph])

P. D. Mannheim, *Is dark matter fact or fantasy? – clues from the data*, International Journal of Modern Physics D **28**, 1944022 (2019). (arXiv:1903.11217 [astro-ph.GA])

P. D. Mannheim, *Determining the normalization of the quantum field theory vacuum, with implications for quantum gravity*, arXiv:2301.13029 [hep-th], Classical and Quantum Gravity **40**, 205007 (2023).

Key reference: S. Weinberg: *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, 1972

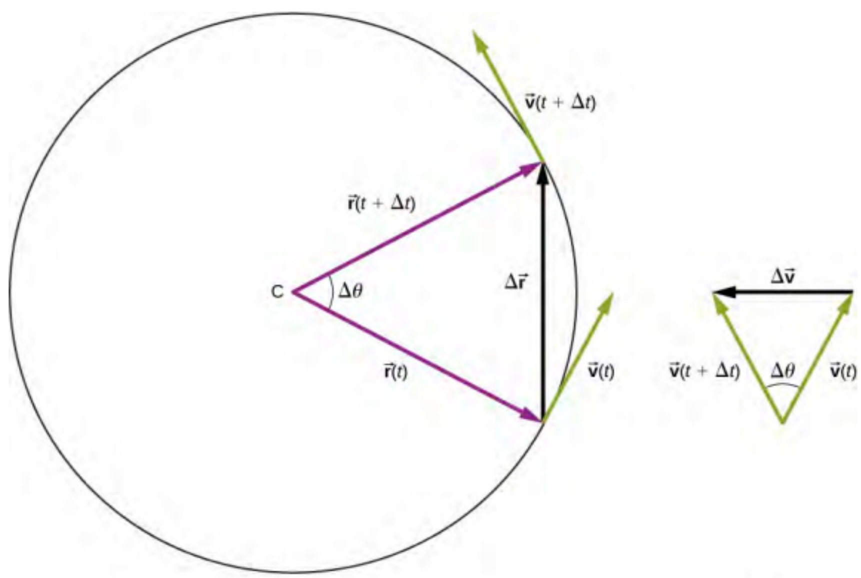
1 Outline

- Newton's Laws of Motion and Gravity: successes and shortcomings.
- No ether, but what would Michelson and Morley have concluded if they did the experiment in the vertical.
- Special Relativity: how to include gravity.
- Curvilinear coordinates: accelerating observers in flat space in the absence of gravity. Mach both right and wrong.
- Newton's Laws of Motion in an accelerating coordinate system in the absence of gravity.
- Need for Levi-Civita connection (Christoffel symbols). Role of metric. Geodesics. Equivalence Principle.
- Riemann curvature. Examples of curved spaces.
- Newton's Laws of Motion in an accelerating coordinate system in the presence of gravity.
- Newton's Law of Gravity in an accelerating coordinate system in the presence of gravity.
- v^2/c^2 gravitational corrections to Newton's Law of Gravity.
- Einstein Equations and gravitational Poisson equation in an accelerating coordinate system.
- Exact all order Schwarzschild solution and black holes.
- Einstein gravity: successes and shortcomings.
- The dark matter problem.
- The dark energy problem.
- The quantum gravity problem.
- Does gravity know about quantum mechanics: Chandrasekhar mass, Cosmic Microwave Background.
- Colella-Overhauser-Werner Experiment.
- Where did the zero-point energy go?

2 Newton's Laws of Motion and Gravity

Newton's Laws of Motion

- (1) constant velocity even if no force. The first law of modern physics, and not just chronologically but also foundationally.
- (2) if force then $\mathbf{F} = m\mathbf{a}$.
- (3) action and reaction equal and opposite.
- First law replaces Aristotle $\mathbf{F} = 0$ implies $\mathbf{v} = 0$.
- Second law requires force even if only a change in direction of velocity and no change in magnitude. Thus circular motion about a center requires a force in direction of change in velocity, i.e. force toward center not along tangent. Ball on a string or planetary orbits. If we replace $\mathbf{v}$ by $\mathbf{v} + \mathbf{v}_0$ where $\mathbf{v}_0$ is a time independent constant, then still have $\mathbf{F} = m\mathbf{a}$. Galilean invariance.
- Third law is conservation of momentum, which generalizes to inelastic processes such as photon or graviton emission.



Newton's Law of Gravity

$$\mathbf{F} = m\mathbf{a} = \frac{mMG_N}{r^2}\hat{\mathbf{a}}, \qquad \hat{\mathbf{a}} = \frac{\mathbf{a}}{a}. \tag{2.1}$$

Recognizes an ordered phenomenon in nature. Universal, all systems use same G_N.

For motion of a particle falling toward the center of the earth, mass of particle drops out and Galileo's Law that all particles fall with same acceleration is recovered. However Newton was concerned that maybe law should be

$$\mathbf{F} = m_i\mathbf{a} = \frac{m_g MG_N}{r^2}\hat{\mathbf{a}} \tag{2.2}$$

with inertial m_i and gravitational m_g, and then mass would not drop out. If masses drop out we have

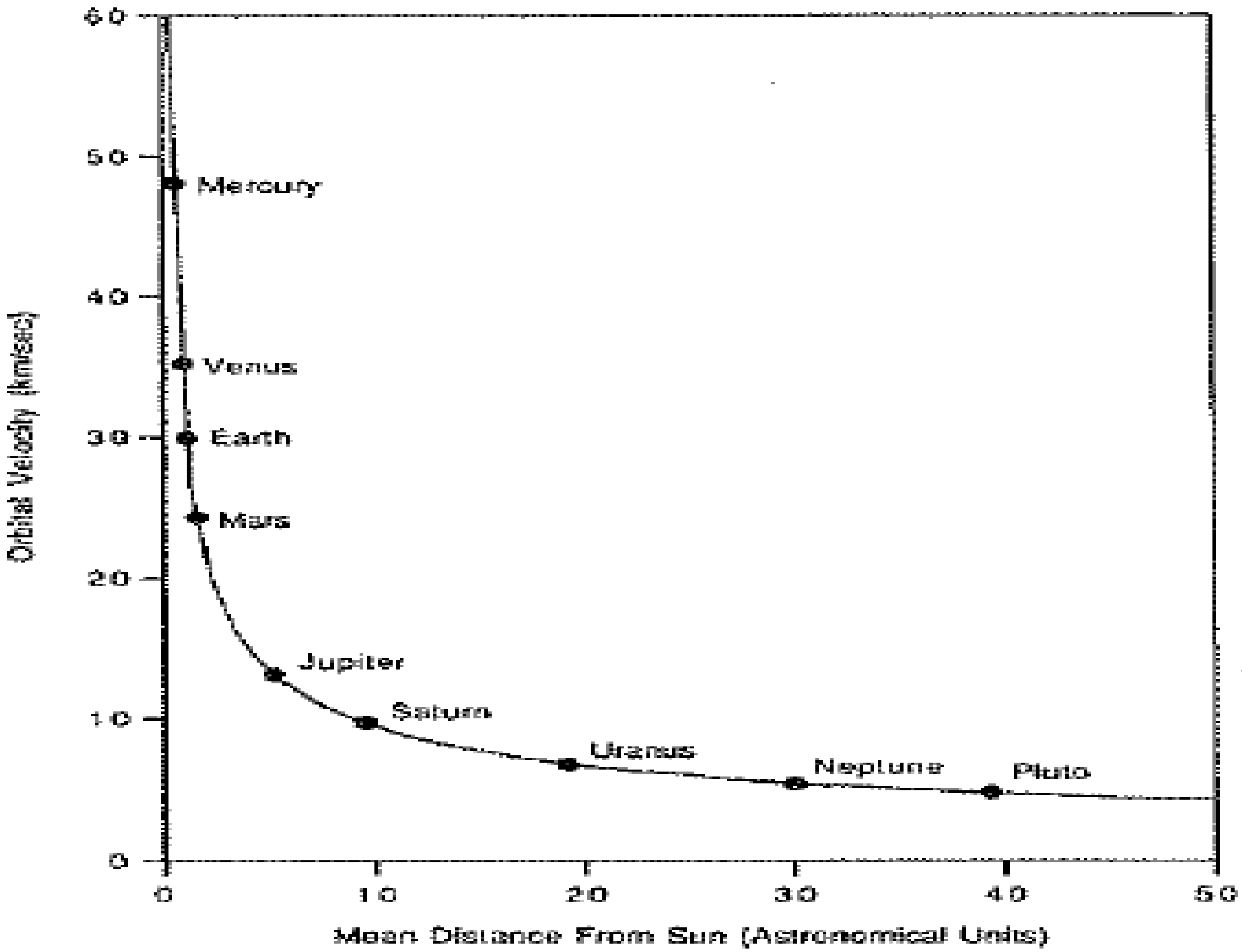
$$\mathbf{a} = \frac{MG_N}{r^2}\hat{\mathbf{a}}. \tag{2.3}$$

If we know G_N (measured by Cavendish) can determine mass of the earth $M_\oplus$ and the mass of the sun $M_\odot$.

Experiment (Eotvos) showed that inertial m_i and gravitational m_g are equal. A key signpost for Einstein.

Successes and Shortcomings

For circular motion $a = v^2/r = MG_N/r^2$ toward the sun. Like ball on a string, but no string. Then get Kepler's Laws; Orbits are ellipses with sun at a focus of ellipse, $v^2 = M_\odot G_N/r$, equal areas in equal times. $T^2 = 4\pi r^3/M_\odot G_N$. Replaces Ptolemy.



Keplerian expectation for planetary orbital velocities – Mercury and Uranus problems

The concern of Mach

Consider

$$\ddot{\mathbf{x}} = 0, \tag{2.4}$$

in an inertial frame.

Set

$$\mathbf{x}' = \mathbf{x} + \frac{1}{2}\mathbf{g}t^2, \tag{2.5}$$

then

$$\ddot{\mathbf{x}}' = \mathbf{g} \tag{2.6}$$

in a noninertial frame.

Mach: local physical laws are determined by the large-scale structure of the universe, i.e. interaction between local and global fixes inertial frames. Mach both wrong and right.

But is gravity just an inertial force? I.e. real or fake?

3 Special Relativity

Maxwell equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0},\quad \boldsymbol{\nabla}\times\boldsymbol{B}-\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}=\mu_0\boldsymbol{J},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0. \tag{3.1}$$

Note: 4+4

In vacuum obtain wave equations

$$\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}-\boldsymbol{\nabla}^2\boldsymbol{E}=0,\qquad \mu_0\epsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}-\boldsymbol{\nabla}^2\boldsymbol{B}=0, \tag{3.2}$$

i.e.,

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}-\boldsymbol{\nabla}^2\boldsymbol{E}=0,\qquad \frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}-\boldsymbol{\nabla}^2\boldsymbol{B}=0,\qquad c=\frac{1}{(\mu_0\epsilon_0)^{1/2}}. \tag{3.3}$$

Thus unify electromagnetism with light and determine the velocity of light.

Problems:

(1) In which frame do we measure c?

(2) Charge at rest produces an electrostatic field. Charge in uniform motion produces a magnetic field. But if keep charge at rest and have an observer move past the charge with a uniform velocity, then what does observer see - electric or magnetic? Sees both (cf. Lorentz force $\mathbf{F} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B}$). Thus $\mathbf{E}$ and $\mathbf{B}$ fields have no independent meaning.

(3) If keep on accelerating with $\mathbf{F} = m\mathbf{a}$ could we eventually go faster than light?

(4) Maxwell equations are not Galilean invariant. They are Lorentz invariant. For x, y, z, t there are three rotations around x, y and z axes, such as

$$x' = \cos\theta x + \sin\theta y, \qquad y' = -\sin\theta x + \cos\theta y, \qquad z' = z \tag{3.4}$$

that leave

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \tag{3.5}$$

invariant. But also three boosts that mix tx, ty and tz, such as

$$ct' = \sinh\theta x + \cosh\theta ct, \qquad x' = \sinh\theta ct + \cosh\theta x, \qquad z' = z, \quad \sinh\theta = v/c(1 - v^2/c^2)^{1/2}, \quad \cosh\theta = 1/(1 - v^2/c^2)^{1/2} \tag{3.6}$$

and leave

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{3.7}$$

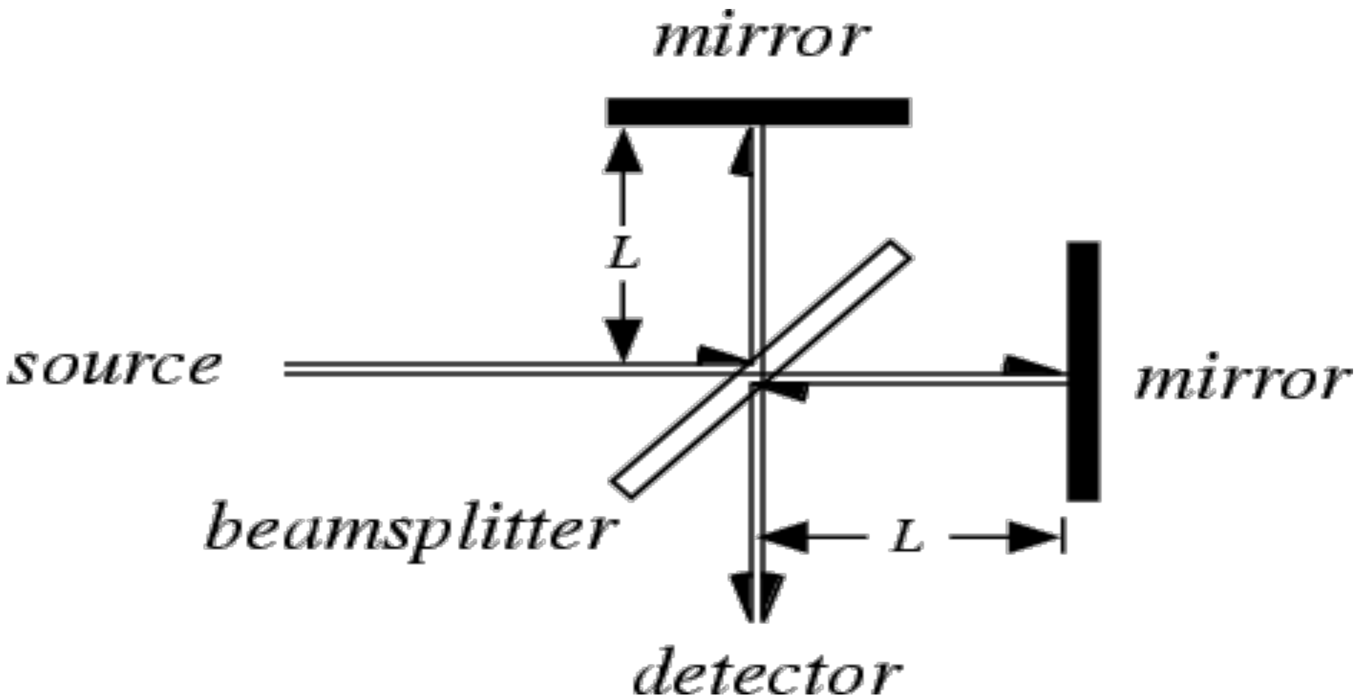
invariant. Similarly,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \boldsymbol{\nabla}'^2, \tag{3.8}$$

while $\mathbf{E}$ and $\mathbf{B}$ mix. Newton's Laws are not Lorentz invariant.

(5) Wave carries energy and momentum. How is this transported. Maxwell answer: there is a mechanical ether.

4 Michelson-Morley Experiment



Michelson-Morley Experiment

If one beam moves parallel to ether and other arm perpendicular to ether should see a fringe shift at detector. None seen. So no ether. However, the vacuum is not what it used to be. It has come a long way since Bernoulli. Quantum field theory vacuum is an ether, just not a mechanical one. We can create particles out of the vacuum, and now we have dark energy.

However, if Michelson and Morley had performed the experiment in the vertical, then light in horizontal arm would sag because of gravitational bending of light (key result of General Relativity), and they would have concluded that they had found the ether.

Einstein: light does not obey $\mathbf{v} \rightarrow \mathbf{v} + \mathbf{v}_0$. Rather it obeys $c \rightarrow c$. But velocity is space over time. So both space and time must vary – Lorentz contraction $L' = L(1 - v^2/c^2)^{1/2}$, and time dilation $t' = t/(1 - v^2/c^2)^{1/2}$. Now observers moving with **uniform** velocity can all describe the same physics. Generalizes Newton' First Law to velocities up to velocity of light.

5 But what happens to Newton's Second Law of Motion?

Einstein: there can only be one invariance law in physics, so make Newton's Laws of Motion be Lorentz invariant, the relativity principle. But relativity requires four-vectors not three-vectors and only have the three-vector $d\mathbf{x}/dt$. So introduce contravariant four-vectors

$$x^\mu = (ct, x, y, z), \quad dx^\mu = (cdt, dx, dy, dz) \tag{5.1}$$

and covariant four-vectors

$$x_\mu = (-ct, x, y, z), \quad dx_\mu = (-cdt, dx, dy, dz). \tag{5.2}$$

We can now form an invariant, the proper time

$$ds^2 = -dx^\mu dx_\mu = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{5.3}$$

with summation over repeated index μ. Thus we now introduce $u^\mu = dx^\mu/ds$, $u_\mu = dx_\mu/ds$ with

$$\frac{dx^\mu}{ds} = \left(\frac{1}{(1-v^2/c^2)^{1/2}}, \frac{v_x}{c(1-v^2/c^2)^{1/2}}, \frac{v_y}{c(1-v^2/c^2)^{1/2}}, \frac{v_z}{c(1-v^2/c^2)^{1/2}}\right), \quad \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \tag{5.4}$$

and obtain

$$u^\mu u_\mu = \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -1, \tag{5.5}$$

i.e., four from three via a constraint.

To take care of the minus sign we introduce a rank two symmetric **METRIC** tensor $\eta_{\mu\nu} = \eta_{\nu\mu}$, with ten independent components

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

and take μ to range over $0, 1, 2, 3$, i.e., t, x, y, z. Then set $x_\mu = \eta_{\mu\nu}x^\nu$, $dx_\mu = \eta_{\mu\nu}dx^\nu$, $s^2 = -\eta_{\mu\nu}x^\mu x^\nu$, $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$.

This seems to be just an inconvenient nuisance. However, Einstein turns $\eta_{\mu\nu}$ into gravity.

Introduce four-momentum

$$p^\mu = mcu^\mu = (E/c, \boldsymbol{p}) = \left(\frac{mc}{(1 - v^2/c^2)^{1/2}}, \frac{m\boldsymbol{v}}{(1 - v^2/c^2)^{1/2}} \right). \tag{5.7}$$

It obeys $-p^\mu p_\mu = m^2c^2 = E^2/c^2 - p^2$, i.e., $E^2 = p^2c^2 + m^2c^4$, and thus $E(\mathbf{p} = 0) = mc^2$. For force we only have a three-force $\mathbf{f} = \mathbf{dp}/dt$. So introduce a four-force

$$g^\mu = \left(\frac{\boldsymbol{f} \cdot \boldsymbol{v}}{c^2(1 - v^2/c^2)^{1/2}}, \frac{\boldsymbol{f}}{c(1 - v^2/c^2)^{1/2}} \right). \tag{5.8}$$

Then with $\mathbf{f} = \mathbf{dp}/dt$ we have

$$mc\frac{d^2x^\mu}{ds^2} = \frac{dp^\mu}{ds} = g^\mu. \tag{5.9}$$

Thus now we have Newton's Second Law of Motion in a form that observers moving with any uniform velocity up to that of light can all agree on. And now no observer can go faster than the velocity of light.

Newton's Third Law of Motion is direct: conservation of the total four-momentum: $\sum dp^\mu/ds = 0$.

To take care of the Maxwell fields we introduce an antisymmetric rank two tensor $F^{\mu\nu} = -F^{\nu\mu}$, with six independent components, just as needed for the three $\mathbf{E}$ fields and three $\mathbf{B}$ fields

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{5.10}$$

To take care of the charge and current we introduce a four-vector $J^\mu = (\rho, \mathbf{J})$, with four independent components and write the Maxwell equations with sources in the form

$$\partial_\nu F^{\mu\nu} = J^\mu, \qquad \epsilon^{\mu\nu\sigma\tau}\partial_\nu F_{\sigma\tau} = 0, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{5.11}$$

where $\epsilon^{\mu\nu\sigma\tau}$ is a fully antisymmetric rank four tensor. So all of its indices have to be different and $\epsilon^{0123} = -\epsilon^{1023} = 1$.

The Lorentz force equation with $\mathbf{F} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B}$ generalizes to

$$\frac{dp^\mu}{ds} = eF^{\mu\nu}\frac{dx_\nu}{ds}. \tag{5.12}$$

6 But observers can accelerate

Consider a free particle obeying

$$\frac{d^2\xi^\alpha}{ds^2} = 0, \qquad ds^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta. \tag{6.1}$$

Now change to some new coordinates x^μ so that the ξ^α depend on the x^μ. (I.e., $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$.) Thus we obtain

$$\frac{d}{ds}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{ds}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{ds^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{6.2}$$

Now multiply by $\partial x^\lambda/\partial\xi^\alpha$, and using the product rule

$$\frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu \tag{6.3}$$

we obtain

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{6.4}$$

where we have introduced the affine connection $\Gamma^\lambda_{\mu\nu}$ defined by

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}. \tag{6.5}$$

Similarly we can write the proper time as

$$ds^2 = -\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta = -\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu = -g_{\mu\nu}dx^\mu dx^\nu, \tag{6.6}$$

where

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}. \tag{6.7}$$

For polar coordinates for instance $-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$, so

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{6.8}$$

Thus the metric can depend on the coordinates, i.e., curvilinear coordinates.

We need to be able to remove all trace of the original ξ^α coordinates and write the connection entirely in terms of the x^μ coordinate system. To this end we evaluate

$$\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \eta_{\alpha\beta}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\lambda}\frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\nu \partial x^\lambda} \\ &= \Gamma^\rho_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\rho_{\lambda\nu}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\eta_{\alpha\beta} \\ &= \Gamma^\rho_{\lambda\mu}g_{\rho\nu} + \Gamma^\rho_{\lambda\nu}g_{\rho\mu}. \end{aligned} \tag{6.9}$$

Thus we obtain

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = \Gamma^\rho_{\lambda\mu}g_{\rho\nu} + \Gamma^\rho_{\lambda\nu}g_{\rho\mu} + \Gamma^\rho_{\mu\lambda}g_{\rho\nu} + \Gamma^\rho_{\mu\nu}g_{\rho\lambda} - \Gamma^\rho_{\nu\mu}g_{\rho\lambda} - \Gamma^\rho_{\nu\lambda}g_{\rho\mu} = 2\Gamma^\rho_{\lambda\mu}g_{\rho\nu}. \tag{6.10}$$

Now introduce an inverse metric that obeys

$$g^{\nu\sigma}g_{\kappa\nu} = \delta^\sigma_\kappa. \tag{6.11}$$

Thus we obtain

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{6.12}$$

With this form for the connection

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{6.13}$$

is known as the geodesic equation. All accelerating observers agree on this. No need for Mach.

7 Polar coordinate example

With $ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$ and

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix} \tag{7.1}$$

we obtain

$$\begin{aligned} &\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^r_{\phi\phi} = -r\sin^2\theta, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \\ &\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta. \end{aligned} \tag{7.2}$$

Equations of motion that follow from

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{7.3}$$

are

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = 0, \qquad \ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} = 0, \qquad \ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{7.4}$$

when $v^2/c^2 \ll 1$.

When $\theta = \pi/2$ these equations integrate to

$$\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} = 0, \qquad r^2\dot{\phi} = J, \qquad \ddot{r} - r\dot{\phi}^2 = \ddot{r} - \frac{J^2}{r^3} = 0, \qquad \frac{\dot{r}^2}{2} + \frac{J^2}{2r^2} = E, \tag{7.5}$$

to give conservation of angular momentum J and energy E.

8 Transformations

Rotation: write

$$x'^1 = x^1 \cos\theta + x^2 \sin\theta, \quad x'^2 = -x^1 \sin\theta + x^2 \cos\theta, \quad x'^3 = x^3 \tag{8.1}$$

in form $x'^i = R^i{}_j x^j$, i.e.,

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \partial x'^1/\partial x^1 & \partial x'^1/\partial x^2 & \partial x'^1/\partial x^3 \\ \partial x'^2/\partial x^1 & \partial x'^2/\partial x^2 & \partial x'^2/\partial x^3 \\ \partial x'^3/\partial x^1 & \partial x'^3/\partial x^2 & \partial x'^3/\partial x^3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{8.2}$$

We can also write $R^i{}_j$ as

$$R^i{}_j = \frac{\partial x'^i}{\partial x^j}. \tag{8.3}$$

Any quantity $A(x)$ that transforms as $A(x') = A(x)$ is known as a scalar, or rank zero tensor (cf. $(x^1)^2 + (x^2)^2 + (x^3)^2 = (x'^1)^2 + (x'^2)^2 + (x'^3)^2$). Any quantity $A^i(x)$ that transforms as $A^{i\prime}(x') = R^i{}_j A^j(x)$ is known as a vector, or rank one tensor (cf. dipole moment). Any quantity $A^{ij}(x)$ that transforms as $A'^{ij}(x') = R^i{}_k R^j{}_\ell A^{k\ell}(x)$ is known as a rank two tensor (cf. quadrupole moment or Maxwell stress tensor).

Lorentz transformation: write

$$x'^0 = x^0 \cosh\theta + x^1 \sinh\theta, \quad x'^1 = x^0 \sinh\theta + x^1 \cosh\theta, \quad x'^2 = x^2, \quad x'^3 = x^3 \tag{8.4}$$

in form $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$, i.e.,

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \partial x'^0/\partial x^0 & \partial x'^0/\partial x^1 & \partial x'^0/\partial x^2 & \partial x'^0/\partial x^3 \\ \partial x'^1/\partial x^0 & \partial x'^1/\partial x^1 & \partial x'^1/\partial x^2 & \partial x'^1/\partial x^3 \\ \partial x'^2/\partial x^0 & \partial x'^2/\partial x^1 & \partial x'^2/\partial x^2 & \partial x'^2/\partial x^3 \\ \partial x'^3/\partial x^0 & \partial x'^3/\partial x^1 & \partial x'^3/\partial x^2 & \partial x'^3/\partial x^3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{8.5}$$

We can also write $\Lambda^\mu{}_\nu$ as

$$\Lambda^\mu{}_\nu = \frac{\partial x^{\mu\prime}}{\partial x^\nu}. \tag{8.6}$$

Any quantity $A(x)$ that transforms as $A(x') = A(x)$ is known as a scalar, or rank zero tensor (cf. $\eta_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = (-(x'^0)^2 + (x'^1)^2 + (x'^2)^2 + (x'^3)^2 = g_{\mu\nu} x'^\mu x'^\nu)$. Any quantity $A^\mu(x)$ that transforms as $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$ is known as a vector, or rank one tensor (cf. electromagnetic current J^μ). Any quantity $A^{\mu\nu}(x)$ that transforms as $A'^{\mu\nu}(x') = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\rho A^{\kappa\rho}(x)$ is known as a rank two tensor (cf. $F^{\mu\nu}$, $g^{\mu\nu}$, or energy-momentum tensor $T^{\mu\nu}$).

In the event that every element of $\Lambda^{\mu}{}_{\nu}$ is independent of the x^{μ} the transformations are **LINEAR**, and are changes involving uniform velocity observers, just like Newton's First Law of Motion. In this case the metric is independent of the coordinates.

In the event that any element of $\Lambda^{\mu}{}_{\nu}$ depends on the x^{μ} the transformations are **NONLINEAR** (cf. $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$), and are changes involving nonuniform velocity observers, such as accelerating observers. These transformations are known as **GENERAL COORDINATE TRANSFORMATIONS**. In this case the metric depends on the coordinates.

While quantities such as dx^{λ}/ds and $g_{\mu\nu}$ transform as vectors and rank two tensors under general coordinate transformations, neither d^2x^{λ}/ds^2 or the connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right) \tag{8.7}$$

transform as vectors or rank three tensors under general coordinate transformations. However, as we now show the geodesic with the specific relative weight

$$\frac{d^2x^{\lambda}}{ds^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0 \tag{8.8}$$

is a general coordinate vector.

Given an arbitrary $g_{\mu\nu}$ how can we determine whether or not it is a transformed $\eta_{\mu\nu}$. Answer given by Riemann tensor

$$R^{\lambda}{}_{\mu\nu\kappa} = \partial_{\kappa}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\kappa} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\eta\nu}. \tag{8.9}$$

and it is a rank four general coordinate tensor. Hence, if even as few as just one component of $R^{\lambda}{}_{\mu\nu\kappa}$ does not vanish in any given coordinate system, $R^{\lambda}{}_{\mu\nu\kappa}$ does not vanish in any coordinate system. Riemann: a space is flat if and only if all components of $R^{\lambda}{}_{\mu\nu\kappa} = 0$. Thus $R^{\lambda}{}_{\mu\nu\kappa} \neq 0$ means space is intrinsically curved. Einstein: This is **GRAVITY**.

9 Tensor analysis

Given that in a coordinate system x^μ we have

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}, \tag{9.1}$$

then in a coordinate system x'^μ we have

$$\begin{aligned}
\Gamma'^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x'^\mu \partial x'^\nu} \\
&= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial \xi^\alpha}{\partial x^\sigma}\right) \\
&= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\left(\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial^2 \xi^\alpha}{\partial x^\sigma \partial x^\kappa} + \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}\frac{\partial \xi^\alpha}{\partial x^\sigma}\right).
\end{aligned} \tag{9.2}$$

Thus

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\Gamma^\rho_{\sigma\kappa} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}. \tag{9.3}$$

The first term is what is required of a tensor, the second term is not. Thus $\Gamma'^\lambda_{\mu\nu}$ is not a rank three tensor.

Noting that

$$\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \delta^\lambda_\nu, \tag{9.4}$$

on differentiating with respect to x'^μ we obtain

$$\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma} = 0, \tag{9.5}$$

so that

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\Gamma^\rho_{\sigma\kappa} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\Gamma^\rho_{\sigma\kappa} - \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}, \tag{9.6}$$

and

$$\Gamma'^\lambda_{\mu\nu}\frac{dx'^\mu}{ds}\frac{dx'^\nu}{ds} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{dx^\sigma}{ds}\frac{dx^\kappa}{ds}\Gamma^\rho_{\sigma\kappa} - \frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}. \tag{9.7}$$

For the acceleration we note that

$$\frac{d^2 x'^\lambda}{ds^2} = \frac{d}{ds}\left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{dx^\rho}{ds}\right) = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{d^2 x^\rho}{ds^2} + \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\kappa}\frac{dx^\rho}{ds}\frac{dx^\kappa}{ds}. \tag{9.8}$$

Thus finally we obtain

$$\frac{d^2 x'^\lambda}{ds^2} + \Gamma'^\lambda_{\mu\nu}\frac{dx'^\mu}{ds}\frac{dx'^\nu}{ds} = \frac{\partial x'^\lambda}{\partial x^\rho}\left(\frac{d^2 x^\rho}{ds^2} + \Gamma^\rho_{\sigma\kappa}\frac{dx^\sigma}{ds}\frac{dx^\kappa}{ds}\right). \tag{9.9}$$

Thus the geodesic equation is a general coordinate vector equation.

Thus for Lorentz force

$$m\left(\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) = eF^\lambda{}_\sigma\frac{dx^\sigma}{ds}. \tag{9.10}$$

10 Covariant derivative of a contravariant vector

Since $\Gamma^{\lambda}_{\mu\nu} = (1/2)g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$ is not a tensor but $g_{\mu\nu}$ is, $\partial_\mu = \partial/\partial x^\mu$ cannot act as a vector (except as we see below when it acts on a scalar). So what do we do with $\partial_\mu V^\nu$ where V^μ is a contravariant vector that obeys

$$V'^{\lambda}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} V^{\rho}(x). \tag{10.1}$$

Differentiating gives

$$\frac{\partial V'^{\lambda}}{\partial x'^{\mu}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial V^{\rho}}{\partial x^{\kappa}} + \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\kappa}}\frac{\partial x^{\kappa}}{\partial x'^{\mu}} V^{\rho}. \tag{10.2}$$

Next, evaluating

$$\begin{aligned}\Gamma'^{\lambda}_{\mu\nu} V'^{\nu} &= \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial x^{\kappa}}{\partial x'^{\mu}}\Gamma^{\rho}_{\sigma\kappa} - \frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\kappa}}\right)\frac{\partial x'^{\nu}}{\partial x^{\tau}} V^{\tau} \\ &= \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\kappa}}{\partial x'^{\mu}}\Gamma^{\rho}_{\sigma\kappa} V^{\sigma} - \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\kappa}} V^{\rho}\end{aligned} \tag{10.3}$$

Thus we obtain

$$\frac{\partial V'^{\lambda}}{\partial x'^{\mu}} + \Gamma'^{\lambda}_{\mu\nu} V'^{\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\kappa}}{\partial x'^{\mu}}\left(\frac{\partial V^{\rho}}{\partial x^{\kappa}} + \Gamma^{\rho}_{\sigma\kappa} V^{\sigma}\right). \tag{10.4}$$

Hence

$$\nabla_\kappa V^{\rho} = \frac{\partial V^{\rho}}{\partial x^{\kappa}} + \Gamma^{\rho}_{\kappa\sigma} V^{\sigma} \tag{10.5}$$

is a rank two tensor.

11 Covariant derivative of a covariant vector

A covariant vector transforms as

$$V'_\nu(x') = \frac{\partial x^\rho}{\partial x'^\nu} V_\rho(x). \tag{11.1}$$

Differentiating gives

$$\frac{\partial V'_\nu}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial V_\rho}{\partial x^\kappa} + \frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\mu} V_\rho. \tag{11.2}$$

Next, evaluating

$$\begin{aligned}
\Gamma'^\lambda_{\mu\nu} V'_\lambda &= \left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\Gamma^\rho_{\sigma\kappa} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right)\frac{\partial x^\tau}{\partial x'^\lambda} V_\tau \\
&= \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\Gamma^\rho_{\sigma\kappa} V_\rho + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho
\end{aligned} \tag{11.3}$$

Thus we obtain

$$\frac{\partial V'_\nu}{\partial x'^\mu} - \Gamma'^\lambda_{\mu\nu} V'_\lambda = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\kappa}{\partial x'^\mu}\left(\frac{\partial V_\rho}{\partial x^\kappa} - \Gamma^\sigma_{\kappa\rho} V_\sigma\right). \tag{11.4}$$

Hence

$$\nabla_\kappa V_\rho = \frac{\partial V_\rho}{\partial x^\kappa} - \Gamma^\sigma_{\kappa\rho} V_\sigma \tag{11.5}$$

is a rank two tensor.

Thus Maxwell equations covariantize to

$$\nabla_\nu F^{\mu\nu} = J^\mu, \qquad \epsilon^{\mu\nu\sigma\tau}\nabla_\nu F_{\sigma\tau} = 0. \tag{11.6}$$

12 Generalizations and the special role of the metric

$$\begin{aligned}\nabla_\lambda T^{\mu\nu} &= \partial_\lambda T^{\mu\nu} + \Gamma^\mu_{\lambda\sigma} T^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma} T^{\mu\sigma},\\ \nabla_\lambda T^\mu{}_\nu &= \partial_\lambda T^\mu{}_\nu + \Gamma^\mu_{\lambda\sigma} T^\sigma{}_\nu - \Gamma^\sigma_{\lambda\nu} T^\mu{}_\sigma,\\ \nabla_\lambda T_{\mu\nu} &= \partial_\lambda T_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} T_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} T_{\mu\sigma}.\end{aligned} \tag{12.1}$$

While rule holds for every index of a general tensor such as for instance $T^{\mu\nu\sigma\tau}{}_{\alpha\beta\gamma\delta}$, there is a special case, viz. no indices. Thus for a scalar S we have $\nabla_\mu S = \partial_\mu S$.

For the metric we have

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma}. \tag{12.2}$$

Now previously we had shown that

$$\begin{aligned}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \eta_{\alpha\beta}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\lambda}\frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\nu \partial x^\lambda}\\ &= \Gamma^\rho_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\rho_{\lambda\nu}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\eta_{\alpha\beta}\\ &= \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\rho\mu}.\end{aligned} \tag{12.3}$$

and shown that this relation is satisfied by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{12.4}$$

Thus the Cartesian coordinate relation $\partial_\lambda \eta_{\mu\nu} = 0$ generalizes to the general coordinate relation

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} = 0. \tag{12.5}$$

Similarly, covariant derivatives are distributive. Thus since we have $\partial_\lambda[V^\mu W^\nu] = \partial_\lambda[V^\mu]W^\nu + V^\mu\partial_\lambda[W^\nu]$ in the Cartesian case, in the general case we have

$$\nabla_\lambda[V^\mu W^\nu] = \nabla_\lambda[V^\mu]W^\nu + V^\mu\nabla_\lambda[W^\mu]. \tag{12.6}$$

Also we have

$$\nabla_\lambda[V_\mu] = \nabla_\lambda[g_{\mu\nu}V^\nu] = \nabla_\lambda[g_{\mu\nu}]V^\nu + g_{\mu\nu}\nabla_\lambda[V^\nu] = g_{\mu\nu}\nabla_\lambda[V^\nu]. \tag{12.7}$$

Thus we can move the metric in and out of covariant derivatives.

Since

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{12.8}$$

we have

$$g^{\mu\sigma}g^{\nu\tau}\nabla_\lambda g_{\mu\nu} = \nabla_\lambda[g^{\mu\sigma}g^{\nu\tau}g_{\mu\nu}] = \nabla_\lambda g^{\sigma\tau} = 0. \tag{12.9}$$

With $V^\mu W_\mu$ being a scalar we have

$$\nabla_\lambda[V^\mu W_\mu] = \partial_\lambda[V^\mu W_\mu] = \partial_\lambda[V^\mu]W_\mu + V^\mu\partial_\lambda[W_\mu], \tag{12.10}$$

but we also have

$$\begin{aligned}
\nabla_\lambda[V^\mu W_\mu] &= \nabla_\lambda[V^\mu]W_\mu + V^\mu\nabla_\lambda[W_\mu] = [\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\nu}V^\nu]W_\mu + V^\mu[\partial_\lambda W_\mu - \Gamma^\nu_{\lambda\mu}W_\nu] \\
&= \partial_\lambda[V^\mu]W_\mu + V^\mu\partial_\lambda[W_\mu] + \Gamma^\mu_{\lambda\nu}V^\nu W_\mu - V^\mu\Gamma^\nu_{\lambda\mu}W_\nu \\
&= \partial_\lambda[V^\mu]W_\mu + V^\mu\partial_\lambda[W_\mu].
\end{aligned} \tag{12.11}$$

We thus see that if the covariant derivative of a contravariant vector contains plus times the connection term, then the covariant derivative of a covariant vector must contain minus times the connection term.

We also use the metric for raising and lowering.

$$V^\mu = g^{\mu\nu} V_\nu, \qquad V_\mu = g_{\mu\nu} V^\nu. \tag{12.12}$$

At any point we can remove the connection. We had shown that

$$\begin{aligned}
\Gamma'^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\kappa}{\partial x'^\mu} \Gamma^\rho_{\sigma\kappa} - \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\sigma \partial x^\kappa} \\
&= \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\kappa}{\partial x'^\mu} \left(\frac{\partial x'^\lambda}{\partial x^\rho} \Gamma^\rho_{\sigma\kappa} - \frac{\partial^2 x'^\lambda}{\partial x^\sigma \partial x^\kappa} \right).
\end{aligned} \tag{12.13}$$

To remove connection at the origin set $x'^\lambda = x^\lambda + (1/2) x^\mu x^\nu [\Gamma^\lambda_{\mu\nu}]_0$ near the origin. This gives

$$\frac{\partial x'^\lambda}{\partial x^\rho} [\Gamma^\rho_{\sigma\kappa}]_0 - \frac{\partial^2 x'^\lambda}{\partial x^\sigma \partial x^\kappa} \to [\Gamma^\lambda_{\sigma\kappa}]_0 - [\Gamma^\lambda_{\sigma\kappa}]_0 = 0. \tag{12.14}$$

13 The determinant

Set $g = -\mathrm{Det}[g_{\mu\nu}]$. For Minkowski with t, x, y, z as the coordinates we have

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^{1/2} = 1; \qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^{1/2} = 1. \tag{13.1}$$

For Minkowski polar with t, r, θ, ϕ as the coordinates we have

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \; g^{1/2} = r^2\sin\theta; \; g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix}, \; g^{1/2} = \frac{1}{r^2\sin\theta}. \tag{13.2}$$

In integrals the measures for the two cases are $\int dtdxdydz$ and $\int r^2\sin\theta dtdrd\theta d\phi$. Thus in both cases we have $\int g^{1/2}d^4x$. So $g^{1/2}$ is the Jacobian. To see why this is we note that

$$g'_{\mu\nu}(x') = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu} g_{\sigma\tau}(x). \tag{13.3}$$

Taking determinants we obtain

$$g' = \left|\frac{\partial x^\sigma}{\partial x'^\mu}\right|\left|\frac{\partial x^\tau}{\partial x'^\nu}\right| g, \qquad g'^{1/2} = J(x, x')g^{1/2}. \tag{13.4}$$

Thus the invariant measure is $\int g^{1/2}d^4x$.

Now we evaluate the contracted

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) = \frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\mu\sigma}. \tag{13.5}$$

Symbolically we need to evaluate $Tr[M^{-1}\partial_\nu M]$. We can work in the diagonal basis in which the eigenvalues of M are on the diagonal. Thus we obtain

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad M^{-1}M' = \begin{pmatrix} 1/\lambda_1 & 0 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 & 0 \\ 0 & 0 & 1/\lambda_3 & 0 \\ 0 & 0 & 0 & 1/\lambda_4 \end{pmatrix} \begin{pmatrix} \lambda'_1 & 0 & 0 & 0 \\ 0 & \lambda'_2 & 0 & 0 \\ 0 & 0 & \lambda'_3 & 0 \\ 0 & 0 & 0 & \lambda'_4 \end{pmatrix}, \qquad \mathrm{Trace}[M^{-1}M'] = \sum_i \frac{\lambda'_i}{\lambda_i} \tag{13.6}$$

Similarly,

$$\mathrm{Det}[M] = \Pi_i \lambda_i, \qquad \partial_\mu \log[\mathrm{Det}[M]] = \sum_i \frac{\lambda'_i}{\lambda_i} = \mathrm{Trace}[M^{-1}M']. \tag{13.7}$$

Thus

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2} \partial_\nu \log g = g^{-1/2} \partial_\nu g^{1/2}. \tag{13.8}$$

Since

$$\nabla_\kappa V^\rho = \frac{\partial V^\rho}{\partial x^\kappa} + \Gamma^{\rho}_{\kappa\sigma} V^\sigma \tag{13.9}$$

we have

$$\nabla_\rho V^\rho = \frac{\partial V^\rho}{\partial x^\rho} + \Gamma^{\rho}_{\rho\sigma} V^\sigma = \frac{\partial V^\rho}{\partial x^\rho} + V^\sigma g^{-1/2} \partial_\sigma g^{1/2} = g^{-1/2} \partial_\sigma \left(g^{1/2} V^\sigma \right). \tag{13.10}$$

For a scalar $\nabla^\sigma S$ is a vector, so that

$$\nabla_\mu \nabla^\mu S = g^{-1/2} \partial_\sigma \left(g^{1/2} g^{\sigma\tau} \partial_\tau S \right). \tag{13.11}$$

Evaluating with $g^{1/2} = r^2 \sin\theta$, $g^{rr} = 1$, $g^{\theta\theta} = 1/r^2$, $g^{\phi\phi} = 1/(r^2 \sin^2\theta)$ for polar coordinates, we obtain

$$\nabla_\mu \nabla^\mu S = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial S}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 S}{\partial \phi^2}. \tag{13.12}$$

14 Gravity- a first look

Having shown that Newton's Laws of Motion have to be modified to make them Lorentz invariant, the obvious thing to do was to stay in flat space and write gravity as an analog of the Lorentz force as written in an accelerating coordinate system in flat space with curvilinear coordinates:

$$m\left(\frac{d^2x^\lambda}{ds^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)=eF^\lambda{}_\sigma\frac{dx^\sigma}{ds},\qquad \Gamma^\lambda_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}\right). \tag{14.1}$$

However that would not lead to $m_i=m_g$. Einstein: if we could pull gravity out of the $\Gamma^\lambda_{\mu\nu}$ term we could get $m_i=m_g$. But then have to replace Lorentz invariance by general coordinate invariance. So look at non-relativistic limit of geodesic

$$m\left(\frac{d^2x^\lambda}{ds^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)=0, \tag{14.2}$$

Taking only the 00 component of $g_{\mu\nu}$ to differ from $\eta_{\mu\nu}$ we obtain

$$ds^2=-g_{\mu\nu}dx^\mu dx^\nu=c^2dt^2[-g_{00}-dr^2/c^2dt^2]=c^2dt^2[-g_{00}-v^2/c^2], \tag{14.3}$$

and thus for $v\ll c$ and $g_{00}=-1+h_{00}$ where h_{00} is small, then to lowest order in h_{00} we can set

$$ds^2=c^2dt^2[1-h_{00}],\qquad \Gamma^r_{00}=-\frac{1}{2}\partial_r h_{00}\qquad m\left[\frac{d^2r}{dt^2}-\frac{c^2}{2}\partial_r h_{00}\right]=0. \tag{14.4}$$

If we now set $h_{00}=-(2/c^2)\phi$ we obtain

$$m\left[\frac{d^2r}{dt^2}+\partial_r\phi\right]=0. \tag{14.5}$$

So finally, if we set $\phi=-MG_N/r$, we obtain Newton's Law of Gravity

$$m\frac{d^2r}{dt^2}=-\frac{mMG_N}{r^2}, \tag{14.6}$$

while establishing that $m_i=m_g=m$.

Concerns: is this metric real or fake. Even if not fake (i.e. cannot get back to $\eta_{\mu\nu}$), can we make it general coordinate invariant, and if so, would we actually recover Newton's Law of Gravity, and could we then get v^2/c^2 correction to Newton. Thus need to ask what Newton's Law of Gravity and the second-order Poisson equation that it satisfies would look like in an accelerated coordinate system. Solution: **curved space**.

15 Examples of curved spaces

$$R^{\lambda}{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\kappa} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\eta\nu}. \tag{15.1}$$

Gauss, Bolyai and Lobachevski

$$ds^2 = dr^2 + r^2 d\theta^2 - dz^2, \qquad z^2 = r^2 + 1, \qquad zdz = rdr$$

$$ds^2 = dr^2 + r^2 d\theta^2 - \frac{r^2 dr^2}{1+r^2} = \frac{dr^2}{1+r^2} + r^2 d\theta^2 = -g_{\mu\nu} dx^\mu dx^\nu. \tag{15.2}$$

$$R_{\lambda\mu\nu\kappa} = -(g_{\mu\nu} g_{\lambda\kappa} - g_{\mu\kappa} g_{\lambda\nu}). \tag{15.3}$$

The 2-dimensional space of (r, θ) has positive signature, but it is embedded in a flat 3-space with Minkowski signature. The 2-space is the space of Gauss, Bolyai and Lobachevski – a space of constant negative curvature, since it is a surface of a hyperboloid in a 3-space, and is thus not flat, since neither a sphere nor a hyperboloid could be transformed into plane. Curvature induced on the 2-space by the embedding. Euclid's fifth axiom: if a line traverses two other lines and the two interior angles that it makes with the two other lines add up to less than 180 degrees, then the two other lines must intersect. It does not hold. So non-Euclidean geometry .

Robertson-Walker

Constrained 4-space

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{15.4}$$

Impose

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2, \qquad x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4 = 0. \tag{15.5}$$

Eliminate x_4

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{[x_1 dx_1 + x_2 dx_2 + x_3 dx_3]^2}{a^2 - x_1^2 + x_2^2 + x_3^2}. \tag{15.6}$$

Rewrite in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \frac{r^2 dr^2}{a^2 - r^2} = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad k = \frac{1}{a^2}. \tag{15.7}$$

Generalize to

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] \tag{15.8}$$

$k > 0$ is 3-space of constant positive curvature, $k = 0$ is 3-space of zero curvature, i.e. 3-flat, $k < 0$ is 3-space of constant negative curvature. $a(t)$ describes the overall temporal evolution of the 3-space. This is the Robertson-Walker metric of modern cosmology associated with an expanding universe.

de Sitter

$$ds^2 = c^2dt^2 - e^{2Hct}\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] = -g_{\mu\nu}dx^\mu dx^\nu. \tag{15.9}$$

$$R_{\lambda\mu\nu\kappa} = H^2(g_{\mu\nu}g_{\lambda\kappa} - g_{\mu\kappa}g_{\lambda\nu}). \tag{15.10}$$

Associated with the inflationary universe, the accelerating universe, and dark energy.

Schwarzschild

$$ds^2 = c^2dt^2\left(1 - \frac{2MG}{c^2r}\right) - \frac{dr^2}{1-2MG/c^2r} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{15.11}$$

Describes Newton's Law of Gravity and gives rise to relativistic corrections, leading to precession of planetary orbits and gravitational bending of light. All nonvanishing components of the Riemann tensor proportional to $-12MG/c^2r^3$.

16 Curvature

What is the status of the Riemann tensor

$$R^{\lambda}{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\kappa} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\eta\nu}. \tag{16.1}$$

Consider $\nabla_\nu V_\mu$. It is a covariant rank two tensor. Thus

$$\nabla_\kappa \nabla_\nu V_\mu = \partial_\kappa \nabla_\nu V_\mu - \Gamma^{\lambda}_{\kappa\nu}\nabla_\lambda V_\mu - \Gamma^{\lambda}_{\kappa\mu}\nabla_\nu V_\lambda. \tag{16.2}$$

Identifying covariant derivatives we can write

$$\begin{aligned}\nabla_\kappa \nabla_\nu V_\mu &= \partial_\kappa\left[\partial_\nu V_\mu - \Gamma^{\lambda}_{\nu\mu}V_\lambda\right] - \Gamma^{\lambda}_{\kappa\nu}\left[\partial_\lambda V_\mu - \Gamma^{\sigma}_{\lambda\mu}V_\sigma\right] - \Gamma^{\lambda}_{\kappa\mu}\left[\partial_\nu V_\lambda - \Gamma^{\sigma}_{\nu\lambda}V_\sigma\right] \\ &= \partial_\kappa\partial_\nu V_\mu - \partial_\kappa\Gamma^{\lambda}_{\nu\mu}V_\lambda - \Gamma^{\lambda}_{\nu\mu}\partial_\kappa V_\lambda - \Gamma^{\lambda}_{\kappa\nu}\left[\partial_\lambda V_\mu - \Gamma^{\sigma}_{\lambda\mu}V_\sigma\right] - \Gamma^{\lambda}_{\kappa\mu}\left[\partial_\nu V_\lambda - \Gamma^{\sigma}_{\nu\lambda}V_\sigma\right].\end{aligned} \tag{16.3}$$

Similarly we can write

$$\nabla_\nu \nabla_\kappa V_\mu = \partial_\nu\partial_\kappa V_\mu - \partial_\nu\Gamma^{\lambda}_{\kappa\mu}V_\lambda - \Gamma^{\lambda}_{\kappa\mu}\partial_\nu V_\lambda - \Gamma^{\lambda}_{\nu\kappa}\left[\partial_\lambda V_\mu - \Gamma^{\sigma}_{\lambda\mu}V_\sigma\right] - \Gamma^{\lambda}_{\nu\mu}\left[\partial_\kappa V_\lambda - \Gamma^{\sigma}_{\kappa\lambda}V_\sigma\right]. \tag{16.4}$$

Thus we obtain

$$\begin{aligned}\nabla_\kappa \nabla_\nu V_\mu - \nabla_\nu \nabla_\kappa V_\mu &= -\partial_\kappa\Gamma^{\lambda}_{\nu\mu}V_\lambda + \partial_\nu\Gamma^{\lambda}_{\kappa\mu}V_\lambda + \Gamma^{\lambda}_{\kappa\mu}\Gamma^{\sigma}_{\nu\lambda}V_\sigma - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\sigma}_{\kappa\lambda}V_\sigma \\ &= -\partial_\kappa\Gamma^{\lambda}_{\nu\mu}V_\lambda + \partial_\nu\Gamma^{\lambda}_{\kappa\mu}V_\lambda + \Gamma^{\eta}_{\kappa\mu}\Gamma^{\lambda}_{\nu\eta}V_\lambda - \Gamma^{\eta}_{\nu\mu}\Gamma^{\lambda}_{\kappa\eta}V_\lambda.\end{aligned} \tag{16.5}$$

Thus finally we obtain

$$\nabla_\kappa \nabla_\nu V_\mu - \nabla_\nu \nabla_\kappa V_\mu = -R^{\lambda}{}_{\mu\nu\kappa}V_\lambda = -R_{\lambda\mu\nu\kappa}V^\lambda. \tag{16.6}$$

We thus establish that covariant derivatives do not commute and that the Riemann tensor is indeed a tensor, i.e., under a general coordinate transformation it transforms as

$$R'_{\lambda\mu\nu\kappa}(x') = \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\frac{\partial x^\delta}{\partial x'^\kappa}R_{\alpha\beta\gamma\delta}(x). \tag{16.7}$$

Thus despite the fact that the connection is not a tensor, the Riemann tensor is indeed a rank four tensor.

17 The significance of the Riemann tensor

$$R^{\lambda}_{\ \mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\eta\kappa} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\eta\nu}, \qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{17.1}$$

If $g_{\mu\nu} = \eta_{\mu\nu}$ then space is flat and $R^{\lambda}_{\ \mu\nu\kappa} = 0$. If $g_{\mu\nu}$ is coordinate equivalent to $\eta_{\mu\nu}$, i.e., if

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta}, \tag{17.2}$$

then $R^{\lambda}_{\ \mu\nu\kappa}$ is still zero. If at least one component of $R^{\lambda}_{\ \mu\nu\kappa}$ is not zero, then metric cannot be coordinate equivalent to a flat metric. Space is then not flat, since $R^{\lambda}_{\ \mu\nu\kappa}$ cannot vanish in any coordinate system. Thus the curved space examples given above really are not flat.

Number of independent components of the Riemann tensor – not 64

Following some algebra rewriting the connection in terms of the metric yields

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu x^\lambda}\right] + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right]. \tag{17.3}$$

Thus we establish

- symmetry: $R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$
- antisymmetry: $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu}$
- cyclicity: $R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$

From the antisymmetry condition the Riemann tensor has to antisymmetric on the first two indices, and antisymmetric on the last two indices. Thus 6x6=36=21+15. But from the symmetry condition it has to be symmetric on the interchange of the first two indices with the last two, so 21. Then the cyclicity condition is a completely antisymmetric condition just one condition, so finally 20 components. In N dimensions would get $N^2(N^2-1)/12$.

The 10-component Ricci tensor and the 1-component Ricci scalar

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = R_{\kappa\mu}, \qquad R = R^{\alpha}{}_{\alpha} = g^{\mu\kappa} R_{\mu\kappa}. \tag{17.4}$$

The Weyl tensor

$$C_{\lambda\mu\nu\kappa} \;=\; R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^{\alpha}{}_{\alpha} \left[g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] - \frac{1}{2} \left[g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right], \tag{17.5}$$

obeys $g^{\lambda\nu} C_{\lambda\mu\nu\kappa} = 0$. It has the property that under a local rescaling of the metric (local conformal transformation) $g_{\mu\nu}(x) \to e^{2\alpha(x)} g_{\mu\nu}(x)$

$$C_{\lambda\mu\nu\kappa} \to \; e^{2\alpha(x)} C_{\lambda\mu\nu\kappa} \tag{17.6}$$

with all derivatives of $\alpha(x)$ dropping out (just like a gauge transformation). Under conformal transformation

$$ds^2 \to e^{2\alpha(x)} ds^2, \quad ds^2 = 0 \to ds^2 = 0, \tag{17.7}$$

so light cone is left invariant.

Geometry is conformal to flat (i.e., $ds^2 = e^{2\alpha(x)}[c^2 dt^2 - dx_1^2 - dx_2^2 - dx_3^2]$), if and only if $C_{\lambda\mu\nu\kappa} = 0$. On introducing the conformal time

$$c d\tau = \int \frac{c dt}{a(t)} \tag{17.8}$$

we can rewrite $k = 0$ Robertson-Walker metric as

$$ds^2 = c^2 dt^2 - a^2(t) \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] = a^2(t) \left[c^2 d\tau^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \right]. \tag{17.9}$$

Thus for Robertson-Walker metric the Weyl tensor vanishes. The dynamics thus only depends on the Ricci tensor and Ricci scalar, just as we will see for cosmology.

18 The Bianchi Identities

As noted above at $x^\mu = 0$ we can make the connection vanish.Thus at that point we obtain

$$\nabla_\eta R_{\lambda\mu\nu\kappa} = \frac{1}{2}\frac{\partial}{\partial_\eta}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu x^\lambda}\right]. \tag{18.1}$$

Evaluation then gives

$$\partial_\eta R_{\lambda\mu\nu\kappa} + \partial_\kappa R_{\lambda\mu\eta\nu} + \partial_\nu R_{\lambda\mu\kappa\eta} = 0. \tag{18.2}$$

Thus in an arbitrary coordinate frame we have

$$\nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\kappa R_{\lambda\mu\eta\nu} + \nabla_\nu R_{\lambda\mu\kappa\eta} = 0. \tag{18.3}$$

On multiplying by $g^{\lambda\nu}$ we obtain

$$\nabla_\eta R_{\mu\kappa} - \nabla_\kappa R_{\mu\eta} + \nabla^\nu R_{\nu\mu\kappa\eta} = 0. \tag{18.4}$$

On multiplying by $g^{\mu\kappa}$ we obtain

$$\nabla_\eta R - \nabla^\mu R_{\mu\eta} - \nabla^\nu R_{\nu\eta} = 0. \tag{18.5}$$

i.e.,

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 0. \tag{18.6}$$

19 The energy-momentum tensor

Consider a general action for a scalar field of the from $I = \int d^4x L(\phi, \partial_\mu \phi)$, with associated Euler-Lagrange equation of motion

$$\partial^\mu \left(\frac{\partial L}{\partial \partial^\mu \phi} \right) - \frac{\partial L}{\partial \phi} = 0. \tag{19.1}$$

Introduce the energy-momentum tensor

$$T_{\mu\nu} = -\partial_\mu \phi \frac{\partial L}{\partial \partial^\nu \phi} + \eta_{\mu\nu} L \tag{19.2}$$

Differentiating and using Euler-Lagrange equation gives

$$\begin{aligned} \partial^\nu T_{\mu\nu} &= -\left(\partial^\nu \partial_\mu \phi\right) \frac{\partial L}{\partial \partial^\nu \phi} - \partial_\mu \phi \partial^\nu \left(\frac{\partial L}{\partial \partial^\nu \phi} \right) + \partial_\mu L \\ &= -\left(\partial^\nu \partial_\mu \phi\right) \frac{\partial L}{\partial \partial^\nu \phi} - \partial_\mu \phi \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial \phi} \partial_\mu \phi + \frac{\partial L}{\partial \partial^\nu \phi} \partial_\mu \partial^\nu \phi = 0. \end{aligned} \tag{19.3}$$

Thus the the energy-momentum tensor is conserved.

For $L = -(1/2)\partial_\mu \phi \partial^\mu \phi - (1/2) m^2 c^2 \phi^2$ (i.e., with $\hbar = 1$), we obtain

$$\begin{aligned} &\partial_\mu \partial^\mu \phi - m^2 c^2 \phi = 0, \\ &T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \left(\frac{1}{2} \partial_\kappa \phi \partial^\kappa \phi + \frac{1}{2} m^2 c^2 \phi^2 \right), \\ &\partial^\mu T_{\mu\nu} = \partial^\mu \partial_\mu \phi \partial_\nu \phi + \partial_\mu \phi \partial^\mu \partial_\nu \phi - \partial_\nu \partial_\kappa \phi \partial^\kappa \phi - m^2 c^2 \partial_\nu \phi \phi = 0. \end{aligned} \tag{19.4}$$

To see the physical meaning of $T_{\mu\nu}$ we note that

$$\begin{aligned} T_{00} &= \dot{\phi}^2 + \frac{1}{2} \left[-\dot{\phi}^2 + \nabla \phi \cdot \nabla \phi + m^2 \phi^2 \right] = \frac{1}{2} \left[\dot{\phi}^2 + \nabla \phi \cdot \nabla \phi + m^2 c^2 \phi^2 \right] \\ &= \text{kinetic energy plus potential energy} \end{aligned} \tag{19.5}$$

20 Perfect fluid

Consider a complex scalar field with action, equation of motion, and energy-momentum tensor of the form

$$\begin{aligned} I_{\mathrm{S}} &= -\int d^4x \frac{1}{2}\left[\partial_\mu \phi^* \partial^\mu \phi + m^2c^2\phi^*\phi\right], \\ \partial_\mu\partial^\mu\phi - m^2c^2\phi &= 0, \qquad \partial_\mu\partial^\mu\phi^* - m^2c^2\phi^* = 0, \\ T_{\mu\nu} &= \frac{1}{2}\left[\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi\right] - \frac{1}{2}\eta_{\mu\nu}\left[\partial^\alpha\phi\partial_\alpha\phi^* + m^2c^2\phi^*\phi\right]. \end{aligned} \tag{20.1}$$

The wave equation has mode solutions of the form $\phi = e^{-i\omega_k t + i\bar{k}\cdot\bar{x}}$, $\phi^* = e^{i\omega_k t - i\bar{k}\cdot\bar{x}}$ where $\omega_k^2/c^2 = \bar{k}^2 + m^2c^2$. The general solution to the wave equation is thus of the form

$$\phi(x) = \sum\left[a_{\bar{k}}e^{-i\omega_k t + i\bar{k}\cdot\bar{x}} + b^*_{\bar{k}}e^{i\omega_k t - i\bar{k}\cdot\bar{x}}\right], \qquad \phi^*(x) = \sum\left[a^*_{\bar{k}}e^{i\omega_k t - i\bar{k}\cdot\bar{x}} + b_{\bar{k}}e^{-i\omega_k t + i\bar{k}\cdot\bar{x}}\right]. \tag{20.2}$$

If we insert this form into $T_{\mu\nu}$ we get a double sum $\sum_{\bar{k}}\sum_{\bar{k}'}$. This is coherent.

However if we add the modes incoherently we only get a single sum $\sum_{\bar{k}}$. I.e., we replace $(A_1 + A_2)^2 = A_1^2 + 2A_1A_2 + A_2^2$ by $A_1^2 + A_2^2$, with no cross term. For a single $a_{\bar{k}}$ mode with $a_{\bar{k}}a^*_{\bar{k}} = c/V\omega_k$ where V is three volume we obtain

$$T_{\mu\nu} = \frac{ck_\mu k_\nu}{V\omega_k}, \tag{20.3}$$

where $k^\mu = (\omega_k/c, k_x, k_y, k_z)$, $k_\mu = (-\omega_k/c, k_x, k_y, k_z)$. Thus for $k^\mu = (\omega_k/c, 0, 0, k)$ and $k^\mu = (\omega_k/c, 0, 0, -k)$ we obtain

$$T_{\mu\nu}(k_z) = \begin{pmatrix} \omega_k/cV & 0 & 0 & k/V \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k/V & 0 & 0 & ck^2/V\omega_k \end{pmatrix}, \qquad T_{\mu\nu}(-k_z) = \begin{pmatrix} \omega_k/cV & 0 & 0 & -k/V \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k/V & 0 & 0 & ck^2/V\omega_k \end{pmatrix} \tag{20.4}$$

Adding them together incoherently gives

$$T_{\mu\nu}(z) = T_{\mu\nu}(k_z) + T_{\mu\nu}(-k_z) = \begin{pmatrix} 2\omega_k/cV & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2ck^2/V\omega_k \end{pmatrix} \tag{20.5}$$

Repeating for modes in the k_x, $-k_x$, k_y, $-k_y$, directions we obtain

$$T_{\mu\nu} = T_{\mu\nu}(x) + T_{\mu\nu}(y) + T_{\mu\nu}(z) = \begin{pmatrix} 6\omega_k/cV & 0 & 0 & 0 \\ 0 & 2ck^2/V\omega_k & 0 & 0 \\ 0 & 0 & 2ck^2/V\omega_k & 0 \\ 0 & 0 & 0 & 2ck^2/V\omega_k \end{pmatrix} \tag{20.6}$$

Defining the energy density $\rho = 6\omega_k/V$, pressure $p = 2c^2k^2/V\omega_k$, and the fluid velocity $u^\mu = (1, 0, 0, 0)$, $u_\mu = (-1, 0, 0, 0)$ as normalized to the timelike $u^\mu u_\mu = \eta_{\mu\nu}u^\mu u^\nu = -1$, we obtain the symmetric

$$T_{\mu\nu} = \begin{pmatrix} \rho/c & 0 & 0 & 0 \\ 0 & p/c & 0 & 0 \\ 0 & 0 & p/c & 0 \\ 0 & 0 & 0 & p/c \end{pmatrix} = \frac{1}{c}\left[(\rho + p)u_\mu u_\nu + p\eta_{\mu\nu}\right]. \tag{20.7}$$

We can now covariantize and obtain the symmetric general coordinate rank two tensor

$$T_{\mu\nu} = T_{\nu\mu} = \frac{1}{c}\left[(\rho + p)u_\mu u_\nu + pg_{\mu\nu},\right], \qquad u^\mu u_\mu = g_{\mu\nu}u^\mu u^\nu = -1, \tag{20.8}$$

where u^μ is a 4-vector and ρ and p are general coordinate scalars. Also $T_{\mu\nu}$ is conserved, and thus obeys

$$\nabla_\mu T^{\mu\nu} = 0. \tag{20.9}$$

21 The Einstein equations

We had seen that we could introduce a weak gravitational potential ϕ by setting $g_{00} = -1 - 2\phi/c^2$, with the point particle action taking the form

$$I = mc \int ds = mc \int [-g_{00}c^2dt^2 + dx^2]^{1/2} = mc^2 \int dt \left[1 + \frac{\phi}{c^2} + \frac{v^2}{2c^2}\right] = \int L dt \tag{21.1}$$

at low velocity. This yields a Lagrangian and Euler-Lagrange equation of the form

$$L = mc^2 + m\phi + \frac{mv^2}{2}, \qquad m\ddot{x} = m\frac{d\phi}{dx} \tag{21.2}$$

viz. the standard nonrelativistic Lagrangian and equation of motion for a particle moving in a gravitational potential. Thus metric is the gravitational field.

We need an equation that fixes the potential. Thus need to generalize $\nabla^2\phi = 4\pi G\rho$ and write it in an accelerated coordinate system. We have an immediate problem: since $\nabla_\kappa g_{\mu\nu} = 0$ we cannot build out of covariant derivative of the metric. However, Ricci tensor is a second derivative function of the metric and it is a tensor. Moreover from Bianchi identity we find

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \qquad \nabla_\mu G^{\mu\nu} = 0. \tag{21.3}$$

Now $T^{\mu\nu}$ obeys $\nabla_\mu T^{\mu\nu} = 0$. Thus we are led to the Einstein equations

$$-\frac{c^3}{8\pi G_N}G_{\mu\nu} = T_{\mu\nu}. \tag{21.4}$$

Thus ten equations for ten unknown components of $g_{\mu\nu}$, which can be solved once a $T_{\mu\nu}$ is specified. However, four general coordinate transformations. Compensated for by four components of the Bianchi identity.

22 The second-order Poisson equation

Try a weak gravity static, spherically symmetric line element of the form

$$ds^2 = c^2 dt^2 (1 + h(\rho)] - (1 + j(\rho)][d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2] \tag{22.1}$$

where $h(\rho)$ and $j(\rho)$ are small. For this metric the nonzero components of the Einstein tensor evaluate to lowest order in $h(\rho)$ and $j(\rho)$ to

$$G_{00} = j'' + \frac{2}{\rho} j', \qquad G_{rr} = -\frac{1}{\rho}[j' + h'], \qquad G_{\theta\theta} = \frac{G_{\phi\phi}}{\sin^2\theta} = -\frac{\rho^2}{2}[j'' + h''] - 2\rho[j' + h'], \tag{22.2}$$

where the prime denotes $d/d\rho$. In the nonrelativistic limit with a mattter energy-momentum tensor $(\rho_M/c + p_M/c)u_\mu u_\nu + (p_M/c)\eta_{\mu\nu}$ with $p_M \ll \rho_M$, the Einstein equations reduce to

$$\begin{aligned} G_{00} &= j'' + \frac{2}{\rho} j' = -\frac{8\pi G_N}{c^4}\rho_M, \\ G_{rr} &= -\frac{1}{\rho}[j' + h'] = 0, \\ G_{\theta\theta} &= -\frac{\rho^2}{2}[j'' + h''] - 2\rho[j' + h'] = 0. \end{aligned} \tag{22.3}$$

Solution is

$$j + h = 0, \qquad h'' + \frac{2}{\rho} h' = \nabla^2 h = \frac{8\pi G_N}{c^4}\rho_M. \tag{22.4}$$

Thus with $h = 2\phi/c^2$ we finish up with

$$\nabla^2\phi = \frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2\frac{d\phi}{d\rho}\right] = \frac{4\pi G_N}{c^2}\rho_M, \tag{22.5}$$

which we recognize as the second-order Poisson equation. For a source of radius ρ_0 and $\rho_M = Mc^2/[4\pi\rho_0^3/3]$, then with $\phi(0) = 0$, in $\rho > \rho_0$ we obtain

$$\int_0^\rho \rho^2 d\rho \nabla^2\phi = \rho^2\frac{d\phi}{d\rho}\bigg|_0^\rho = MG_N, \qquad \phi(\rho > \rho_0) = -\frac{MG_N}{\rho}. \tag{22.6}$$

Finally, we can write the line element as

$$ds^2 = c^2dt^2(1 + 2\phi/c^2] - (1 - 2\phi/c^2)][d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2], \tag{22.7}$$

and get a huge bonus: we do not just get Newton, we get the v^2/c^2 correction, just as needed for the orbit of Mercury.

There is actually an exact exterior ($\rho > \rho_0$) solution

$$ds^2 = c^2dt^2\left(\frac{1 - MG_N/2c^2\rho}{1 + MG_N/2c^2\rho}\right)^2 - \left(1 + \frac{MG_N}{2c^2\rho}\right)^4 [d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta d\phi^2]. \tag{22.8}$$

With $\rho = \frac{1}{2}\left[r - MG_N/c^2 + (r^2 - 2MG_N r/c^2)^{1/2}\right]$, $r = \rho(1 + MG_N/2c^2\rho)^2$ it is coordinate equivalent to the exact solution found by Schwarzschild:

$$ds^2 = c^2dt^2\left(1 - \frac{2MG_N}{c^2 r}\right) - dr^2\left(1 - \frac{2MG_N}{c^2 r}\right)^{-1} - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{22.9}$$

a solution which is singular at $r = 2MG_N/c^2$, the Schwarzschild radius of a black hole.

23 The Schwarzschild solution

For a static, spherically symmetric source such as a star we take as line element

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{23.1}$$

so that the metric and its inverse are given by

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/B(r) & 0 & 0 & 0 \\ 0 & 1/A(r) & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix}. \tag{23.2}$$

The nonvanishing components of the connection $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$ are given by

$$\begin{aligned}
&\Gamma^r_{\theta\theta} = -\frac{r}{A(r)}, \qquad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A(r)}, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \\
&\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
&\Gamma^r_{rr} = \frac{A'(r)}{2A(r)}, \qquad \Gamma^r_{tt} = \frac{B'(r)}{2A(r)}, \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{B'(r)}{2B(r)},
\end{aligned} \tag{23.3}$$

where the prime denotes the derivative with respect to r. The nonzero components of $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}$, where $R_{\mu\kappa} = \partial_\kappa\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta}$, are given by

$$\begin{aligned}
G_{tt} &= -\frac{B}{r^2} + \frac{B}{r^2A} - \frac{BA'}{rA^2}, \\
G_{rr} &= \frac{A}{r^2} - \frac{1}{r^2} - \frac{B'}{rB}, \\
G_{\theta\theta} &= \frac{G_{\phi\phi}}{\sin^2\theta} = -\frac{r^2B''}{2AB} + \frac{r^2A'B'}{4A^2B} + \frac{r^2B'^2}{4AB^2} - \frac{rB'}{2AB} + \frac{rA'}{2A^2}.
\end{aligned} \tag{23.4}$$

For the energy-momentum tensor $T_{\mu\nu} = \frac{1}{c}\left[(\rho+p)u_\mu u_\nu + p g_{\mu\nu},\right]$, with velocity $u^\mu u_\mu = g_{\mu\nu}u^\mu u^\nu = -1$, the only nonzero components of the velocity are $u^0 = B^{-1/2}$, $u_0 = -B^{1/2}$. Thus we have

$$T_{\mu\nu} = \begin{pmatrix} B(r)\rho(r)/c & 0 & 0 & 0 \\ 0 & A(r)p(r)/c & 0 & 0 \\ 0 & 0 & r^2 p(r)/c & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta p(r)/c \end{pmatrix}. \tag{23.5}$$

The Einstein equations $-(c^3/8\pi G_N)G_{\mu\nu} = T_{\mu\nu}$ thus give us

$$\begin{aligned} &-\frac{B}{r^2} + \frac{B}{r^2 A} - \frac{BA'}{rA^2} = -\frac{8\pi G_N}{c^4}B\rho. \\ &\frac{A}{r^2} - \frac{1}{r^2} - \frac{B'}{rB} = -\frac{8\pi G_N}{c^4}Ap, \\ &-\frac{r^2B''}{2AB} + \frac{r^2A'B'}{4A^2B} + \frac{r^2B'^2}{4AB^2} - \frac{rB'}{2AB} + \frac{rA'}{2A^2} = -\frac{8\pi G_N}{c^4}r^2 p. \end{aligned} \tag{23.6}$$

The third equation is not independent of the first two since $\nabla_\nu G^{\mu\nu} = 0$. Multiplying the first equation by A/B and adding it to the second equation gives

$$\frac{1}{r}\left(\frac{B'}{B} + \frac{A'}{A}\right) = \frac{8\pi G_N}{c^4}A(\rho+p). \tag{23.7}$$

Inserting this equation back into the second Einstein equation then gives

$$\frac{A'}{rA} + \frac{A}{r^2} - \frac{1}{r^2} = \frac{8\pi G_N}{c^4}A\rho, \qquad \frac{rA'}{A^2} + 1 - \frac{1}{A} = \frac{8\pi G_N}{c^4}r^2\rho = \frac{d}{dr}\left(r - \frac{r}{A}\right). \tag{23.8}$$

Integrating $\int dr$, identifying $4\pi\int_0^r dr r^2\rho = M(r)c^2$ and setting $A(0) = 1$ we obtain

$$\frac{1}{A} = 1 - \frac{2M(r)G_N}{c^2 r}, \qquad \frac{B'}{B} = -\frac{A'}{A} + \frac{8\pi G_N}{c^4}Ar(\rho+p) = \frac{A}{r} - \frac{1}{r} + \frac{8\pi G_N}{c^4}Ap. \tag{23.9}$$

Thus for a source of radius r_0 and mass $4\pi\int_0^{r_0} dr r^2\rho = Mc^2$, so that $\rho(r > r_0) = 0$, $p(r > r_0 = 0)$, we obtain

$$A(r < r_0) = \left(1 - \frac{2M(r)G_N}{c^2 r}\right)^{-1}, \qquad A(r > r_0) = \left(1 - \frac{2MG_N}{c^2 r}\right)^{-1}, \qquad B(r > r_0) = \frac{1}{A(r > r_0)} = 1 - \frac{2MG_N}{c^2 r}. \tag{23.10}$$

This is the Schwarzschild vacuum solution exterior to a localized source. To solve for $B(r < r_0)$ we need to know $p(r)$,

23.1 There has to be a source

If we insert $A'/A = -B'/B$, i.e., $A = 1/B$, into $G_{\theta\theta}$, then in the vacuum we obtain

$$G_{\theta\theta} = -\frac{r^2}{2AB}\left[B'' + \frac{2}{r}B'\right] = -\frac{r^2}{2}\nabla^2 B = -\frac{r^2}{2}\nabla^2\left(1 - \frac{2MG_N}{c^2 r}\right) = -\frac{MG_N r^2}{4\pi c^2}\delta^3(\bar{r}). \tag{23.11}$$

so not a vacuum solution at $r = 0$. Thus only an exterior solution at $r > r_0$, and needs some mass in interior region to support it.

23.2 Some numbers

For the sun

$$R^S_\odot = \frac{2M_\odot G_\mathrm{N}}{c^2} = 2.96 \times 10^5 \text{ cm}. \tag{23.12}$$

Radius of the sun $R_\odot = 6.96 \times 10^{10}$ cm. So the solar Schwarzschild radius is deep within the sun.

23.3 Black holes

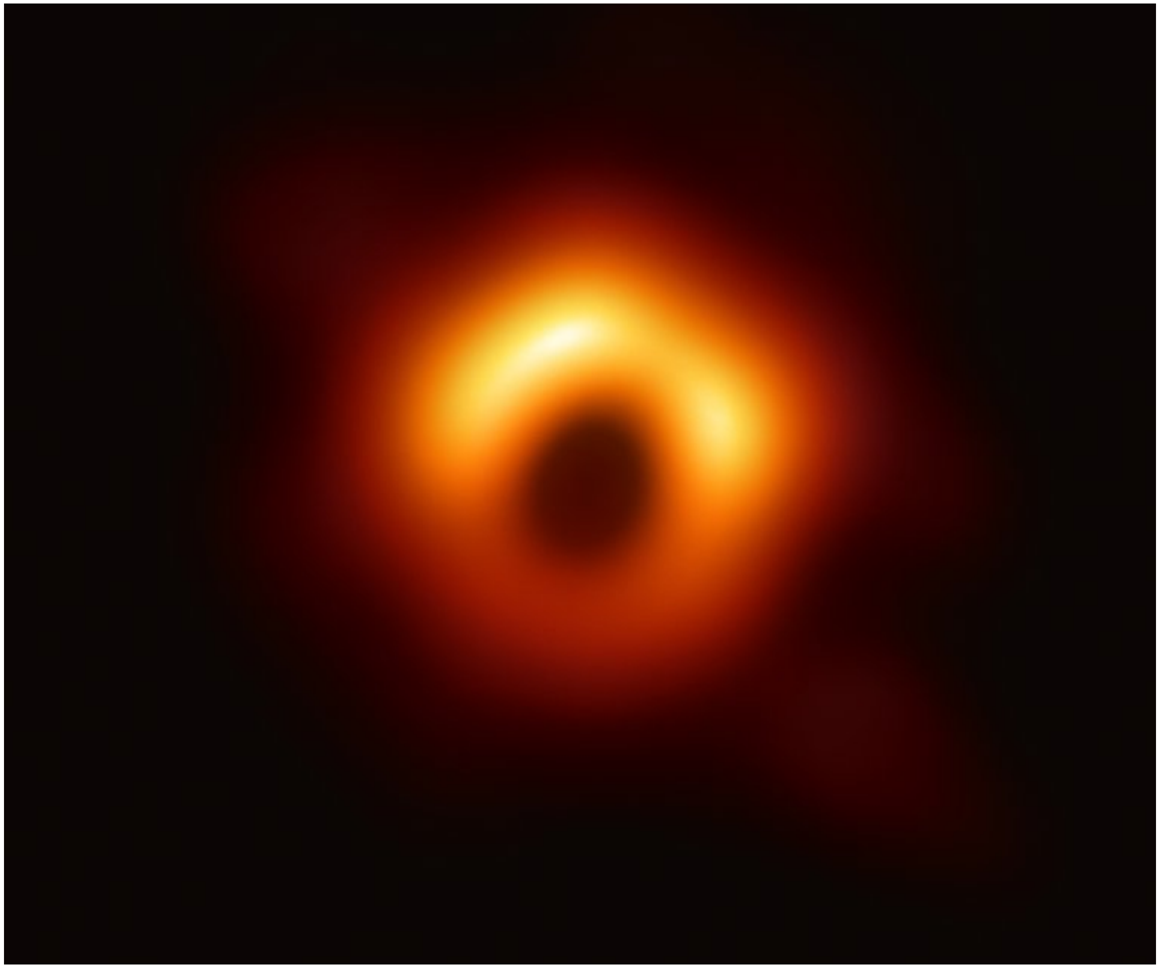
But if the Schwarzschild radius R_S is greater or equal to the radius R_0 of a system we have a black hole, so classically light cannot escape. Except it can quantum-mechanically, viz. Hawking radiation. r_S acts as horizon

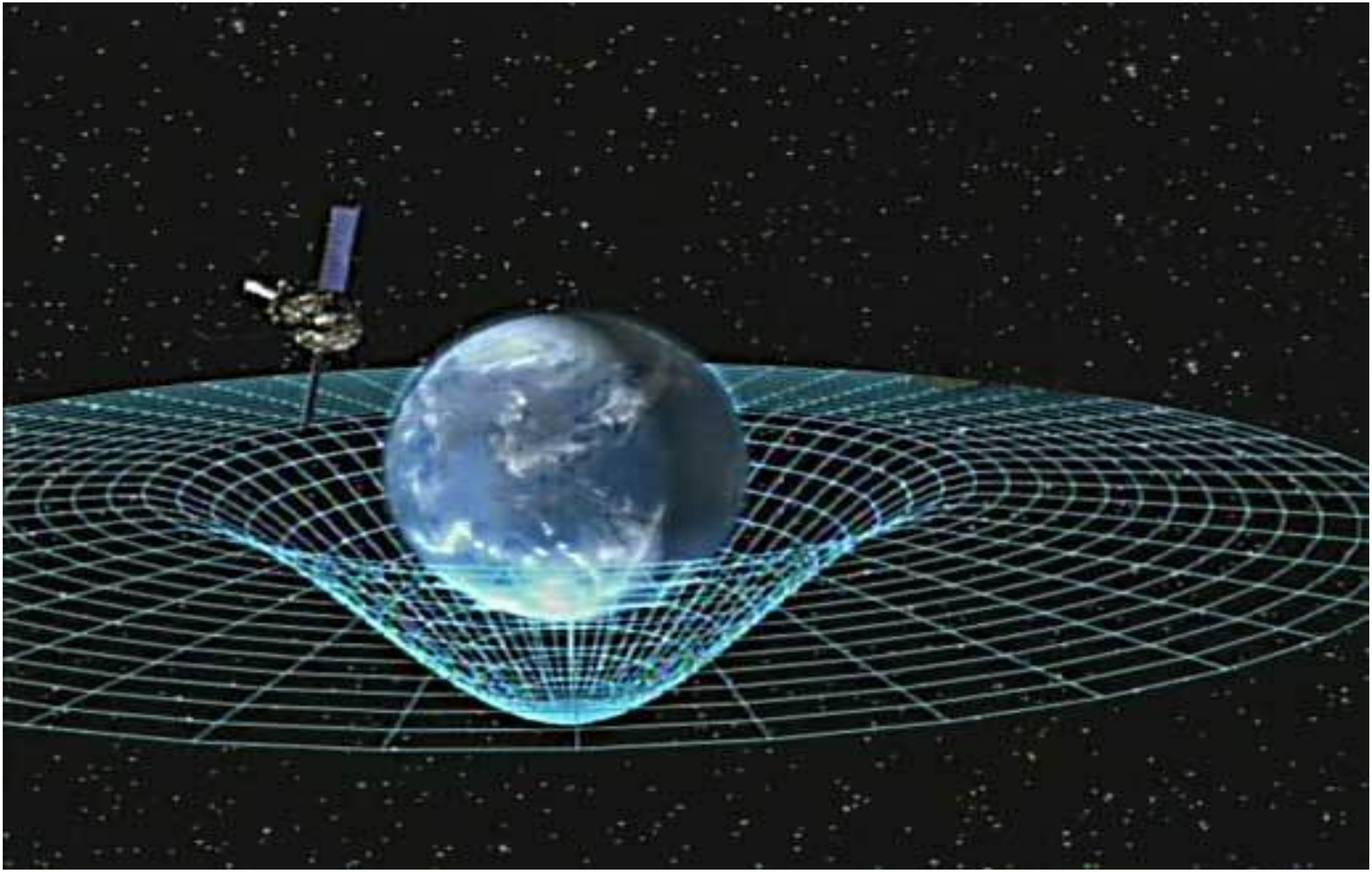
The density of a black hole whose Schwarzschild radius is equal to its radius is given by

$$\rho = \frac{M}{(4\pi/3)R_0^2} = \frac{M}{(4\pi/3)R_S^2} = \frac{3M}{4\pi}\left(\frac{c^2}{2MG_N}\right)^3. \tag{23.13}$$

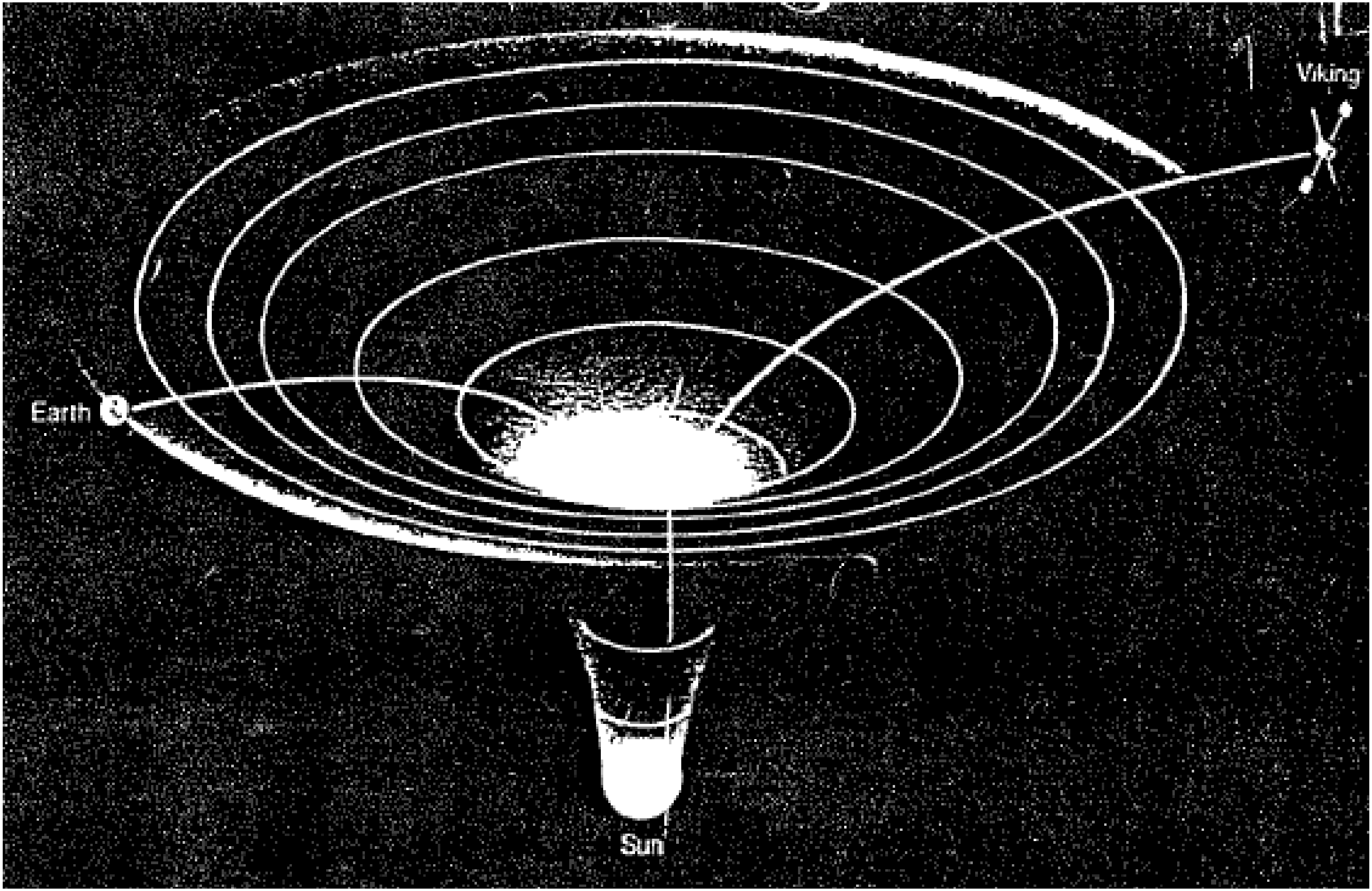
For one solar mass get $\rho = 10^{16}$ gm.cm^{-3}. The density of the proton is of order $M_p/R_p^3 = 10^{-24}/10^{-39} = 10^{15}$ gm.cm^{-3}. Thus a solar mass black hole has nuclear density right throughout the star. If we increase the black hole mass to $10^7 M_\odot$ density becomes 10 gm.cm^{-3}, not far from the density of water - quite counterintuitive.

Evidence for black holes: centers of active galactic nuclei, centers of spirals, gravitational waves seen at LIGO, event horizon telescope.

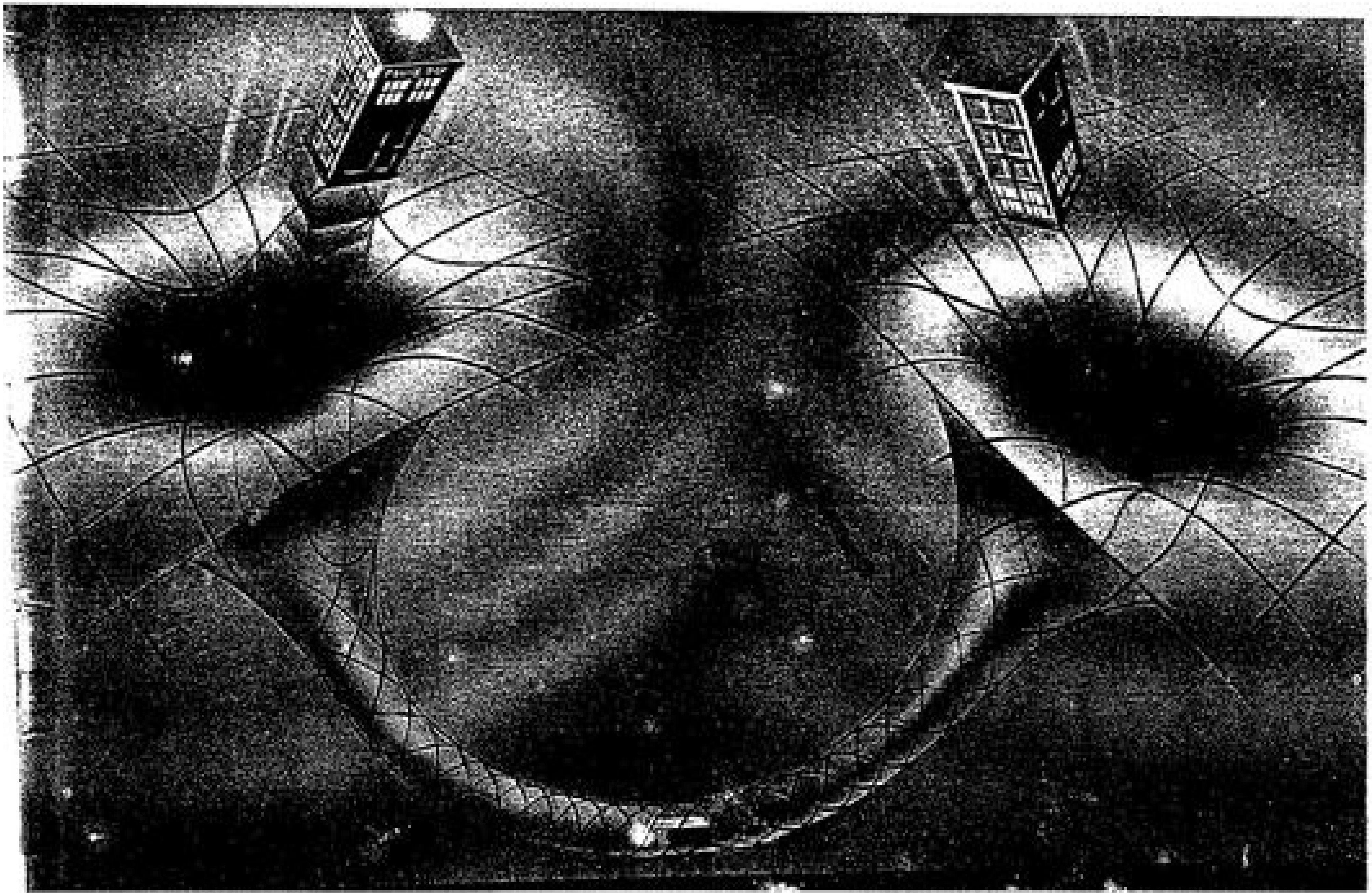




Light Bending



Light Bending



Tardis through the wormhole

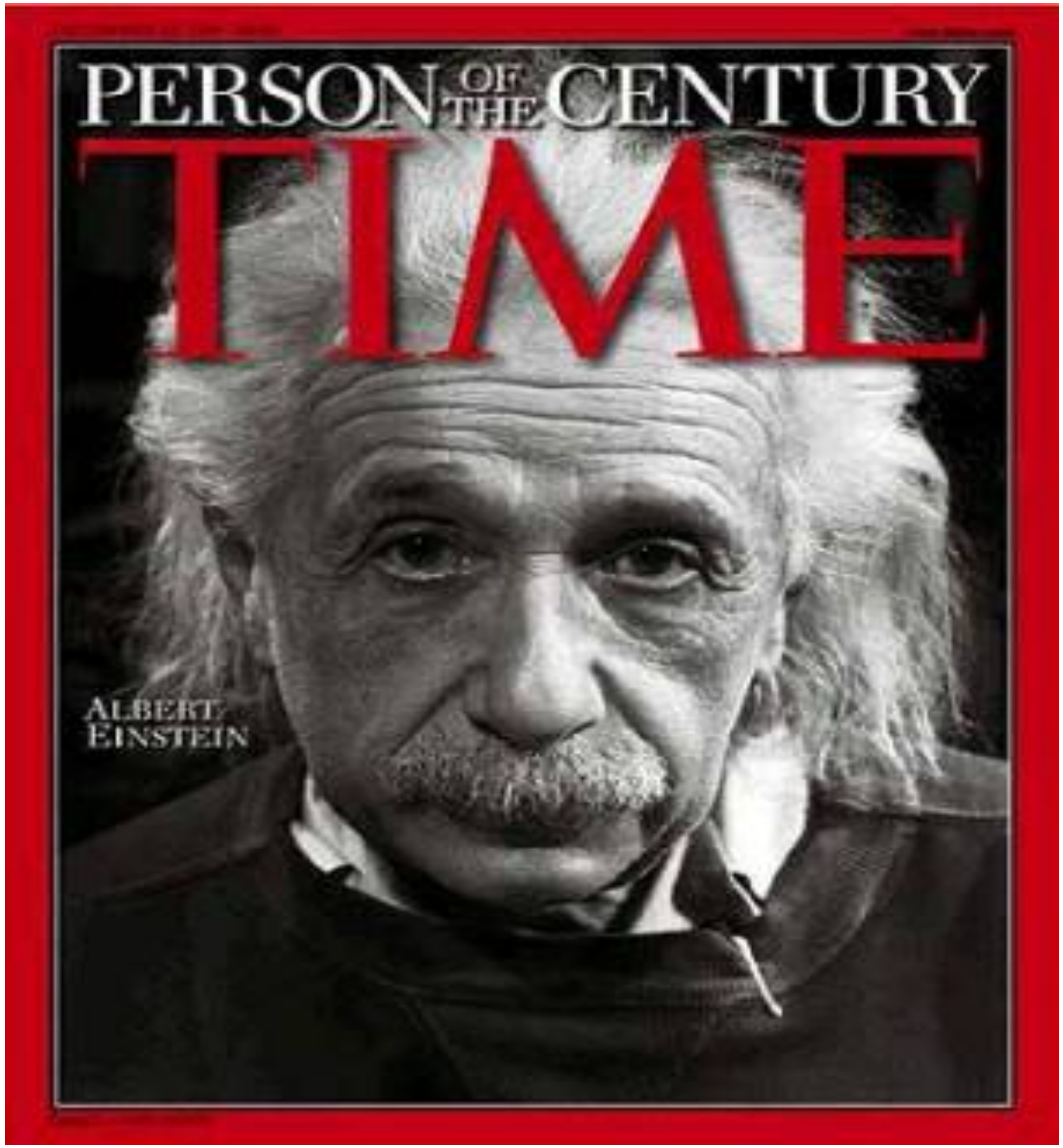


Figure 1: **In December 1999 Time magazine designates Albert Einstein the man of the twentieth century**

24 Geodesics

For the general static, spherically symmetric metric of the form $d\tau^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$ the four equations of motion contained in $d^2x^\lambda/d\tau^2 + \Gamma^\lambda_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = 0$ take the form

$$\begin{aligned} c\frac{d^2t}{d\tau^2} + \frac{cB'}{B}\frac{dt}{d\tau}\frac{dr}{d\tau} &= 0, \\ \frac{d^2r}{d\tau^2} + \frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{d\tau}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{c^2B'}{2A}\left(\frac{dt}{d\tau}\right)^2 &= 0, \\ \frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 &= 0, \\ \frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} + 2\frac{\cos\theta}{\sin\theta}\frac{d\phi}{d\tau}\frac{d\theta}{d\tau} &= 0, \end{aligned} \tag{24.1}$$

with the prime denoting differentiation with respect to r. Equatorial plane solutions can be found in which θ is fixed to $\theta = \pi/2$, with the equations of motion for the three other coordinates reducing to

$$\begin{aligned} c\frac{d^2t}{d\tau^2} + \frac{cB'}{B}\frac{dt}{d\tau}\frac{dr}{d\tau} &= 0, \\ \frac{d^2r}{d\tau^2} + \frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2 - \frac{r}{A}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{c^2B'}{2A}\left(\frac{dt}{d\tau}\right)^2 &= 0, \\ \frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau}, &= 0. \end{aligned} \tag{24.2}$$

The first and last of these equations take the form

$$\frac{d}{d\tau}\left[\log\left(\frac{d\phi}{d\tau}\right) + \log r^2\right] = 0, \qquad \frac{d}{d\tau}\left[\log\left(\frac{cdt}{d\tau}\right) + \log B\right] = 0 \tag{24.3}$$

On choosing convenient integration constants they have first integrals of the form

$$r^2\frac{d\phi}{d\tau} = J, \qquad c\frac{dt}{d\tau} = \frac{1}{B(r)}. \tag{24.4}$$

Inserting these relations into the radial derivative equation gives

$$\frac{d^2r}{d\tau^2} + \frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2 - \frac{J^2}{Ar^3} + \frac{B'}{2AB^2} \;=\; 0, \tag{24.5}$$

with integral

$$A(r)\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} \;=\; -E, \tag{24.6}$$

where E is an integration constant, equal to the energy per unit mc^2. Thus mc^2-Energy= kinetic energy, so $1 - E > 0$. Evaluating the line element gives $ds^2 = Ed\tau^2$. Thus $1 > E > 0$ for massive particles and $E = 0$ for massless ones. Using $cdt/d\tau = 1/B(r)$ we can eliminate τ and obtain

$$\frac{r^2}{c}\frac{d\phi}{dt} = JB(r), \qquad \frac{A(r)}{c^2B^2(r)}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E, \qquad ds^2 = EB^2(r)c^2dt^2. \tag{24.7}$$

As a check we note that for a circular orbit the second equation in (24.2) gives

$$\frac{v^2}{r^2} = \left(\frac{d\phi}{dt}\right)^2 = \frac{c^2}{2r}B'(r) = \frac{2MG_N}{2r^3}, \qquad \frac{v^2}{r} = \frac{MG_N}{r^2}. \tag{24.8}$$

From (24.4) and (24.6) we obtain for the orbit

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2B(r)} = -\frac{E}{J^2}. \tag{24.9}$$

so that

$$\phi = \pm\int dr \frac{JB^{1/2}(r)A^{1/2}(r)}{r[r^2 - Er^2B(r) - J^2B(r)]^{1/2}}. \tag{24.10}$$

For bound orbits there are turning points where $dr/d\phi = 0$, i.e., where

$$r^2(1 - EB) - J^2B = 0 \tag{24.11}$$

With $B = 1 - 2\beta/r$ and $1 - E > 0$, $\beta > 0$ we have

$$
\begin{aligned}
&r^2(r - Er + 2E\beta) - J^2(r - 2\beta) = 0, \qquad r^3 + \frac{2\beta E r^2}{1-E} - \frac{J^2 r}{1-E} + \frac{2\beta J^2}{1-E} = 0, \\
&(r - a_1)(r - a_2)(r - a_3) = r^3 - r^2(a_1 + a_2 + a_3) + r(a_i a_2 + a_2 a_3 + a_3 a_1) - a_1 a_2 a_3 = 0.
\end{aligned}
\tag{24.12}
$$

Thus $a_1 + a_2 + a_3 < 0$, $a_i a_2 + a_2 a_3 + a_3 a_1 < 0$, $a_1 a_2 a_3 < 0$. From $a_1 a_2 a_3 < 0$ either one negative root or three. From $a_i a_2 + a_2 a_3 + a_3 a_1 < 0$ three negative excluded. Thus one negative and two positive roots. But the radial coordinate is positive. Thus only two relevant positive roots.

Thus two turning points, r_+ and r_-, apogee and perigee. Hence the orbit is an ellipse. Introduce semilatus rectum $2/L = 1/r_+ + 1/r_-$. Can solve exactly, find orbit precesses at rate of $6\pi M_\odot G_N/c^2 L$ radians per revolution. For Mercury, $L = 5.53 \times 10^{12}$ cm, Get $\Delta\phi = 0.104$ seconds per revolution. Then 43.03 seconds per century, just as required.

To understand what is happening consider the weak gravity Schwarzschild line element and point particle action

$$
\begin{aligned}
ds^2 &= c^2 dt^2 \left(1 - \frac{2\beta}{r}\right) - dr^2 \left(1 + \frac{2\beta}{r}\right) = c^2 dt^2 \left[1 - \frac{2\beta}{r} - \frac{v^2}{c^2} - \frac{2v^2\beta}{c^2 r}\right], \\
I &= mc \int ds = mc^2 \int dt \left[1 - \frac{\beta}{r} - \frac{v^2}{2c^2} - \frac{v^2\beta}{c^2 r}\right].
\end{aligned}
\tag{24.13}
$$

Now Newtonian v^2/c^2 is equal to β/r. Thus correction term is is of order $v^2\beta/c^2 r = v^4/c^4$ or equivalently of order β^2/r^2. Hence precession of planetary orbits. (If just β/r then no precession.)

Moreover, if $v = c$ get

$$
ds^2 = c^2 dt^2 \left[1 - \frac{\beta}{r} - 1 - \frac{\beta}{r}\right]. \tag{24.14}
$$

i.e., equal amounts of time dilation and Lorentz contraction causing gravitational bending of light. Full calculation of $\phi(r)$ for unbound orbit. Find for light just grazing the sun get $\Delta\phi = M_\odot G_N/R_\odot$=1.75 seconds, just as observed.

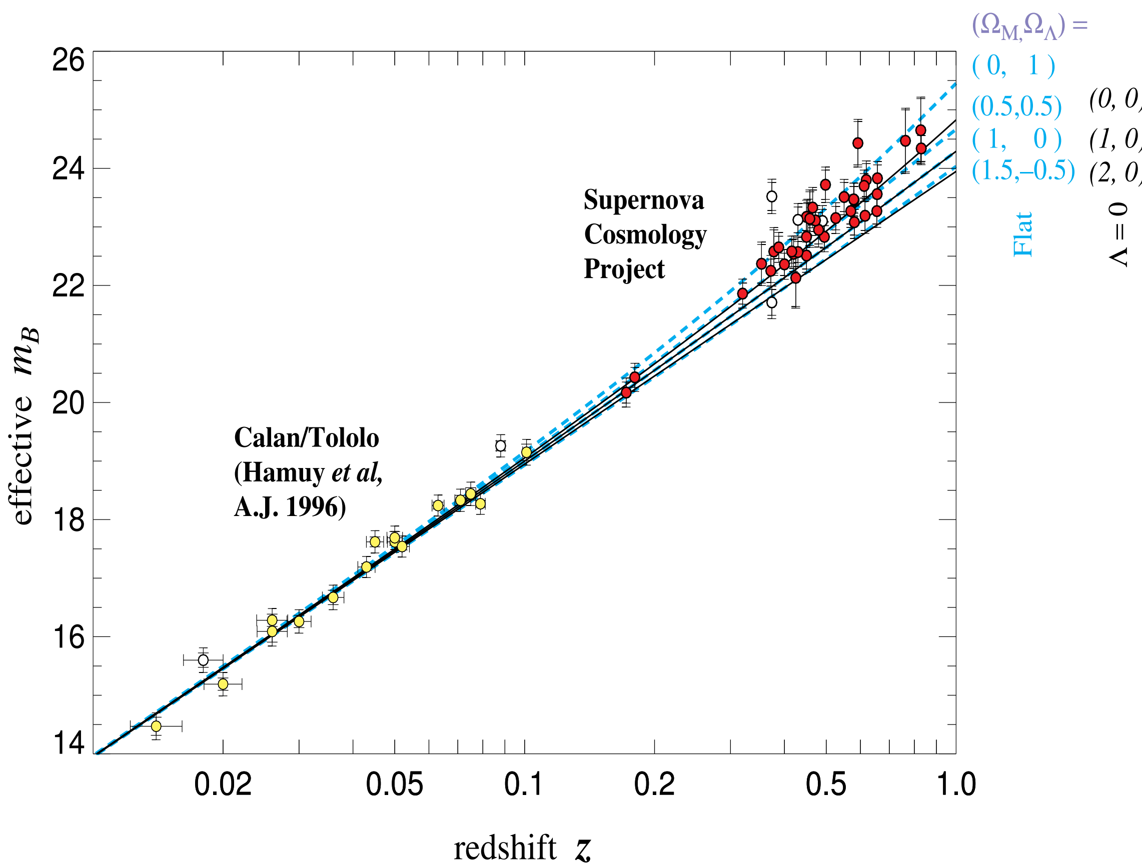
Third classic test: gravitational redshift. Also confirmed.

25 Cosmology

25.1 Hubble flow

Hubble identified a systematic behavior in galaxies: they were all redshifted with respect to us, i.e., moving away from us, and had velocities of the form $v = HD$, where D is the distance from us and H is a constant.

Rationale: No point is special. Consider three equally spaced points A, B, C on a straight line. Let B have a velocity v with respect to A. Let C have a velocity v with respect to B. Then C has a velocity 2v with respect to A, and is twice as far from A as B is. Thus $v = HD$. Supernovae data as log plot:



If no point is special then universe is homogeneous and isotropic. So line element is as in (15.8):

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] \tag{25.1}$$

As space expands the axes on a grid expand with it, so coordinates of points do not change, viz. comoving. However actual distance between points does change since grid itself expands with expansion radius $a(t)$. Here k is the spatial 3-curvature, $k > 0$ closed surface, $k = 0$ flat, $k < 0$ open surface.

Redshift: Put ourselves at origin of coordinates. Consider a light signal that leaves a point distance r_1 from us at time t_1 and reaches us at time t_0. It travels on null geodesic

$$c\int_{t_1}^{t_0}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{(1-kr^2)^{1/2}} \tag{25.2}$$

Consider a second signal that leaves r_1 at a slightly later time $t_1 + \delta t_1$ and reaches us at $t_0 + \delta t_0$. It travels on

$$c\int_{t_1+\delta t_1}^{t_0+\delta t_0}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{(1-kr^2)^{1/2}}. \tag{25.3}$$

Thus

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}, \qquad \frac{\delta t_1}{\delta t_0} = \frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)}. \tag{25.4}$$

Define redshift

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\nu_1}{\nu_0} - 1 = \frac{a(t_0)}{a(t_1)} - 1. \tag{25.5}$$

25.2 Newtonian cosmology

Consider a static, spherically symmetric universe of density ρ, total mass $M = (4\pi/3)\rho r^3$, with a particle of mass m, velocity v at a distance r. Its energy is

$$U = \frac{1}{2}mv^2 - \frac{mMG_N}{r} = \frac{1}{2}mv^2 - \frac{4\pi m G_N \rho r^3}{3r} \tag{25.6}$$

Now set $v = Hr$. This gives

$$U = \frac{m}{2}H^2r^2 - \frac{4\pi m G_N \rho r^2}{3} = \frac{m}{2}r^2\left(H^2 - \frac{8\pi}{3}G_N\rho\right) = \frac{4\pi m G_N r^2}{3}\left(\rho_c - \rho\right). \tag{25.7}$$

This allows us to define a critical density $\rho_c = 3H^2/8\pi G_N$ of order 10^{-29} gm.cm^{-3}. Thus if $\rho > \rho_c$ U iis negative and particle is bound. If. $\rho < \rho_c$ particle escapes, i.e., unbound.

If $\rho = \rho_c$ then

$$v = \frac{dr}{dt} = \left(\frac{2MG_N}{r}\right)^{1/2}, \qquad (2MG_N)^{1/2}t = \frac{2}{3}r^{3/2}, \qquad 2MG_N = rv^2 = H^2r^3 = \frac{9}{4}2H^2MG_Nt^2. \tag{25.8}$$

Thus we get

$$t = \frac{2}{3H} \tag{25.9}$$

With $H_0 = 72\,\mathrm{km\,sec^{-1}\,Mpc^{-1}} = 2.4\times10^{-18}$ sec, $1/H_0 = 4\times10^{17}$sec, current age of universe $=2.67\times10^{17}$ sec. Approximately 10 billion years.

26 General relativistic cosmology

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] = c^2dt^2 - a^2(t)\tilde{g}_{ij}dx^idx^j. \tag{26.1}$$

Nonzero components of the affine connections, the Ricci tensor and Einstein tensor are (c=1)

$$\Gamma^0_{ij} = a\dot{a}\tilde{g}_{ij}, \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^I_j, \qquad \Gamma^i_{jk} = \frac{1}{2}(\tilde{g}^{-1})^{i\ell}\left[\partial_j\tilde{g}_{k\ell} + \partial_k\tilde{g}_{j\ell} - \partial_\ell\tilde{g}_{jk}\right],$$
$$R_{00} = \frac{3\ddot{a}}{a}, \qquad R_{ij} = -(a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij}. \qquad R = -\frac{6(a\ddot{a}+\dot{a}^2+k)}{a^2},$$
$$G_{00} = -\frac{3(\dot{a}^2+k)}{a^2}, \qquad G_{ij} = \tilde{g}_{ij}\left[2a\ddot{a} + \dot{a}^2 + k\right]. \tag{26.2}$$

Because of the homogeneous and isotropic geometry the energy-momentum tenser must be a perfect fluid with ρ and p only being functions of t, and with $u^\mu = (1,0,0,0)$, $u_\mu = (-1,0,0,0)$. Thus with $T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}$ the Einstein equations take the form

$$-\frac{3(\dot{a}^2+k)}{a^2} = -8\pi G_N\rho, \qquad 2a\ddot{a} + \dot{a}^2 + k = -8\pi G_N a^2 p. \tag{26.3}$$

so that, with $H = \dot{a}/a$ (cf. $v/r = H$), we obtain

$$\dot{a}^2 + k = \frac{8\pi G_N}{3}a^2\rho, \qquad H^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho, \qquad \ddot{a} = -\frac{4\pi G_N}{3}a(\rho+3p). \tag{26.4}$$

The first of these equations is known as the Friedmann equation, the second is analogous to Newtonian cosmology with same $\rho_c = 3H^2/8\pi G_N$, and the third says that even with $\dot{a} > 0$, nonetheless $\ddot{a}$ is negative (deceleration) if $\rho + 3p > 0$, i.e., slowing down expansion.

Covariant conservation of $T^{\mu\nu}$ gives

$$\partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\sigma}T^{\sigma 0} + \Gamma^0_{\mu\sigma}T^{\mu\sigma} = 0, \qquad \partial_0 T^{00} + \Gamma^i_{i0}T^{00} + \Gamma^0_{ij}T^{ij} = 0, \qquad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho+p) = 0. \tag{26.5}$$

Three key solutions, relativistic, nonrelativistic and cosmological constant:

$$(1): \quad p = \frac{\rho}{3}, \quad \rho = \frac{A}{a^4}, \qquad (2): \quad p = 0, \quad \rho = \frac{B}{a^3}, \qquad (3): \quad p = -\rho, \qquad \rho = \Lambda = \text{constant}. \tag{26.6}$$

Known as radiation era, matter era, cosmological constant era (viz. inflation).

26.1 Cosmological implications

Some typical solutions with $k = 0$.

$$\rho = \frac{B}{a^3}, \qquad \dot{a}^2 = \frac{8\pi G_N B}{3a}, \qquad a(t) = (6\pi G_N B)^{1/2} t^{2/3},$$
$$\rho = \frac{A}{a^4}, \qquad \dot{a}^2 = \frac{8\pi G_N A}{3a^2}, \qquad a(t) = \left(\frac{32\pi G_N A}{3}\right)^{1/2} t^{1/2},$$
$$\rho = \Lambda, \qquad \dot{a}^2 = \frac{8\pi G_N \Lambda a^2}{3}, \qquad a(t) = e^{Ht}, \qquad H = \left(\frac{8\pi G_N \Lambda}{3}\right)^{1/2}. \tag{26.7}$$

Some typical radiation era solutions for any k

$$\rho = \frac{A}{a^4}, \qquad \dot{a}^2 + k = \frac{8\pi G_N A}{3a^2}, \qquad a^2(t) = \left(\frac{32\pi G_N A}{3}\right)^{1/2} t - kt^2 \tag{26.8}$$

If $k = 0$ or $k < 0$ then $a(t)$ expands forever. If $k > 0$ then $a(t)$ reaches a maximum at $t = (8\pi G_N A/3)^{1/2}/k$, after which $a(t)$ contracts. $H = \dot{a}/a$ blows up at $t = 0$, the big bang.. For black body $\rho = A/a^4 = a_B T^4$. Thus $a(t) = \mu/T$ where $\mu = (A/a_B)^{1/4}$, so early universe is hot, and universe cools as it expands. Thus approximately radiation until last scattering of photons and baryons (at around 3000° when temperature becomes too low to ionize atoms) and matter era since then until today (around 3°). After last scattering radiation propagates as a free black body. If switch from radiation era to matter era iabruptly at t_L then $9Bt_L^{1/3} = 16A$.

Successes: Hubble flow, cosmic microwave background (CMB), primordial nucleosynthesis, baryon acoustic fluctuations in the CMB.

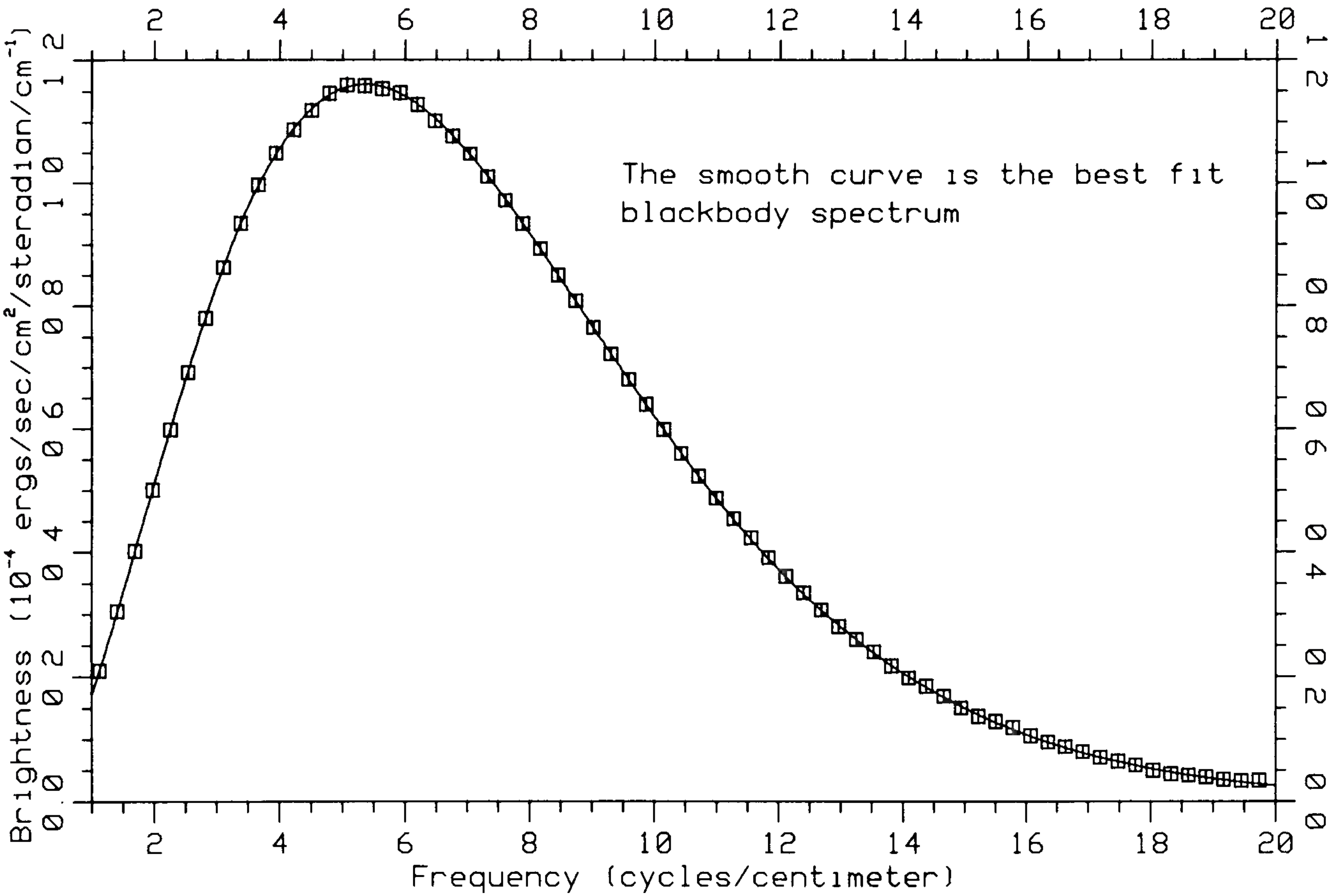


FIG. 2.—Preliminary spectrum of the cosmic microwave background from the FIRAS instrument at the north Galactic pole, compared to a blackbody. Boxes are measured points and show size of assumed 1% error band. The units for the vertical axis are 10^{-4} ergs s^{-1} cm^{-2} sr^{-1} cm.

26.2 Horizon problem

In the $k=0$ case for the line element $ds^2 = c^2dt^2 - a^2(t)[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2]$ the proper radial distance (the horizon) is $d_H = a(t)\int dr$. The last scattering sky is at time t_L since the big bang and at a distance r_L from us. A radial null geodesic that reaches us at time t_0 and $r=0$ from the last scattering sky is of the form

$$\int_0^{r_L} dr = \int_{t_L}^{t_0} \frac{dt}{a(t)}, \qquad r_L = \frac{3}{(6\pi G_N B)^{1/2}}[t_0^{1/3} - t_L^{1/3}] \approx \frac{3}{(6\pi G_N B)^{1/2}} t_0^{1/3}. \tag{26.9}$$

A yardstick on the sky at time t_L that fills the sky subtends an angle π, and at our current time has an approximate proper angular diameter $d_A = a(t_L)\pi r_L$.

A null signal that sets out at $t=0$ at a distance r_0 from us travels to last scattering according to

$$\int_{r_L}^{r_0} dr = \int_0^{t_L} \frac{dt}{a(t)}, \qquad r_0 - r_L = \left(\frac{3}{8\pi G_N A}\right)^{1/2} t_L^{1/2}, \tag{26.10}$$

and has an associated proper distance $d_H = a(t_L)t_L^{1/2}(3/8\pi G_N A)^{1/2}$. Thus we get

$$\frac{d_H}{d_A} = \left(\frac{3}{8\pi G_N A}\right)^{1/2} t_L^{1/2}\frac{(6\pi G_N B)^{1/2}}{3\pi} t_0^{-1/3} = \frac{2}{3\pi}\left(\frac{t_L}{t_0}\right)^{1/3} = \frac{2}{3\pi}\left(\frac{T_0}{T_L}\right)^{1/2} = \text{a few degrees} \tag{26.11}$$

So last scattering sky should not have thermalized, and yet it has to one part in 10^5.

26.3 Flatness problem

Measure baryon density ρ_B today to be about $0.01\rho_c$. If $k = 0$ should be equal to ρ_c. With evolution equation $\dot{a}^2 + k = (8\pi G/3)a^2\rho$, at big bang singularity the infinity in $\dot{a}^2$ balances singularity in $(8\pi G/3)a^2\rho$ no matter what k is. So structure of initial universe is same as that of a $k = 0$ universe. The matter density now redshifts for 10 billion years and yet is still close to a universe with $k = 0$ today. Chance of getting such a universe today evaluates to one part in 10^{60}. Thus have to fine tune initial conditions to have universe evolve into what we see today. Now if $k = 0$ would not need to fine tune initial conditions since then $\rho = \rho_c$ in every epoch. But current $\rho = 0.01\rho_c$. This is the flatness problem.

26.4 Cosmological constant problem

As universe cools it goes through electroweak phase transition at around $10^{15}\ {}^\circ K$. This releases a free energy of order $T^4 = 10^{60}$. But current temperature is of order a few degrees. So energy in vacuum energy (viz. cosmological constant) is of 10^{60} times that of energy in ordinary matter. A total disaster unless quenched.

26.5 Accelerating universe problem

High redshift supernova data show gravity has a **repulsive** component.

26.6 Quantum gravity problem

Quantum graviton loops are infinite. A total disaster unless quenched.

26.7 General nature of the problems

If $a(t) = t^n$ then $\dot{a} = nt^{n-1}$, $\ddot{a} = n(n-1)t^{n-2}$, $\int dt/a(t) = t^{1-n}/(1-n)$, $\int dt/a(t) = \log t$ if $n = 1$.

If $n < 1$ then $\dot{a}$ blows up at $t = 0$, $\ddot{a} < 0$, $\int dt/a(t)$ is finite at $t = 0$. Initial singularity and flatness problem, deceleration, and horizon problem

If $n \geq 1$ then $\dot{a}$ does not blow up at $t = 0$, $\ddot{a} > 0$, $\int dt/a(t)$ blows up at $t = 0$, No initial singularity, no flatness problem, acceleration, no horizon problem.

26.8 Inflation and the dark matter problem

Can solve horizon problem if $n \geq 1$. Brout, Englert and Gunzig (1978), Kazanas (1979), Starobinsky (1979), Guth (1980) showed that if $a(t) = e^{Ht}$ then no horizon problem.

Guth: $a(t) = e^{Ht}$ also solves fine tuning flatness problem. Early universe inflates very rapidly, $a(t)$ becomes so big that universe is effectively flat. (Technically curvature does not gravitate very much.). Then e^{Ht} switches off and we have $a(t) = t^{1/2}$ and then $a(t) = t^{2/3}$. But if $k = 0$ then $\rho = \rho_c$. But luminous baryon density is only 1 per cent of critical. So what is the remaining 99 per cent – dark matter, So search for it began, and after forty years none found as of yet. As well as cosmological dark matter, also need dark matter for galaxies and clusters of galaxies.

To check if the 99 per cent is there, study accelerating universe high redshift supernova. Ruled out, find need for only 30 percent dark matter, as it decelerates. Need something additional, to give acceleration: 70 per cent dark energy (viz. the cosmological constant cannot be ignored). So inflation does not solve fine tuning problem. Only fixes the sum of dark matter energy density and dark energy, but not their redshift-dependent ratio. Nonetheless, inflationary fluctuations work very well for fluctuations in the CMB.

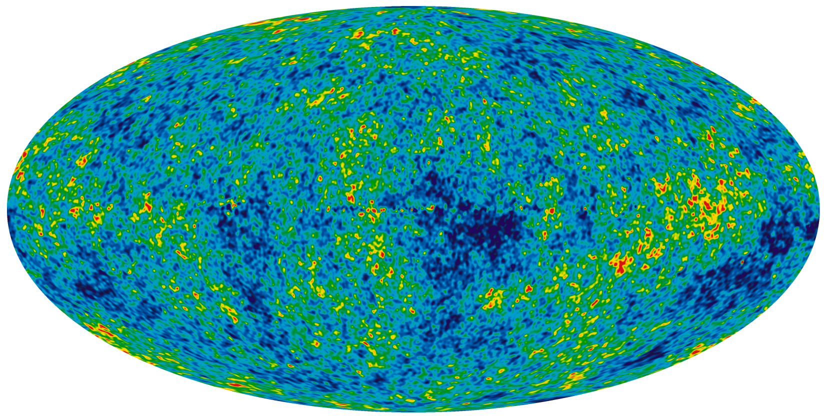


Figure 2: The variation in temperature is of order 10^{-5}. Small departure from uniform expanding Hubble flow.

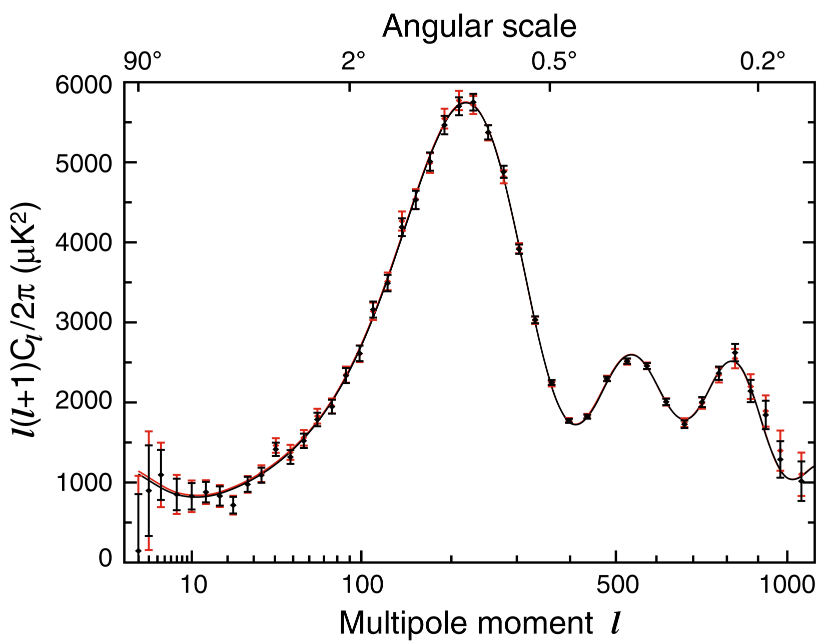


Figure 3: 30 percent dark matter 70 percent dark energy fit to angular momentum decomposition of the CMB fluctuations

27 Quantum Gravity

27.1 Does gravity actually know about quantum mechanics? – Experimental considerations

Before discussing how one might quantize gravity we need to discuss whether we need to. Macroscopically, there are two established sources of gravity that are intrinsically quantum-mechanical: (i) the Pauli degeneracy pressure of white dwarf stars with Chandrasekhar mass $M_{\mathrm{CH}} \sim (\hbar c/G)^{3/2}/m_p^2$, and (ii) the energy density $\rho = \pi^2 k_{\mathrm{B}}^4 T^4/15c^3\hbar^3$ and pressure $p = \rho/3$ of the cosmic microwave background black-body radiation in cosmology.

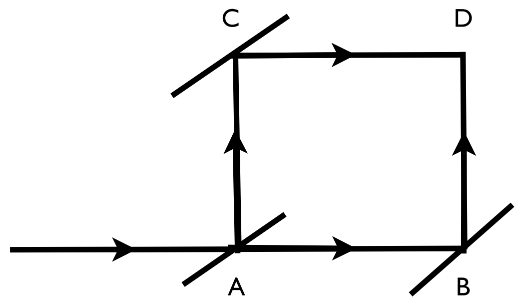


Figure 4: Colella-Overhauser-Werner experiment

Microscopically, the Colella-Overhauser-Werner experiment (1975) shows that the quantum-mechanical phase of the wave function of a neutron of mass m and velocity v is modified as it traverses the gravitational field g of the earth. In the vertical $ABCD$ interferometer CD lies vertically above AB. The incoming neutron beam splits at A with one component traveling horizontally to B and the other component traveling vertically to C. The components at B and C are then reflected so that they interfere at D. With CD being at higher gravitational potential than AB, interference fringes are seen at D. With a change in the action of the neutron being of the form $\Delta I = -mgH^2/v$ (i.e., change compared to the $ABCD$ interferometer lying in the horizontal), the phase shift is given by $\Delta\phi_{\rm COW} = \Delta I/\hbar = -mgH^2/v\hbar$ (see e.g. Mannheim 1998), where $AB = BD = DC = CA = H$, yielding an observable fringe shift at D even though H is only of the order of centimeters and m is the minuscule mass of a neutron — it is just that ΔI is not small on the scale of $\hbar$.

Thus gravity can measure the actual value of the stationary action and thus can measure the mass m, even though it drops out of the classical geodesic. The quantum-mechanical version of the equivalence principle is thus that the inertial and gravitational de Broglie wavelengths $\hbar/m_i v$ and $\hbar/m_g v$ are equal (i.e., interference in the horizontal and vertical). On measuring a nonvanishing fringe shift, Colella, Overhauser and Werner provided the first laboratory evidence of its kind that shows that gravity knows about quantum mechanics.

27.2 Theoretical considerations and concerns

In equations such as the Einstein equations:

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)=T_{\rm M}^{\mu\nu}, \tag{27.1}$$

we note that if the Einstein tensor is to be equal to the matter field energy-momentum tensor, then either both sides are classical c-numbers or both sides are quantum-mechanical q-numbers. Otherwise, if the gravity side were to be classical while the matter side were to be quantum mechanical, then the quantum $T_{\rm M}^{\mu\nu}$ would have to be equal to a c-number in every single field configuration imaginable, which is impossible. Moreover, from gravitational experiments described above we know that the source of gravity is quantum-mechanical, and not only that, we know that gravity knows it. **Hence gravity must be quantized.**
However, since these gravitational experiments are not sensitive to quantum gravity effects themselves (graviton loops), for phenomenological purposes it is conventional to use a hybrid in which we keep gravity classical but take c-number matrix elements of its source, to give

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)=\langle T_{\rm M}^{\mu\nu}\rangle. \tag{27.2}$$

But $\langle T_{\rm M}^{\mu\nu}\rangle$ involves products of fields at the same point, so it is not finite. Thus we additionally subtract off the infinite zero-point part and take the source to be the normal-ordered $\langle T_{\rm M}^{\mu\nu}\rangle_{\rm FIN}=\langle T_{\rm M}^{\mu\nu}\rangle-\langle T_{\rm M}^{\mu\nu}\rangle_{\rm DIV}$ instead:

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)=\langle T_{\rm M}^{\mu\nu}\rangle_{\rm FIN}. \tag{27.3}$$

It is in this subtracted form that the standard applications of gravity are made. Thus in $\Sigma(a^{\dagger}a+1/2)\hbar\omega$ we keep the $\Sigma a^{\dagger}a\hbar\omega$ term but ignore the $\Sigma(1/2)\hbar\omega$ zero-point energy density term in $\langle T_{\rm M}^{00}\rangle$, precisely as is done in determining the Chandrasekhar mass or the black body contribution to cosmology. Also we ignore the zero-point pressure in the spatial $\langle T_{\rm M}^{ij}\rangle$.

As of today there is no known justification for using (27.3), with this subtracted hybrid not having been derived from a fundamental theory or having been shown to be able to survive quantum gravitational corrections. Moreover, it is this very hybrid that is used in cosmology, and the cosmological constant problem is then the phenomenological need to make $\langle T^{00}_{\rm M}\rangle_{\rm FIN}$ be small. Absent a first-principles derivation of

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)=\langle T^{\mu\nu}_{\rm M}\rangle-\langle T^{\mu\nu}_{\rm M}\rangle_{\rm DIV}=\langle T^{\mu\nu}_{\rm M}\rangle_{\rm FIN}, \tag{27.4}$$

this is not the right starting point for attacking the cosmological constant problem.

There is no apparent reason why the zero point energy density of the matter sector should not gravitate. Moreover, while one only needs to consider energy differences in flat space physics, in gravity one has to consider energy itself, with the hallmark of Einstein gravity being that gravity couples to everything. Hence for gravity one cannot ignore zero-point contributions. And if we throw them away then gravity does not know where the zero of energy is – and that is what creates the cosmological constant problem.

Moreover, even if one does start with the subtracted hybrid, then the order G contribution to gravity is given by the flat spacetime $\langle\Omega|T^{\mu\nu}_{\rm M}|\Omega\rangle_{\rm FIN}$, with Lorentz invariance allowing a finite flat spacetime $\langle\Omega|T^{\mu\nu}_{\rm M}|\Omega\rangle_{\rm FIN}$ to be of the generic form $-\Lambda g_{\mu\nu}$. Thus even if one ignores the matter sector zero-point energy contributions one still has a vacuum energy problem; with the standard strong, electromagnetic, and weak interactions typically then generating a huge such Λ. We thus recognize two types of vacuum problem, zero-point and $-\Lambda g_{\mu\nu}$ problems.

Moreover, if we take gravity to be quantum-mechanical (as we must), it too will have divergent zero-point contributions. However, in order to be able to discuss gravitational zero-point contributions, we need a consistent quantum gravity, This leads us (Mannheim 2017) to conformal gravity, viz. gravity based on the square of the Weyl tensor, viz.

$$I_{\mathrm{W}} = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \tag{27.5}$$

where

$$C_{\lambda\mu\nu\kappa} \;=\; R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^{\alpha}{}_{\alpha} \left[g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] - \frac{1}{2} \left[g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right], \tag{27.6}$$

and α_g is a dimensionless gravitational coupling constant.
As has been shown in Mannheim 2017, rather than the divergent zero-point energy density in the gravity sector being yet another vacuum energy problem, instead it is its interplay with the matter field zero-point contribution that actually leads to a solution to the cosmological constant problem. Thus the cosmological constant problem arises entirely due to ignoring how $\langle T_{\mathrm{M}}^{\mu\nu} \rangle_{\mathrm{FIN}}$ got to be finite in the first place, and then using $R^{\mu\nu} - (1/2) g^{\mu\nu} R^{\alpha}{}_{\alpha} = -8\pi G \langle T_{\mathrm{M}}^{\mu\nu} \rangle_{\mathrm{FIN}}$ as the starting point, with gravity actually being quantized by the source that it is coupled to, rather than by being quantized on its own.

As discussed in Mannheim 2017 theory conformal gravity recovers Schwarzschild on solar system distances, generates linear and quadratic potentials in galaxies that remove the need for galactic dark matter, solves the cosmological horizon, flatness, accelerating universe and cosmological constant problems without any fine tuning or any need for any cosmological dark matter, and provides a consistent renormalizable, unitary theory of quantum gravity in four spacetime dimensions, the only dimensions for which we have any evidence.

SUMMARY

All the big problems have a common origin: The extrapolation of standard Newton-Einstein Gravity beyond its solar system origins.

1. Continue to galaxies get **dark matter problem**
2. Continue to cosmology get the **cosmological constant/dark energy problem**
3. Continue to quantum field theory far off the mass shell get the **renormalization** and **vacuum zero-point energy problems**

The Standard Solution: Supersymmetry, Extra Dimensions, String Theory, The Multiverse, The Anthropic Principle. No evidence for any of them. But until recently no evidence against any of them either.

Recent evidence against supersymmetry. Not found at the LHC. Should have been found in same energy region as the recently found Higgs boson.

Solution: Change the extrapolation: get conformal gravity. All these problems are solved, with no need for any of the dark fixes. Quantum conformal gravity is ghost free and unitary (Bender and Mannheim, PRL 100, 110402 (2008); PRD 78, 025022 (2008)). Through scale invariance with anomalous dimensions Higgs boson is a naturally dynamical fermion-antifermion bound state (Mannheim, Prog. Part. Nucl. Phys. 94, 125 (2017)), so no hierarchy problem.

MORAL OF THE STORY

At the beginning of the 20th century studies of black-body radiation on microscopic scales led to a paradigm shift in physics. Thus it could that at the beginning of the 21st century studies of phenomena such as black-body radiation, this time on macroscopic cosmological scales, might be presaging a paradigm shift all over again.