QUANTUM MECHANICS OFF THE BEATEN TRACK

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We discuss some non-conventional options for quantum theory. We show that not only do Hamiltonians not need to be Hermitian to have real eigenvalues, they do not even need to be diagonalizable or possess a complete spectrum of energy eigenstates. We discuss how to formulate quantum mechanics in such cases. We show that it is not always possible to write the momentum operator as the familiar derivative operator $-id/dx$ where x is real, and discuss what one should then do. Following work by Bender and Mannheim [Phys. Rev. Lett. 100, 110402 (2008)] we show how these issues are relevant to the construction of a consistent theory of quantum gravity in four spacetime dimensions. In this theory there is no observable graviton.

P. D. Mannheim, *Extension of the CPT Theorem to non-Hermitian Hamiltonians and Unstable States*, Phys. Lett. B 753, 286 (2016). (arXiv:1512.03736 [quant-ph]).

P. D. Mannheim, *Antilinearity Rather than Hermiticity as a Guiding Principle for Quantum Theory*, J. Phys. A 51, 315302 (2018). (arXiv:1512.04915 [hep-th]).

1 ANTILINEARITY VERSUS HERMITICITY

(1). Hermiticity implies the reality of eigenvalues

$$|H - \lambda I| = 0, \quad |H^\dagger - \lambda^* I| = 0, \quad |H - \lambda^* I| = 0, \quad \lambda = \lambda^*. \tag{1}$$

(2). Hermiticity implies conservation of probability

$$i\partial_t|n\rangle = H|n\rangle, \quad -i\partial_t\langle n| = \langle n|H^\dagger, \quad i\partial_t\langle n|n\rangle = \langle n|(H - H^\dagger)|n\rangle = 0 \tag{2}$$

(3). Is the converse true? Does reality of eigenvalues and conservation of probability imply Hermiticity.

(4). Answer no. Bender and Boettcher (1998): $H = p^2 + ix^3$ has all eigenvalues real.

(5). So if Hermiticity is only sufficient, what is the necessary condition: H has to have an antilinear symmetry A that effects $AiA^{-1} = -i$. For non-relativistic physics $A = PT$ (Bender), for relativistic physics $A = CPT$ (Mannheim). P is parity, T is time reversal, C is charge conjugation.

(6). So what is the necessary and sufficient condition? Hamiltonian must possess an antilinear symmetry, and the eigenstates of the Hamiltonian must be eigenstates of the antilinear operator.

For finite-dimensional systems that obey $[H, PT] = 0$ with a diagonalizable H, one can always construct a $\mathcal{C}$ operator that obeys $\mathcal{C}^2 = I$, $[H, \mathcal{C}] = 0$. Necessary and sufficient condition for real eigenvalues (Bender and Mannheim 2010) is that $[\mathcal{C}, PT] = 0$.

2 HOW ANTILINEAR SYMMETRY WORKS

Consider the eigenvector equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle = E|\psi(t)\rangle. \tag{3}$$

Replace the parameter t by $-t$ and then multiply by some general antilinear operator A:

$$i\frac{\partial}{\partial t}A|\psi(-t)\rangle = AHA^{-1}A|\psi(-t)\rangle = E^*A|\psi(-t)\rangle. \tag{4}$$

If H has an antilinear symmetry so that $AHA^{-1} = H$, then

$$HA|\psi(-t)\rangle = E^*A|\psi(-t)\rangle. \tag{5}$$

(1) (Wigner in study of time reversal): Energies can be real and have eigenfunctions that obey $A|\psi(-t)\rangle = |\psi(t)\rangle$,

(2) or energies can appear in complex conjugate pairs that have conjugate eigenfunctions ($|\psi(t)\rangle \sim \exp(-iEt)$ and $A|\psi(-t)\rangle \sim \exp(-iE^*t)$).

As to the converse, suppose we are given that the energy eigenvalues are real or appear in complex conjugate pairs. In such a case not only would E be an eigenvalue but E^* would be too. Hence, we can set $HA|\psi(-t)\rangle = E^*A|\psi(-t)\rangle$ in (4), and obtain

$$(AHA^{-1} - H)A|\psi(-t)\rangle = 0. \tag{6}$$

Then if the eigenstates of H are complete, (6) must hold for every eigenstate, to yield $AHA^{-1} = H$ as an operator identity, with H thus having an antilinear symmetry.

3 A SIMPLE EXAMPLE

$$N = \begin{pmatrix} C+A & iB \\ iB & C-A \end{pmatrix}, \tag{7}$$

where A, B and C are all real. The matrix N is not Hermitian but does have a PT symmetry if we set $P = \sigma_3$ and $T = K$ where K effects complex conjugation, $[H, KP] = 0$, $PKHKP = H$. The eigenvalues of N are given by

$$\Lambda_{\pm} = C \pm (A^2 - B^2)^{1/2}, \tag{8}$$

and they are real if $A^2 \geq B^2$ and in a complex conjugate pair if $A^2 < B^2$, just as required of a non-Hermitian but PT-symmetric matrix.

In addition, if $A = B$ the matrix N only has one eigenvector despite having two solutions to $|M - \lambda I| = 0$ (both with $\lambda = C$), viz.

$$\begin{pmatrix} C+A & iA \\ iA & C-A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} Ca + Aa + iAb \\ iAa + Ca - Ab \end{pmatrix} = C \begin{pmatrix} a \\ b \end{pmatrix} \tag{9}$$

only has one eigenvector, viz

$$b = ia, \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ ia \end{pmatrix} \tag{10}$$

and thus cannot be diagonalized by a similarity transformation. It is thus a non-diagonalizable, Jordan-block matrix. This particular Jordan-block situation is a case where the Hamiltonian is manifestly non-diagonalizable and thus manifestly non-Hermitian and yet all eigenvalues are real.

Three options for antilinear symmetry:

(1) Energies real and eigenvectors complete.

(2) Energies in complex pairs and eigenvectors complete.

(3) Energies real but eigenvectors incomplete.

Antilinearity richer than Hermiticity.

For decays: there is a transition between the two members of the complex conjugate pair, one decaying in time ($e^{-iE_Rt-E_It}$), the other growing ($e^{-iE_Rt+E_It}$). As population of decaying one decreases population of growing one increases. If Hamiltonian is Hermitian there should be no decays, just stationary states with real energies.

$$N = \begin{pmatrix} C+A & iB \\ iB & C-A \end{pmatrix}. \tag{11}$$

Introduce

$$S = \frac{1}{2(A^2-B^2)^{1/4}} \begin{pmatrix} (A+B)^{1/2}+(A-B)^{1/2} & i[(A+B)^{1/2}-(A-B)^{1/2}] \\ -i[(A+B)^{1/2}-(A-B)^{1/2}] & (A+B)^{1/2}+(A-B)^{1/2} \end{pmatrix}, \tag{12}$$

$$S^{-1} = \frac{1}{2(A^2-B^2)^{1/4}} \begin{pmatrix} (A+B)^{1/2}+(A-B)^{1/2} & -i[(A+B)^{1/2}-(A-B)^{1/2}] \\ i[(A+B)^{1/2}-(A-B)^{1/2}] & (A+B)^{1/2}+(A-B)^{1/2} \end{pmatrix}, \tag{13}$$

$$V = \frac{1}{(A^2-B^2)^{1/2}} \begin{pmatrix} A & iB \\ -iB & A \end{pmatrix}, \quad V^{-1} = \frac{1}{(A^2-B^2)^{1/2}} \begin{pmatrix} A & -iB \\ iB & A \end{pmatrix}, \tag{14}$$

and they effect

$$\begin{aligned} SNS^{-1} = N' &= \begin{pmatrix} C+(A^2-B^2)^{1/2} & 0 \\ 0 & C-(A^2-B^2)^{1/2} \end{pmatrix}, \\ VNV^{-1} &= \begin{pmatrix} C+A & -iB \\ -iB & C-A \end{pmatrix} = N^{\dagger} \end{aligned} \tag{15}$$

Thus can diagonalize N as long as $A \neq B$, and can construct a V that effects $VNV^{-1} = N^{\dagger}$, just as is characteristic of a matrix with an antilinear symmetry, and with $A = B$ being Jordan block. If $VNV^{-1} = N^{\dagger}$ then N and $N^{\dagger}$ have common set of eigenvalues, so real or in complex pairs.

So now let us look at the eigenvectors.

When $A^2 > B^2$ the left- and right-eigenvectors that obey $\langle L_\pm| = \langle R_\pm|V$ are given by

$$
\begin{aligned}
R_+ &= \frac{1}{2(A^2-B^2)^{1/4}}\begin{pmatrix} (A+B)^{1/2}+(A-B)^{1/2} \\ i[(A+B)^{1/2}-(A-B)^{1/2}] \end{pmatrix},\\
R_- &= \frac{1}{2(A^2-B^2)^{1/4}}\begin{pmatrix} -i[(A+B)^{1/2}-(A-B)^{1/2}] \\ (A+B)^{1/2}+(A-B)^{1/2} \end{pmatrix},\\
L_+ &= \frac{1}{2(A^2-B^2)^{1/4}}\left((A+B)^{1/2}+(A-B)^{1/2}, \quad i[(A+B)^{1/2}-(A-B)^{1/2}] \right),\\
L_- &= \frac{1}{2(A^2-B^2)^{1/4}}\left(-i[(A+B)^{1/2}-(A-B)^{1/2}], \quad (A+B)^{1/2}+(A-B)^{1/2} \right), \qquad (16)
\end{aligned}
$$

and these eigenvectors are normalized according to the positive definite $\langle L_n|R_m\rangle = \langle R_n|V|R_m\rangle = \delta_{m,n}$, i.e. according to $L_\pm R_\pm = 1$, $L_\mp R_\pm = 0$. In addition N and the identity I can be reconstructed as

$$N = |R_+\rangle\Lambda_+\langle L_+| + |R_-\rangle\Lambda_-\langle L_-|, \quad I = |R_+\rangle\langle L_+| + |R_-\rangle\langle L_-|, \qquad (17)$$

to thus be diagonalized in the left-right basis.

While N is not Hermitian, when $A^2 > B^2$, N can be brought to a Hermitian form. It is thus Hermitian in disguise, i.e. it does not look Hermitian but it can be brought to a Hermitian form by a similarity transformation.

To appreciate the point, consider $H^\dagger = H$ and transform with $H' = SHS^{-1}$ where $SS^\dagger \neq I$. We obtain

$$H'^\dagger = S^{-1\dagger}H^\dagger S^\dagger = S^{-1\dagger}HS^\dagger = S^{-1\dagger}S^{-1}H'SS^\dagger = [SS^\dagger]^{-1}H'SS^\dagger \neq H'. \tag{18}$$

Thus unless S is unitary $H'^\dagger$ is not equal to H', with the $H_{ij} = H^*_{ji}$ Hermiticity condition being a condition that is not preserved under a general similarity transformation. Thus if one starts with some general H' that does not obey $H' = H'^\dagger$, it might be similarity equivalent to a Hermitian H but one does not know a priori. It only will be similarity equivalent to a Hermitian H if the eigenvalues of H' are all real and the eigenspectrum is complete. And the necessary condition for that to be the case is that H' possess an antilinear symmetry.

In general one should define Hermitian in a basis-independent way, taking it to mean that eigenvalues are real and that eigenbasis is complete. Then there will always exist a basis in which $H_{ij} = H^*_{ji}$.

When $A^2 - B^2$ is negative the eigenvalues are in a complex pair.

When $A^2 = B^2$ (viz. $\Lambda_\pm = C$),

$$N = \begin{pmatrix} C+A & iB \\ iB & C-A \end{pmatrix} \to N = \begin{pmatrix} C+A & iA \\ iA & C-A \end{pmatrix}, \quad \Lambda_\pm = C \pm (A^2 - B^2)^{1/2} \to C, \tag{19}$$

we note that even though all of $L_\pm$, $R_\pm$ become singular at $A^2 = B^2$, N still has left- and right-eigenvectors L and R that are given up to an arbitrary normalization by

$$L = \begin{pmatrix} 1 & i \end{pmatrix}, \quad R = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad LN = CN, \quad NR = CR, \tag{20}$$

and no matter what that normalization might be, they obey the **zero norm** condition characteristic of Jordan-block matrices:

$$LR = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0. \tag{21}$$

Even though the eigenspectrum of N is incomplete, the vector space on which it acts is still complete. One can take the extra states to be

$$L' = \begin{pmatrix} 1 & -i \end{pmatrix}, \quad R' = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{22}$$

with $L'R' = 0$, so that R and R' span the space on which N acts to the right, while L and L' span the space on which N acts to the left.

4 FURTHER INSIGHT INTO NON-DIAGONALIZABLE HAMILTONIANS

Consider the two oscillator system obeying the fourth-order derivative

$$(\partial_t^2 + \omega_1^2)(\partial_2^2 + \omega_2^2)\psi(t) = 0 \tag{23}$$

There are two solutions

$$\psi_1(t) = e^{i\omega_1 t}, \quad \psi_2(t) = e^{i\omega_2 t}, \tag{24}$$

together with their conjugates.

Now let $\omega_1 = \omega_2 = \omega$. The system now obeys

$$(\partial_t^2 + \omega^2)^2\psi(t) = 0 \tag{25}$$

and the two solutions collapse onto

$$\frac{1}{2}\left[\psi_1(t) + \psi_2(t)\right] \to \psi(t) = e^{i\omega t}. \tag{26}$$

So where did the other solution go? Set $\omega_1 = \omega + \epsilon$, $\omega_2 = \omega - \epsilon$

$$\frac{1}{2\epsilon}\left[\psi_1(t) - \psi_2(t)\right] = \frac{1}{2\epsilon}e^{i\omega t}\left[e^{i\epsilon t} - e^{-i\epsilon t}\right] \to \frac{1}{2\epsilon}e^{i\omega t}\left[1 + i\epsilon t - (1 - i\epsilon t)\right] = ite^{i\omega t}. \tag{27}$$

Thus the second solution is **NOT** stationary (not an eigenstate of $i\partial_t$). Hence it is **NOT** an eigenstate of the Hamiltonian. This is why the Hamiltonian is **NOT** diagonalizable, it is missing an eigenstate. As for the eigenstate it does have, its norm is zero. Thus in the limit both $\psi_1(t) = e^{i\omega_1 t}$ and $\psi_2(t) = e^{i\omega_2 t}$ become unobservable.

The conformal gravity theory is a fourth-order derivative theory of gravity, and even though it has classical gravitational waves **its quantized version has no observable graviton**.

5 PROBABILITY CONSERVATION

Consider a right eigenstate of H in which H acts to the right as $i\partial_t|R(t)\rangle = H|R(t)\rangle$ with solution $|R(t)\rangle = \exp(-iHt)|R(0)\rangle$. The Dirac norm

$$\langle R(t)|R(t)\rangle = \langle R(0)|\exp(iH^\dagger t)\exp(-iHt)|R(0)\rangle \tag{28}$$

is not time independent if H is not Hermitian, and would not describe unitary time evolution. However, this only means that the Dirac norm is not unitary, not that no norm is unitary.

Since $i\partial_t|R(t)\rangle = H|R(t)\rangle$ only involves ket vectors, there is some freedom in choosing bra vectors. So let us introduce a more general scalar product $\langle R(t)|V|R(t)\rangle$ with some time-independent linear operator V. We find

$$i\frac{\partial}{\partial t}\langle R_j(t)|V|R_i(t)\rangle = \langle R_j(t)|(VH - H^\dagger V)|R_i(t)\rangle. \tag{29}$$

Thus if we set

$$VH - H^\dagger V = 0, \tag{30}$$

then scalar products will be time independent and probability is conserved.

For the converse we note if we are given that all V scalar products are time independent, then if the set of all $|R_i(t)\rangle$ is complete we would obtain $VH - H^\dagger V = 0$ as an operator identity. The condition $VH - H^\dagger V = 0$ is thus both **necessary and sufficient** for the time independence of the V scalar products $\langle R(t)|V|R(t)\rangle$.

Now if $VH - H^\dagger V = 0$, we can set $VH|\psi\rangle = EV|\psi\rangle = H^\dagger V|\psi\rangle$. Consequently H and $H^\dagger$ have the same set of eigenvalues, i.e. for every E there is an E^*. (When V is invertible, this also follows from $H^\dagger = VHV^{-1}$, an isospectral similarity transformation.) Energy eigenvalues are thus either real or in complex conjugate pairs.

Consequently, H must have an antilinear symmetry.

Consider

$$-i\frac{\partial}{\partial t}\langle R|V = \langle R|H^\dagger V = \langle R|VH. \tag{31}$$

Can identify $\langle L| = \langle R|V$ as left-eigenvector, and thus inner product is $\langle L|R\rangle$, and operator matrix elements are $\langle L|\hat{O}|R\rangle$.

6 THE TAKEAWAY

Antilinear symmetry of a Hamiltonian implies that energies are real or in complex pairs.

Energies real or in complex pairs implies that Hamiltonian has an antilinear symmetry.

REALITY OF EIGENVALUES IMPLIES ANTILINEARITY NOT HERMITICITY

Probability conservation implies that Hamiltonian has an antilinear symmetry.

Antilinear symmetry of a Hamiltonian implies that probability is conserved.

CONSERVATION OF PROBABILITY IMPLIES ANTILINEARITY NOT HERMITICITY

Since a Hamiltonian cannot have more eigenvectors than right and left ones, $\langle L|R\rangle = \langle R|V|R\rangle$ is most general inner product one can use.

One never needs to impose Hermiticity. One only needs to impose antilinear symmetry.

But is any particular antilinear symmetry to be preferred. Study implications of Lorentz invariance.

So consider as an example

$$I_{\rm S}=\int d^4x\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi-m^2\phi^2] \tag{32}$$

with Hamiltonian

$$H=\int d^3x\frac{1}{2}[\dot{\phi}^2+\vec{\nabla}\phi\cdot\vec{\nabla}\phi+m^2\phi^2]. \tag{33}$$

Solutions to the wave equation obey

$$\phi(\vec{x},t)=\Sigma[a(\vec{k})\exp(-i\omega_kt+i\vec{k}\cdot\vec{x})+a^\dagger(\vec{k})\exp(+i\omega_kt-i\vec{k}\cdot\vec{x})],\qquad \omega_k^2=\vec{k}^2+m^2 \tag{34}$$

and the Hamiltonian is given by

$$H=\Sigma\frac{1}{2}[\vec{k}^2+m^2]^{1/2}[a^\dagger(\vec{k})a(\vec{k})+a(\vec{k})a^\dagger(\vec{k})]. \tag{35}$$

If $m^2>0$ all energies are real, and both H and $\phi(\vec{x},t)$ are Hermitian. However, if $m^2=-n^2<0$, now

$$\omega_k^2=\vec{k}^2-n^2, \tag{36}$$

the $k<n$ energies come in complex conjugate pairs and neither H nor $\phi(\vec{x},t)$ is Hermitian.

Instead H is CPT symmetric, and $\phi(\vec{x},t)$ is CPT even. Despite this, the standard derivation of the CPT theorem would have identified $I_{\rm S}=\int d^4x[\partial_\mu\phi\partial^\mu\phi+n^2\phi^2]/2$ as being a Hermitian theory. **But it is not, and one cannot tell by inspection.** One needs to solve the theory and get the solutions first. Nonetheless, in both the $m^2>0$ and $m^2<0$ cases $\phi(\vec{x},t)$ is a CPT even field and H is CPT invariant (since m^2 is real), and is something that one can tell by inspection. Thus CPT symmetry is input, and H and $\phi(\vec{x},t)$ will only be Hermitian for certain values of parameters (reminiscent of our two-dimensional example where $E_{\pm}=C\pm(A^2-B^2)^{1/2}$).

Another example: the Pais-Uhlenbeck (PU) oscillator with Hamiltonian

$$H_{\rm PU}=\frac{p_x^2}{2\gamma}+p_zx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2-\frac{\gamma}{2}\omega_1^2\omega_2^2z^2. \tag{37}$$

Energies $E(n_1,n_2)=(n_1+1/2)\omega_1+(n_2+1/2)\omega_2$ are real if ω_1 and ω_2 are real. However, now set $\omega_1=\alpha+i\beta$, $\omega_2=\alpha-i\beta$ with α and β real. Energies now come in complex conjugate pairs and yet this necessarily non-Hermitian Hamiltonian is given by the seemingly Hermitian

$$H_{\rm PU}=\frac{p_x^2}{2\gamma}+p_zx+\gamma(\alpha^2-\beta^2)x^2-\frac{\gamma}{2}(\alpha^2+\beta^2)z^2. \tag{38}$$

Hermiticity never needs to be postulated, with it being output in those cases in which it is found to occur.

Probability conservation and complex Lorentz invariance entail CPT invariance not Hermiticity.

7 RELATION OF PT AND CPT TO COMPLEX LORENTZ TRANSFORMATIONS

On coordinates PT implements $x^\mu \to -x^\mu$, and thus so does CPT since the coordinates are charge conjugation even. With a boost in the x_1-direction implementing $x_1' = x_1\cosh\xi + t\sinh\xi$, $t' = t\cosh\xi + x_1\sinh\xi$, with complex $\xi = i\pi$ we obtain

$$\begin{aligned}
\Lambda^0{}_1(i\pi):&\qquad x_1 \to -x_1, \qquad t\to -t,\\
\Lambda^0{}_2(i\pi):&\qquad x_2 \to -x_2, \qquad t\to -t,\\
\Lambda^0{}_3(i\pi):&\qquad x_3 \to -x_3, \qquad t\to -t,\\
\pi\tau = \Lambda^0{}_3(i\pi)\Lambda^0{}_2(i\pi)\Lambda^0{}_1(i\pi):&\qquad x^\mu \to -x^\mu.
\end{aligned} \tag{39}$$

Complex $\pi\tau$ implements the linear part of a PT and CPT transformation on coordinates.

With $\Lambda^0{}_i(i\pi)$ implementing $e^{-i\pi\gamma^0\gamma_i/2} = -i\gamma^0\gamma_i$ for Dirac gamma matrices, on introducing

$$\hat{\pi}\hat{\tau} = \hat{\Lambda}^0{}_3(i\pi)\hat{\Lambda}^0{}_2(i\pi)\hat{\Lambda}^0{}_1(i\pi), \tag{40}$$

we obtain

$$\hat{\pi}\hat{\tau}\psi_1(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_1(-x), \qquad \hat{\pi}\hat{\tau}\psi_2(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_2(-x). \tag{41}$$

Thus up to an overall complex phase, quite remarkably we recognize this transformation as acting as none other than the **linear** part of a CPT transformation since $\hat{C}\hat{P}\hat{T}[\psi_1(x)+i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)]$.

Thus CPT is naturally associated with the complex Lorentz group. Complex Lorentz invariance plus probability conservation implies CPT invariance without requiring Hermiticity (Mannheim2018).

8 What do we Mean by Hermitian, and When is it Different from Self-adjoint.

In a given basis $H = H^\dagger$ means $H_{ij} = H^*_{ji}$. Apply a similarity transformation. $H' = SHS^{-1}$. Get

$$(H')^\dagger = (S^{-1})^\dagger H^\dagger S^\dagger = (S^{-1})^\dagger H S^\dagger. \tag{42}$$

Only equals H' if

$$(S^{-1})^\dagger H S^\dagger = SHS^{-1}, \tag{43}$$

i.e. if

$$S^{-1}(S^{-1})^\dagger H S^\dagger S = H. \tag{44}$$

Not obeyed in general if S is not unitary, i.e. if $S^\dagger \neq S^{-1}$. Arbitrary S thus transforms to a non-orthonormal skew basis, with $H_{ij} = H^*_{ji}$ being a nonlinear relation that only holds in certain bases. Thus what we mean by Hermitian is that we can find a basis in which $H_{ij} = H^*_{ji}$. **Basis independent definition: Hermitian means all eigenvalues real and eigenfunctions complete.**

In contrast, a commutation relation is preserved under a similarity transformation (even with antilinear operators), with $[A', B'] = [A, B]$. Antilinear symmetry is thus basis independent.

For a second-order differential operator D in the form $D = -p(x)d^2/dx^2 - p'(x)d/dx + q(x)$ that acts on wave functions $\phi(x)$, $\psi(x)$, one can show (Green's theorem) that

$$\int_a^b dx[\phi^* D\psi - (\psi^* D\phi)^*] = \int_a^b dx[\phi^* D\psi - [D\phi]^*\psi] = \left[p\psi\phi^{*\prime} - p\phi\psi^{*\prime}\right]_a^b \tag{45}$$

Self-adjointness requires the vanishing of the surface term. Then get standard definition of Hermitian.

To define a commutator $[\hat{x}, \hat{p}] = i\hbar$, we need to specify a basis on which it acts. Can set $\hat{p} = -i\hbar d/dx$ only when acting on a good, i.e. normalizable, test function according to

$$\left[\hat{x}, -i\hbar \frac{d}{dx}\right] \psi(x) = i\hbar\psi(x). \tag{46}$$

Thus for a harmonic oscillator $\hat{H} = \hat{p}^2 + \hat{x}^2$ for instance we have the following two solutions:

$$\left[-\frac{d^2}{dx^2} + x^2\right] e^{-x^2/2} = e^{-x^2/2}, \qquad \left[-\frac{d^2}{dx^2} + x^2\right] e^{+x^2/2} = -e^{+x^2/2} \tag{47}$$

Of them only the $e^{-x^2/2}$ wave function is normalizable (cf. vanishing of the surface term), with $\int dx \psi^*(x)\psi(x)$ being finite. And when acting on it we can indeed represent $\hat{p}$ as $\hat{p} = -i\hbar d/dx$. Here x is real and we are working in the coordinate basis in which $\hat{x}$ is Hermitian, has real eigenvalues x, and is diagonal in this basis.

$$\left[-\frac{d^2}{dx^2}+x^2\right]e^{-x^2/2}=e^{-x^2/2},\quad \left[-\frac{d^2}{dx^2}+x^2\right]e^{+x^2/2}=-e^{+x^2/2} \tag{48}$$

But what of the $e^{+x^2/2}$ solution. It is not normalizable and we cannot represent $\hat{p}=-i\hbar d/dx$ when acting on it since cannot throw away the surface term in an integration by parts. However suppose we make x pure imaginary. Then $e^{+x^2/2}$ is normalizable on the imaginary axis. Thus we can take both $\hat{x}$ and $\hat{p}$ to be anti-Hermitian and represent $[\hat{x},\hat{p}]=i\hbar$ as $[-i\hat{x},i\hat{p}]=i\hbar$. This is equivalent to the similarity transformation $\hat{S}=\exp(-\pi\hat{p}\hat{x}/2)$ that effects

$$\hat{S}\hat{p}\hat{S}^{-1}=i\hat{p}=\hat{q},\quad \hat{S}\hat{x}\hat{S}^{-1}=-i\hat{x}=\hat{y}, \tag{49}$$

while preserving both the commutation relation $[\hat{x},\hat{p}]=[\hat{y},\hat{q}]=i$ and the eigenvalues of a Hamiltonian $\hat{H}(\hat{x},\hat{p})$ that is built out of $\hat{x}$ and $\hat{p}$. We thus have

$$\left[\hat{y},\hbar\frac{d}{dy}\right]\psi(y)=i\hbar\psi(y), \tag{50}$$

and now $e^{+x^2/2}=e^{-y^2/2}$ is a good test function. Thus $e^{-x^2/2}$ is a good test function when x is real, while $e^{+x^2/2}$ is a good test function when x is pure imaginary. When x is pure imaginary we can set

$$[\hat{p}^2+\hat{x}^2]e^{+x^2/2}=-[\hat{q}^2+\hat{y}^2]e^{-y^2/2}=\left[\frac{d^2}{dy^2}-y^2\right]e^{-y^2/2}=-e^{-y^2/2}. \tag{51}$$

Thus while the eigenvalues of $\hat{p}^2+\hat{x}^2$ would be positive if $\hat{p}$ and $\hat{x}$ are both Hermitian, the eigenvalues of $\hat{p}^2+\hat{x}^2$ would be negative if $\hat{p}$ and $\hat{x}$ are both anti-Hermitian.

Thus for $\hat{H}=\hat{p}^2+\hat{x}^2$, while both $\hat{x}$ and $\hat{p}$ might be Hermitian (or self-adjoint) when acting on their own eigenstates that does not make them Hermitian when acting on the eigenstates of $\hat{H}$. Ditto $\hat{H}=\hat{p}^2+i\hat{x}^3$. This is the secret of PT.

Now we can always make a similarity transformation through any angle such as $\hat{S} = \exp(\theta \hat{p}\hat{x})$ that effects $\hat{S}\hat{p}\hat{S}^{-1} = \hat{p}\exp(-i\theta)$, $\hat{S}\hat{x}\hat{S}^{-1} = \hat{x}\exp(i\theta)$. Ordinarily this is not of any significance since we work with Hermitian operators that have normalizable wave functions on the real axis, and we have no need to go into the complex plane. But if the wave functions are not normalizable on the real axis, we may be able to continue into a so-called "Stokes wedge" in the complex plane where they then are normalizable, and cross over a "Stokes line" that divides the two regions ($\theta = \pi/4$ in the harmonic oscillator case). This is what happens with $\hat{H} = \hat{p}^2 + i\hat{x}^3$.

However, independent of whether or not a Hamiltonian might be Hermitian, if it has an antilinear symmetry it must be self-adjoint in some Stokes wedge in the complex plane. And in such wedges one must use the $\langle L|R\rangle$ norm.

Now this is true no matter whether energies are all real, whether some or all energies come in complex conjugate pairs, or whether the Hamiltonian is a non-diagonalizable Jordan-block Hamiltonian. These latter two cases represent Hamiltonians that are not Hermitian but are self-adjoint.

The art of the PT symmetry program is to find the appropriate Stokes wedges in the complex plane.

Theorem: Antilinear symmetry implies self-adjointness, while self-adjointness implies antilinearity.

Thus as with self-adjointness, Hermiticity is determined not by the form of the operators (i.e. not by inspection) but by the boundary conditions.

9 Conformal Gravity and Pais-Uhlenbeck Oscillator

Conformal gravity is a fourth-order derivative theory of gravity with action $I_{\rm W}=-\alpha_g\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}$, where $C_{\lambda\mu\nu\kappa}$ is the Weyl tensor. When coupled to Einstein gravity, and linearized around flat spacetime according to $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, one can find a gauge (the conformal gauge) in which the fluctuation equations are diagonal in the (μ,ν) indices and can be associated with the generic scalar field action I_S, propagator $D(k^2)$, and Hamiltonian $H=\int d^3xT_{00}$, where

$$
\begin{aligned}
I_S &= \frac{1}{2}\int d^4x\Big[\partial_{\mu}\partial_{\nu}\phi\partial^{\mu}\partial^{\nu}\phi-(M_1^2+M_2^2)\partial_{\mu}\phi\partial^{\mu}\phi+M_1^2M_2^2\phi^2\Big],\\
D(k^2) &= \frac{1}{(k^2-M_1^2)(k^2-M_2^2)}=\frac{1}{M_1^2-M_2^2}\Big(\frac{1}{k^2-M_1^2}-\frac{1}{k^2-M_2^2}\Big),
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
T_{00} &= \pi_0\dot{\phi}+\frac{1}{2}\pi_{00}^2+\frac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2-\frac{1}{2}M_1^2M_2^2\phi^2-\frac{1}{2}\pi_{ij}\pi^{ij}+\frac{1}{2}(M_1^2+M_2^2)\phi_{,i}\phi^{,i},\\
\pi^{\mu} &= \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}-\partial_{\lambda}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}\right),\qquad \pi_{\mu\lambda}=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}.
\end{aligned}
\tag{53}
$$

The relative minus sign in $D(k^2)$ suggests that the theory contains states of negative norm. To find out whether or not this is the case, following Bender and Mannheim (2008) we explicitly construct the Hilbert space.

To see what is involved we note that on setting $\omega_1=(\bar{k}^2+M_1^2)^{1/2}$, $\omega_2=(\bar{k}^2+M_2^2)^{1/2}$ and dropping the spatial dependence, the action reduces to the quantum-mechanical Pais-Uhlenbeck oscillator model action

$$
I_{\rm PU}=\frac{\gamma}{2}\int dt\left[\ddot{z}^2-\left(\omega_1^2+\omega_2^2\right)\dot{z}^2+\omega_1^2\omega_2^2z^2\right],
\tag{54}
$$

and with $x=\dot{z}$, $[z,p_z]=i$, $[x,p_x]=i$, the Hamiltonian is given by (Mannheim and Davidson (2000))

$$
H_{\rm PU}=\frac{p_x^2}{2\gamma}+p_zx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2-\frac{\gamma}{2}\omega_1^2\omega_2^2z^2.
\tag{55}
$$

When ω_1 and ω_2 are both taken to be real and positive, all the eigenvalues of $H_{\rm PU}$ are real. When $M_1^2 = M_2^2$ and $\omega_1 = \omega_2$, $H_{\rm PU}$, and thus the Hamiltonian associated with the pure conformal gravity $I_{\rm W}$, are Jordan-block (the partial fraction decomposition of $D(k^2)$ in (52) becomes undefined. When $\omega_1 = \alpha + i\beta$, $\omega_1 = \alpha - i\beta$, the energy eigenvalues appear in complex conjugate pairs. In all the three cases $H_{\rm PU}$ has an antilinear symmetry.

The wave function associated with $E = (\omega_1 + \omega_2)$ is of the form $(x = \dot{z})$

$$\psi(z,x) \;=\; \exp\Big[\frac{\gamma}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2 + i\gamma\omega_1\omega_2 zx - \frac{\gamma}{2}(\omega_1+\omega_2)x^2\Big]. \tag{56}$$

The wave function associated with $E = 2\omega$ where $\omega_1 = \omega_2 = \omega$ is of the form

$$\psi(z,x) \;=\; \exp\Big[\gamma\omega^3 z^2 + i\gamma\omega^2 zx - \gamma\omega x^2\Big]. \tag{57}$$

The wave function associated with $E = 2\alpha$ where $\omega_1 = \alpha + i\beta$, $\omega_1 = \alpha - i\beta$ $(\alpha > 0)$ is of the form

$$\psi(z,x) \;=\; \exp\Big[\gamma\alpha(\alpha^2+\beta^2)z^2 + i\gamma(\alpha^2+\beta^2)zx - \gamma\alpha x^2\Big]. \tag{58}$$

None of these wave functions is normalizable on the real z axis, but all are normalizable on the pure imaginary z axis. Thus all are self-adjoint in appropriate Stokes wedges that contain the imaginary z axis. We cannot use the Dirac norm in these wedges. Instead we must use the $\langle L|R\rangle$ norm, and it is found (Bender and Mannheim) to never be negative.

The propagator is not given by $\langle\Omega_R|T(\phi(x)\phi(0))|\Omega_R\rangle$ but by $\langle\Omega_L|T(\phi(x)\phi(0))|\Omega_R\rangle = \langle\Omega_R|VT(\phi(x)\phi(0))|\Omega_R\rangle$ instead. Thus one cannot identify a c-number propagator such $D(k^2) = 1/k^4$ with a quantum field theory matrix element until one first constructs the appropriate Hilbert space.

For conformal gravity, it is the V operator that generates the relative minus sign in $D(k^2)$ in (52) and not any Hilbert space negative norm structure. In consequence conformal gravity is ghost free and unitary. With it also being renormalizable (its coupling constant α_g being dimensionless), it provides a consistent quantum gravitational theory, one constructed in the four spacetime dimensions for which there is evidence.

10 The Hermiticity Puzzle – Where Does Hermiticity Come From?

If we introduce a path integral

$$W(J) = \int D[\phi] e^{i[I_S(\phi)+J\phi]} \tag{59}$$

everything is classical. Thus no reference to any Hilbert space and no a priori justification for taking the quantum Hamiltonian to be Hermitian, since that is a quantum statement.

Consider action

$$I = \int_0^t dt\dot{x}^2, \quad x(t=0) = x_i, \quad x(t=T) = x_f, \quad \ddot{x} = 0, \quad x_{CL}(t) = \frac{(x_f - x_i)t}{T} + x_i \tag{60}$$

Let us derive the classical path by a variational principle. Consider arbitrary path between fixed end points

$$x(t) = x_{CL}(t) + \sum_n a_n \sin\left(\frac{n\pi t}{T}\right) \tag{61}$$

We obtain

$$I = \frac{(x_f - x_i)^2}{T} + \sum_n \frac{a_n^2 n^2 \pi^2}{2T}. \tag{62}$$

So take derivative with respect to each a_n, and find minimum action is when all a_n are zero, and then stationary solution is $x = x_{CL}(t)$.

But what do we do with all the a_n? Do they play a role in physics? Feynman: integrate then back up. We obtain

$$\Pi_n \int_{-\infty}^{\infty} da_n e^{iI/\hbar} = \langle x(t=0)|x(t=T)\rangle = \langle x(t=0)|e^{-i\hat{H}T}|x(t=0)\rangle. \tag{63}$$

Global approach to quantum mechanics. But where is Hermiticity?

One can implement CPT on every classical path, and thus if path integral is CPT invariant, then the associated quantum theory will be CPT invariant too, regardless of whether or not the quantum Hamiltonian might be Hermitian.

Making the path integral CPT invariant is actually non-trivial for gauge theories, since need a rule to know to use combination $i\partial_\mu - eA_\mu$ in path integral rather than the in a sense more natural, but not viable, purely real derivative $\partial_\mu - eA_\mu$. Even though in quantum theory it is $i\partial_\mu$ that is Hermitian rather than ∂_μ, the path integral does not know this. However, classically it is $i\partial_\mu$ that is CPT even, just like eA_μ. It is thus CPT symmetry that forces $i\partial_\mu - eA_\mu$ in the path integral.

Now $W(J)$ generates the c-number quantum theory Green's functions, but how do we know that we can associate them with the matrix element of a q-number operator of the form $\langle \Omega_R | T(\phi(x)\phi(0)) | \Omega_R \rangle$ rather than with $\langle \Omega_L | T(\phi(x)\phi(0)) | \Omega_R \rangle = \langle \Omega_R | VT(\phi(x)\phi(0)) | \Omega_R \rangle$ instead. So how does Hermiticity come into physics.

Answer: Path integral always exists with some real or complex measure if it is CPT invariant (analog of antilinearity implies self-adjointness). The underlying quantum Hamiltonian is Hermitian if path integral exists with a real measure, in which case the left vacuum is the conjugate of the right vacuum and $V = I$. However, if, in analog to wave functions, the path integral only exists if we need to continue the measure into the complex plane, then the underlying quantum theory is of the antilinear Bender type.

Antilinear CPT symmetry thus has primacy over Hermiticity, and it is antilinearity, not Hermiticity, that should be taken as a guiding principle for quantum theory.

11 SUMMARY

WE NEVER NEED TO POSTULATE HERMITICITY – ONLY NEED ANTILINEARITY.

HAMILTONIANS THAT HAVE AN ANTILINEAR SYMMETRY CAN BE HERMITIAN AS WELL. HERMITIAN ONLY IF PATH INTEGRAL EXISTS WITH REAL MEASURE.

ANTILINEAR SYMMETRY FOLLOWS FROM THE CONSERVATION OF PROBABILITY AND COMPLEX LORENTZ INVARIANCE ALONE.

ANTILINEAR CPT SYMMETRY HAS PRIMACY OVER HERMITICITY, AND IT IS CPT NOT HERMITICITY THAT SHOULD BE TAKEN AS A GUIDING PRINCIPLE FOR QUANTUM THEORY.

Lorentz transformations are of the form $\Lambda = e^{iw^{\mu\nu}M_{\mu\nu}}$ with six angles $w^{\mu\nu} = -w^{\nu\mu}$ and six Lorentz generators $M_{\mu\nu} = -M_{\nu\mu}$ that obey

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(-\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} + \eta_{\nu\sigma}M_{\rho\mu}). \quad (64)$$

Under a Lorentz transformation the line element transforms as

$$x^\alpha \eta_{\alpha\beta} x^\beta \rightarrow x^\alpha \widetilde{\Lambda} \eta_{\alpha\beta} \Lambda x^\beta, \quad (65)$$

(tilde denotes transpose), with $\widetilde{\Lambda} = e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}}$. Given the Lorentz algebra one has $e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}}\eta_{\alpha\beta} = \eta_{\alpha\beta}e^{-iw^{\mu\nu}M_{\mu\nu}}$ (i.e. Minkowski metric orthogonal), with the line element thus being invariant. While this analysis familiarly holds for real $w^{\mu\nu}$, since $w^{\mu\nu}$ plays no explicit role in it, the analysis equally holds if $w^{\mu\nu}$ is **complex**.

For a general spin zero Lagrangian where $w^{\mu\nu}M_{\mu\nu}$ acts as

$$w^{\mu\nu}(x_\mu p_\nu - x_\nu p_\mu) = 2w^{\mu\nu}x_\mu p_\nu.$$

Under an infinitesimal Lorentz transformation the action $I = \int d^4x L(x)$ transforms as

$$\delta I = 2w^{\mu\nu}\int d^4x x_\mu \partial_\nu L(x) = 2w^{\mu\nu}\int d^4x \partial_\nu [x_\mu L(x)], \tag{66}$$

to thus be a total derivative and thus be left invariant. However the change will be a total derivative even if $w^{\mu\nu}$ is complex. So again we see that we have invariance under **complex** Lorentz transformations.

For Majorana spinors ψ under a Lorentz transformation we have

$$\widetilde{\psi}\gamma^0\psi \to \tilde{\psi}e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}}\gamma^0 e^{iw^{\mu\nu}M_{\mu\nu}}\psi = \tilde{\psi}\gamma^0 e^{-iw^{\mu\nu}M_{\mu\nu}}e^{iw^{\mu\nu}M_{\mu\nu}}\psi = \widetilde{\psi}\gamma^0\psi. \tag{67}$$

So once again we see that we have invariance under **complex** Lorentz transforms and not just under real ones.

For Dirac spinors written as a sum of two Majorana spinors $\psi(x) = \psi_1(x) + i\psi_2(x)$, we find that under $\hat{P}$, $\hat{T}$, and $\hat{C}\hat{P}\hat{T}$

$$\begin{gathered}\hat{P}\psi(\vec{x},t)\hat{P}^{-1} = \gamma^0\psi(-\vec{x},t), \qquad \hat{T}\psi(\vec{x},t)\hat{T}^{-1} = \gamma^1\gamma^2\gamma^3\psi(\vec{x},-t),\\ \hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)],\end{gathered} \tag{68}$$

The last of these relations is central to the derivation of the CPT theorem.

THE TAKEAWAY

Complex Lorentz invariance is just as natural as real Lorentz invariance.

13 RELATION OF PT AND CPT TO COMPLEX LORENTZ TRANSFORMATIONS

On coordinates PT implements $x^\mu \to -x^\mu$, and thus so does CPT since the coordinates are charge conjugation even. With a boost in the x_1-direction implementing $x_1' = x_1 \cosh\xi + t\sinh\xi$, $t' = t\cosh\xi + x_1 \sinh\xi$, with complex $\xi = i\pi$ we obtain

$$
\begin{aligned}
\Lambda^0{}_1(i\pi): &\qquad x_1 \to -x_1, \qquad t \to -t, \\
\Lambda^0{}_2(i\pi): &\qquad x_2 \to -x_2, \qquad t \to -t, \\
\Lambda^0{}_3(i\pi): &\qquad x_3 \to -x_3, \qquad t \to -t, \\
\pi\tau = \Lambda^0{}_3(i\pi)\Lambda^0{}_2(i\pi)\Lambda^0{}_1(i\pi): &\qquad x^\mu \to -x^\mu.
\end{aligned} \tag{69}
$$

Complex $\pi\tau$ implements the linear part of a PT and CPT transformation on coordinates.

With $\Lambda^0{}_i(i\pi)$ implementing $e^{-i\pi\gamma^0\gamma_i/2} = -i\gamma^0\gamma_i$ for Dirac gamma matrices, on introducing

$$
\hat{\pi}\hat{\tau} = \hat{\Lambda}^0{}_3(i\pi)\hat{\Lambda}^0{}_2(i\pi)\hat{\Lambda}^0{}_1(i\pi), \tag{70}
$$

we obtain

$$
\hat{\pi}\hat{\tau}\psi_1(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_1(-x), \qquad \hat{\pi}\hat{\tau}\psi_2(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_2(-x). \tag{71}
$$

Thus up to an overall complex phase, quite remarkably we recognize this transformation as acting as none other than the **linear** part of a CPT transformation since $\hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)]$.

Thus CPT is naturally associated with the complex Lorentz group.

With the Lagrangian density $L(x)$ being spin zero, $\hat{\pi}\hat{\tau}$ effects $\hat{\pi}\hat{\tau}L(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = L(-x)$ up to a phase. We will show below that the phase is one. Thus, with K denoting complex conjugation, when acting on a spin zero Lagrangian we can identify $\hat{C}\hat{P}\hat{T} = K\hat{\pi}\hat{\tau}$. On applying $\hat{\pi}\hat{\tau}$ we obtain

$$\begin{aligned} \hat{C}\hat{P}\hat{T}\int d^4xL(x)[\hat{C}\hat{P}\hat{T}]^{-1} &= K\hat{\pi}\hat{\tau}\int d^4xL(x)\hat{\tau}^{-1}\hat{\pi}^{-1}K \\ &= K\int d^4xL(-x)K = K\int d^4xL(x)K = \int d^4xL^*(x). \end{aligned} \quad (72)$$

Establishing the CPT theorem is thus reduced to showing that $L(x) = L^*(x)$.

	C	P	T	CP	CT	PT	CPT
$\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}\gamma^0\psi$	-	+	+	-	-	+	-
$\bar{\psi}\gamma^i\psi$	-	-	-	+	+	+	-
$\bar{\psi}\gamma^0\gamma^5\psi$	+	-	+	-	+	-	-
$\bar{\psi}\gamma^i\gamma^5\psi$	+	+	-	+	-	-	-
$\bar{\psi}i[\gamma^0,\gamma^i]\psi$	-	-	+	+	-	-	+
$\bar{\psi}i[\gamma^i,\gamma^j]\psi$	-	+	-	-	+	-	+
$\bar{\psi}[\gamma^0,\gamma^i]\gamma^5\psi$	-	+	-	-	+	-	+
$\bar{\psi}[\gamma^i,\gamma^j]\gamma^5\psi$	-	-	+	+	-	-	+

Table 1: C, P, and T assignments for fermion bilinears

CPT phase alternates with spin. All spin zero quantities have even CPT. Also all are **real** (Mannheim 2018). Also, because of Lorentz invariance, Lagrangians have to be spin zero. And as we have seen, the action

$$I = \int d^4xL(x)$$

is invariant under complex Lorentz invariance.

	C	P	T	CP	CT	PT	CPT
$\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}\psi\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}\psi\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}i\gamma^5\psi\bar{\psi}i\gamma^5\psi$	+	+	+	+	+	+	+
$\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$	+	+	+	+	+	+	+
$\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\gamma^5\psi$	-	-	+	+	-	-	+
$\bar{\psi}\gamma^\mu\gamma^5\psi\bar{\psi}\gamma_\mu\gamma^5\psi$	+	+	+	+	+	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\psi\bar{\psi}i[\gamma_\mu,\gamma_\nu]\psi$	+	+	+	+	+	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\psi\bar{\psi}[\gamma_\mu,\gamma_\nu]\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\gamma^5\psi\bar{\psi}i[\gamma_\mu,\gamma_\nu]\gamma^5\psi$	+	+	+	+	+	+	+

Table 2: C, P, and T assignments for fermion bilinears and quadrilinears that have spin zero

All spin zero combinations have CPT even and real.

15 PROOF OF THE CPT THEOREM

Since probability conservation requires an antilinear symmetry, we have

$$\begin{aligned} K\hat{\pi}\hat{\tau}\int d^4x L(x)\hat{\tau}^{-1}\hat{\pi}^{-1}K &= K\int d^4x L(-x)K = K\int d^4x L(x)K = \int d^4x L^*(x) \\ &= K\int d^4x L(x)K = \int d^4x L(x), \end{aligned} \tag{73}$$

where we have used K as the antilinear symmetry needed for probability conservation. Thus infer that all the numerical coefficients in $L(x)$ are real, that $L(x) = L^*(x)$, and that $\int d^4x L(x)$ is CPT invariant, **with the CPT theorem thus being extended to non-Hermitian Hamiltonians.**

16 SOME IMPLICATIONS

(1) In the complex conjugate energy case time-independent transitions occur between decaying and growing states. A decay such as $K^+ \to \pi^+\pi^0$ can thus occur if the Hamiltonian has an antilinear symmetry, even though it would be forbidden if the Hamiltonian is Hermitian. Then the CPT theorem in the antilinear case ensures that its rate is equal to that of $K^- \to \pi^-\pi^0$. We thus extend the CPT theorem to unstable states.

(2) In those cases in which charge conjugation is separately conserved (in non-relativistic quantum mechanics C plays no role since one is below the threshold for particle production) CPT reduces to PT, even if the Hamiltonian is not Hermitian. (Even for non-Hermitian Hamiltonians CPT plus C implies PT.) In such cases we recover the non-Hermitian PT program of Bender and collaborators, and thus put the PT symmetry program on a quite firm theoretical foundation.

(3). Can extend Goldstone theorem and Higgs mechanism to non-Hermitian case with CPT symmetry. (Alexandre, Ellis, Millington and Seynaeve, 2018; Mannheim, 2018)

(4) The conformal gravity theory with action $I_{\rm W} = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$ where $C^{\lambda\mu\nu\kappa}$ is the Weyl conformal tensor falls into the non-Hermitian, CPT symmetric category (Bender and Mannheim 2008), and is able to be ghost free and unitary at the quantum level because of it (the $\langle L|R\rangle$ norm is positive definite), to thus provide a fully consistent quantum theory of gravity without any of the string theory need for supersymmetry or extra spacetime dimensions. It has been shown (Mannheim) that conformal gravity solves the dark matter, dark energy and quantum gravity problems. If conformal gravity can replace Einstein gravity then **one of the four fundamental forces is a Bender PT-type theory**.

(3) Our derivation of the CPT theorem leads to $L = L^*$ and thus to $H = H^*$. In contrast, in the standard derivation of the CPT theorem $H = H^\dagger$ is input. Here $H = H^*$ is output, with it being probability conservation plus complex Lorentz invariance that is input. Now in one of the standard derivations of the CPT theorem (see e.g. Weinberg Quantum Field Theory I) one notes that all spin zero multilinears are Hermitian. Then a Hermiticity assumption requires all numerical coefficients be real and the CPT theorem follows. Remarkably then, both types of derivation lead to the very same functional form for the action, with real numerical coefficients in each case. So how can we tell them apart.