COSMOLOGICAL PERTURBATIONS IN EINSTEIN GRAVITY AND CONFORMAL GRAVITY

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We discuss the general structure of cosmological perturbation theory. We discuss gauge invariance and the decomposition theorem. We provide some solvable models in both Einstein gravity and conformal gravity, and in conformal gravity show how quickly fluctuations can build up.

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A. Amarasinghe, T. Liu, D. A. Norman and P. D. Mannheim, *Exact solution to perturbative conformal cosmology from recombination until the current era*, Physical Review D **103**, 104022 (2021). (arXiv:2101.02608 [gr-qc])

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Starting point: the cosmological principle: all points are equivalent

Isotropy: Universe looks same in all directions. Homogeneity: no place is special

Possibilities: An infinite plane, a closed spherical surface, an open hyperboloid

Geometrically: a maximally 3-symmetric space of constant 3-curvature k, with $k = 0$, $k > 0$ or $k < 0$

Universe is expanding with expansion radius $a(t)$ and on largest scales obeys the cosmological principle

With $(i, j, k) = (r, \theta, \phi)$ and 3-metric $\tilde{\gamma}_{ij}$ gives Robertson-Walker line element:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2\right] = c^2 dt^2 - a^2(t)\tilde{\gamma}_{ij} dx^i dx^j \qquad (1.1)$$

and 3-dimensional spatial Riemann tensor on every comoving time slice of the form

$$\tilde{R}_{ijk\ell} = k[\tilde{\gamma}_{jk}\tilde{\gamma}_{i\ell} - \tilde{\gamma}_{ik}\tilde{\gamma}_{j\ell}]. \qquad (1.2)$$

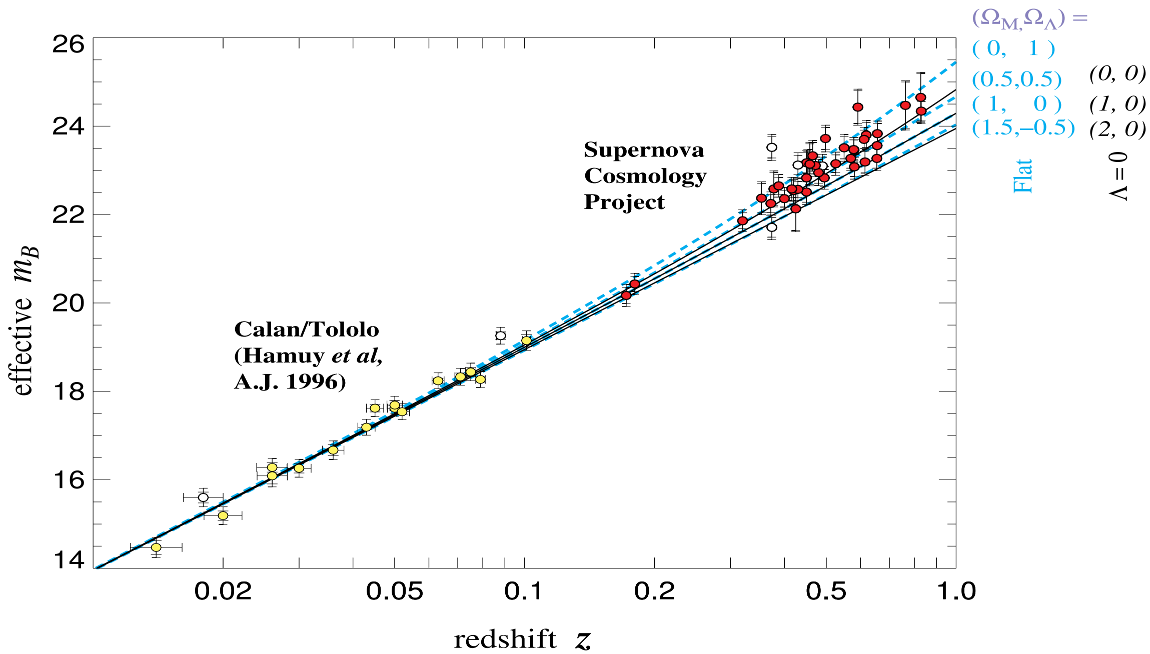
So how good is the cosmological principle?

1.1 Hubble Flow

Hubble identified a systematic behavior in galaxies: they were all redshifted with respect to us, i.e., moving away from us, and had velocities of the form $v = HD$, where D is the distance from us and H is a constant. Rationale: No point is special. Consider three equally spaced points A, B, C on a straight line. Let B have a velocity v with respect to A. Let C have a velocity v with respect to B. Then C has a velocity 2v with respect to A, and is twice as far from A as B is. Thus $v = HD$.

$$\begin{array}{lll} A & B & C \\ A & V_{BA} = v & V_{CB} = v, \quad v_{CA} = 2v \end{array} \tag{1.3}$$

Hubble plot (from Type 1A superernova data) as log plot with slope $H = 75$ Km/sec/Mps, i.e. $H^{-1} = 4 \times 10^{17}$ seconds, a first estimate of the age of the Universe.



1.2 Cosmic Microwave Background (CMB)

Expect and find within small errors CMB to be uniform in every direction, and have the form of a blackbody with energy density $\rho = \pi^2 k_B^4 T^4 / 15 c^3 \hbar^3$ and pressure $p = \rho/3$, to give a current Universe temperature of order $3^\circ K$.

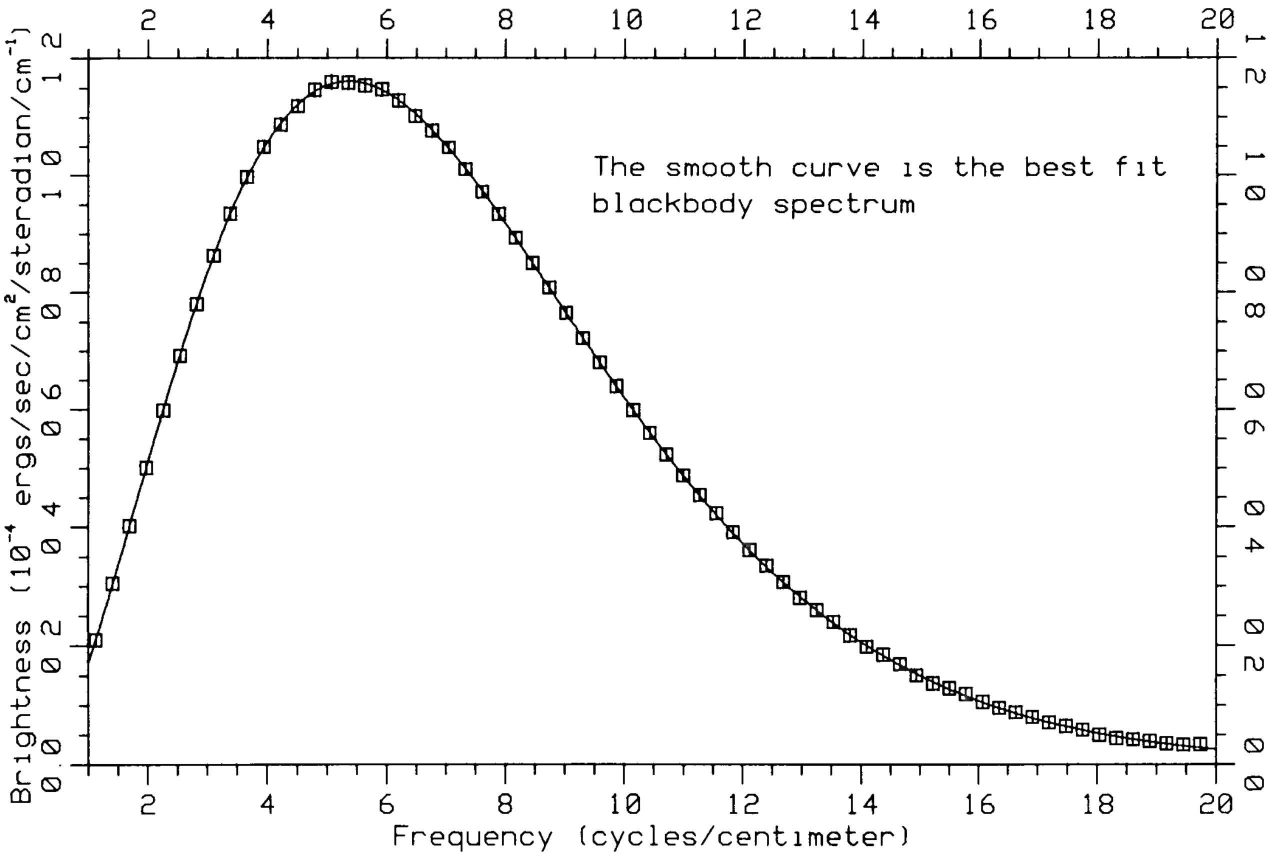


FIG. 2.—Preliminary spectrum of the cosmic microwave background from the FIRAS instrument at the north Galactic pole, compared to a blackbody. Boxes are measured points and show size of assumed 1% error band. The units for the vertical axis are 10^{-4} ergs s^{-1} cm^{-2} sr^{-1} cm.

However on smaller scales (less than 200 or so Megaparsec) we see departures from homogeneity and isotropy, with temperature fluctuations in the CMB being of oder $\Delta T/T = 10^{-5}$. Could these departures be described by a small perturbation to uniformity. So need to develop a theory of cosmological perturbations.

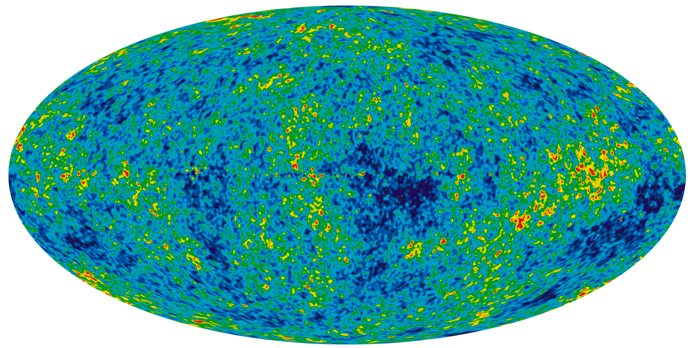


Figure 1: The variation in temperature is of order 10^{-5}. Small departure from uniformly expanding Hubble flow.

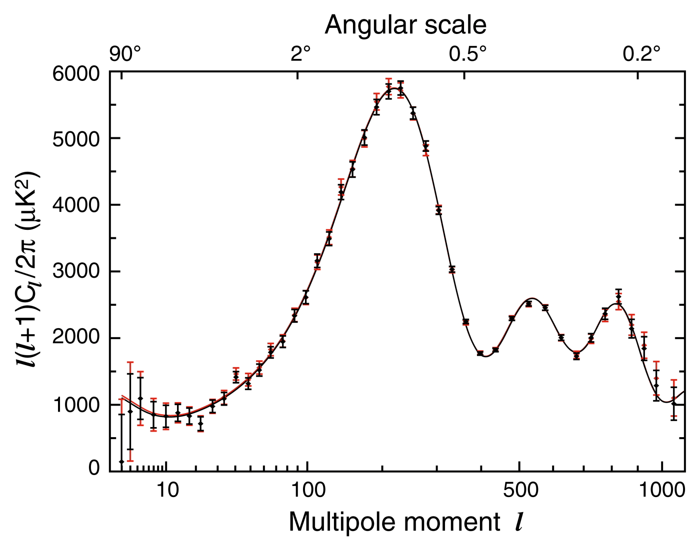


Figure 2: 30 percent dark matter 70 percent dark energy fit to angular momentum decomposition of the CMB fluctuations

2 Cosmological Perturbation Theory

On introducing the conformal time τ and writing the expansion radius as $\Omega(\tau) = a(t)$ according to

$$\tau = \int \frac{dt}{a(t)}, \qquad \Omega(\tau) = a(t), \tag{2.1}$$

we take the background plus fluctuating line element to be of the scalar, vector, tensor (SVT) form

$$\begin{aligned} ds^2 = -(g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\tau)\left[d\tau^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2\right] \\ + \Omega^2(\tau)\left[2\phi d\tau^2 - 2(\tilde{\nabla}_i B + B_i)d\tau dx^i - [-2\psi\tilde{\gamma}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\right], \end{aligned} \tag{2.2}$$

where $\tilde{\nabla}_i = \partial/\partial x^i$ and $\tilde{\nabla}^i = \tilde{\gamma}^{ij}\tilde{\nabla}_j$ (with Latin indices) are defined with respect to the background 3-space metric $\tilde{\gamma}_{ij}$.With

$$\tilde{\gamma}^{ij}\tilde{\nabla}_j V_i = \tilde{\gamma}^{ij}[\partial_j V_i - \tilde{\Gamma}^k_{ij} V_k] \tag{2.3}$$

for any three-vector V_i in a 3-space with 3-space connection $\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{\gamma}^{k\ell}[\partial_i\tilde{\gamma}_{\ell j} + \partial_j\tilde{\gamma}_{\ell i} - \partial_\ell\tilde{\gamma}_{ij}]$, the elements of (2.2) obey

$$\tilde{\gamma}^{ij}\tilde{\nabla}_j B_i = 0, \quad \tilde{\gamma}^{ij}\tilde{\nabla}_j E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\gamma}^{jk}\tilde{\nabla}_k E_{ij} = 0, \quad \tilde{\gamma}^{ij}E_{ij} = 0. \tag{2.4}$$

As written, (2.2) contains ten elements, whose transformations are defined with respect to the background spatial sector as four 3-dimensional scalars (ϕ, B, ψ, E) each with one degree of freedom, two transverse 3-dimensional vectors (B_i, E_i) each with two independent degrees of freedom, and one symmetric 3-dimensional transverse-traceless tensor (E_{ij}) with two degrees of freedom, so $1+1+1+1+2+2+4 = 10$. **Since this a 3-dimensional SVT formalism we will need to establish that it leads to fluctuation equations that are not 3-dimensionally but 4-dimensionally covariant.**

To set up this SVT basis requires some specific asymptotic spatial boundary conditions. Thus on introducing the Green's function D that obeys $\tilde{\nabla}_i\tilde{\nabla}^i D^{(3)}(x,x') = \tilde{\gamma}^{-1/2}\delta^3(x-x')$ (where $\tilde{\gamma}$ is the determinant of $\tilde{\gamma}_{ij}$), we for instance obtain

$$\begin{aligned} h^{0i} = \tilde{\nabla}^i B + B^i, \quad \tilde{\nabla}_i h^{i0} = \tilde{\nabla}_i\tilde{\nabla}^i B, \quad B(x) = \int d^3x'\tilde{\gamma}'^{1/2} D^{(3)}(x,x')\tilde{\nabla}'_j h^{j0}(x'), \\ B^i(x) = h^{0i} - \tilde{\nabla}^i \int d^3x'\tilde{\gamma}'^{1/2} D^{(3)}(x,x')\tilde{\nabla}'_j h^{j0}(x'), \end{aligned} \tag{2.5}$$

and thus require that the integral exist. As we see, the relation between h^{0i} and the transverse B^i and longitudinal $\tilde{\nabla}^i B$ is nonlocal.

3 Relating the SVT and $h_{\mu\nu}$ Fluctuation Bases

For simplicity we take the background metric to be $-\Omega^2(\tau)\eta_{\mu\nu}$ (i.e., $k=0$), and set $h_{\mu\nu}=\Omega^2(\tau)f_{\mu\nu}$, to obtain

$$\begin{aligned} ds^2 &= -[\Omega^2(\tau)\eta_{\alpha\beta}+h_{\alpha\beta}]dx^\alpha dx^\beta \\ &= -\Omega^2(\tau)[\eta_{\alpha\beta}+f_{\alpha\beta}]dx^\alpha dx^\beta = \Omega^2(\tau)\left[d\tau^2-\delta_{ij}dx^i dx^j - f_{00}d\tau^2 - 2f_{0i}d\tau dx^i - f_{ij}dx^i dx^j\right]. \end{aligned} \tag{3.1}$$

Identifying terms gives

$$\begin{aligned} 2\phi &= -f_{00}, \quad B_i+\tilde{\nabla}_i B = f_{0i}, \quad f_{ij} = -2\psi\delta_{ij}+2\tilde{\nabla}_i\tilde{\nabla}_j E+\tilde{\nabla}_i E_j+\tilde{\nabla}_j E_i+2E_{ij}, \\ \delta^{ij}f_{ij} &= -6\psi+2\tilde{\nabla}_i\tilde{\nabla}^i E, \qquad \tilde{\nabla}^j f_{ij} = -2\tilde{\nabla}_i\psi+2\tilde{\nabla}_i\tilde{\nabla}_k\tilde{\nabla}^k E+\tilde{\nabla}_k\tilde{\nabla}^k E_i, \\ \tilde{\nabla}^i\tilde{\nabla}^j f_{ij} &= -2\tilde{\nabla}_k\tilde{\nabla}^k\psi+2\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell E = \frac{4}{3}\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell E+\frac{1}{3}\tilde{\nabla}_k\tilde{\nabla}^k\delta^{ij}f_{ij} = 4\tilde{\nabla}_k\tilde{\nabla}^k\psi+\tilde{\nabla}_k\tilde{\nabla}^k(\delta^{ij}f_{ij}), \end{aligned} \tag{3.2}$$

so that

$$\begin{aligned} 2\phi &= -f_{00}, \qquad B=\int d^3y D^{(3)}(\mathbf{x}-\mathbf{y})\tilde{\nabla}^i_y f_{0i}, \qquad B_i = f_{0i}-\tilde{\nabla}_i B, \\ \psi &= \frac{1}{4}\int d^3y D^{(3)}(\mathbf{x}-\mathbf{y})\tilde{\nabla}^k_y\tilde{\nabla}^\ell_y f_{k\ell}-\frac{1}{4}\delta^{k\ell}f_{k\ell}, \\ E &= \int d^3y D^{(3)}(\mathbf{x}-\mathbf{y})\left[\frac{3}{4}\int d^3z D^{(3)}(\mathbf{y}-\mathbf{z})\tilde{\nabla}^k_z\tilde{\nabla}^\ell_z f_{k\ell}-\frac{1}{4}\delta^{k\ell}f_{k\ell}\right], \\ E_i &= \int d^3y D^{(3)}(\mathbf{x}-\mathbf{y})\left[\tilde{\nabla}^j_y f_{ij}-\tilde{\nabla}^y_i\int d^3z D^{(3)}(\mathbf{y}-\mathbf{z})\tilde{\nabla}^k_z\tilde{\nabla}^\ell_z f_{k\ell}\right], \\ 2E_{ij} &= f_{ij}+2\psi\delta_{ij}-2\tilde{\nabla}_i\tilde{\nabla}_j E-\tilde{\nabla}_i E_j-\tilde{\nabla}_j E_i, \end{aligned} \tag{3.3}$$

with B_i, E_i and E_{ij} then obeying (2.4). (In (3.3) in a symbol such as $\tilde{\nabla}^i_y$ for instance the y indicates that the derivative is taken with respect to the y coordinate.)

As we see, we need to go to fairly high derivatives in order to be able to express each of the SVT components entirely in terms of combinations of components of the $h_{\mu\nu}$.

4 Gauge Structure of the SVT Basis

In order to explore the gauge structure of the SVT basis we implement an infinitesimal 4-dimensional coordinate transformation $\bar{x}_\mu = x_\mu + \epsilon_\mu(x)$. It is convenient to write ϵ_μ in the scalar, vector form, viz.

$$\epsilon_\mu = \Omega^2(\tau) f_\mu, \qquad f_0 = -T, \qquad f_i = L_i + \tilde{\nabla}_i L \qquad \delta^{ij}\tilde{\nabla}_j L_i = \tilde{\nabla}^i L_i = 0. \tag{4.1}$$

With a general coordinate transformation being of the form $\bar{g}^{\mu\nu} = (\partial\bar{x}^\mu/\partial x^\sigma)(\partial\bar{x}^\nu/\partial x^\tau)g^{\sigma\tau}$, to lowest order in ϵ_μ the line element ds^2 and the fluctuations $h_{\mu\nu}$ and $f_{\mu\nu} = \Omega^{-2}(x)h_{\mu\nu}$ transform into

$$\begin{aligned}
ds^2 &= -\Omega^2(\tau)[\eta_{\alpha\beta} + \bar{f}_{\alpha\beta}]d\bar{x}^\alpha d\bar{x}^\beta \\
&= \bar{\Omega}^2(\bar{\tau})\left[(1+2\bar{\phi})d\bar{\tau}^2 - 2(\tilde{\nabla}_i\bar{B} + \bar{B}_i)d\bar{\tau}d\bar{x}^i - [(1-2\bar{\psi})\bar{\delta}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j\bar{E} + \tilde{\nabla}_i\bar{E}_j + \tilde{\nabla}_j\bar{E}_i + 2\bar{E}_{ij}]d\bar{x}^i d\bar{x}^j\right], \\
\bar{h}_{\mu\nu} &= h_{\mu\nu} - \partial_\nu\epsilon_\mu - \partial_\mu\epsilon_\nu + 2\Gamma^\lambda_{\mu\nu}\epsilon_\lambda \\
&= h_{\mu\nu} - \partial_\nu\epsilon_\mu - \partial_\mu\epsilon_\nu + \Omega^{-2}(\tau)[\epsilon_\mu\partial_\nu + \epsilon_\nu\partial_\mu - \epsilon_\lambda\eta_{\mu\nu}\eta^{\lambda\sigma}\partial_\sigma]\Omega^2(\tau), \\
\bar{f}_{\mu\nu} &= f_{\mu\nu} - \partial_\nu f_\mu - \partial_\mu f_\nu - \Omega^{-2}(\tau)f_\lambda\eta_{\mu\nu}\eta^{\lambda\sigma}\partial_\sigma\Omega^2(\tau), \\
\bar{f}_{00} &= f_{00} + 2\dot{T} + \Omega^{-2}(\tau)[T\partial_0 + (L_i + \tilde{\nabla}_i L)\delta^{ij}\partial_j]\Omega^2(\tau), \\
\bar{f}_{0i} &= f_{0i} + \partial_i T - \dot{L}_i - \tilde{\nabla}_i\dot{L}, \\
\bar{f}_{ij} &= f_{ij} - \partial_i(L_j + \tilde{\nabla}_j L) - \partial_j(L_i + \tilde{\nabla}_i L) - \delta_{ij}\Omega^{-2}(\tau)[T\partial_0 + (L_i + \tilde{\nabla}_i L)\delta^{ij}\partial_j]\Omega^2(\tau),
\end{aligned} \tag{4.2}$$

where the dot denotes derivative with respect to τ. Following some algebra we obtain

$$\begin{aligned}
\bar{\phi} &= \phi - \dot{T} - \Omega^{-1}\dot{\Omega}T, \qquad \bar{B} = B + T - \dot{L}, \qquad \bar{\psi} = \psi + \Omega^{-1}\dot{\Omega}T, \qquad \bar{E} = E - L, \\
\bar{B}_i &= B_i - \dot{L}_i, \qquad \bar{E}_i = E_i - L_i, \qquad \bar{E}_{ij} = E_{ij},
\end{aligned} \tag{4.3}$$

and gauge invariant combinations take the form

$$\begin{aligned}
&\bar{\phi} + \bar{\psi} + \dot{\bar{B}} - \ddot{\bar{E}} = \phi + \psi + \dot{B} - \ddot{E}, \qquad \bar{\psi} - \Omega^{-1}\dot{\Omega}(\bar{B} - \dot{\bar{E}}) = \psi - \Omega^{-1}\dot{\Omega}(B - \dot{E}), \\
&\bar{B}_i - \dot{\bar{E}}_i = B_i - \dot{E}_i, \qquad \bar{E}_{ij} = E_{ij},
\end{aligned} \tag{4.4}$$

Thus $1 + 1 + 2 + 2 = 6 = 10 - 4$ just as required.

4.1 Fluctuations Around Flat

As a check, we note that for fluctuations around flat, i.e., $\Omega = 1$, $ds^2 = (-\eta_{\mu\nu} - h_{\mu\nu})dx^\mu dx^\nu$, the gauge invariants are

$$\phi + \dot{B} - \ddot{E}, \qquad \psi, \qquad \bar{B}_i - \dot{\bar{E}}_i = B_i - \dot{E}_i, \qquad \bar{E}_{ij} = E_{ij}, \tag{4.5}$$

and since in a flat background there is no $T_{\mu\nu}$ (and thus no $\delta T_{\mu\nu}$), then from the Einstein equations $-(1/8\pi G_N)G_{\mu\nu} = T_{\mu\nu}$, where the Einstein tensor is given in terms of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R by $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$, the perturbed Einstein tensor $\delta G_{\mu\nu}$ is gauge invariant on its own, and evaluates to

$$\begin{aligned}
\delta G_{00} &= -2\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\psi, \\
\delta G_{0i} &= -2\tilde{\nabla}_i\dot{\psi} + \tfrac{1}{2}\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a(B_i - \dot{E}_i), \\
\delta G_{ij} &= -2\delta_{ij}\ddot{\psi} - \delta^{ab}\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}_a(\phi + \dot{B} - \ddot{E}) + \delta^{ab}\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}_a\psi + \tilde{\nabla}_j\tilde{\nabla}_i(\phi + \dot{B} - \ddot{E}) - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\
&+ \tfrac{1}{2}\tilde{\nabla}_i(\dot{B}_j - \ddot{E}_j) + \tfrac{1}{2}\tilde{\nabla}_j(\dot{B}_i - \ddot{E}_i) - \ddot{E}_{ij} + \delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a E_{ij}, \\
g^{\mu\nu}\delta G_{\mu\nu} &= -\delta G_{00} + \delta^{ij}\delta G_{ij} = 4\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\psi - 6\ddot{\psi} - 2\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a(\phi + \dot{B} - \ddot{E}),
\end{aligned} \tag{4.6}$$

to thus depend on none other than the combinations given in (4.5). It is thus 4-dimensionally gauge invariant. **Thus the SVT basis leads to a fully 4-dimensionally covariant fluctuation equation even though the SVT basis itself is only 3-dimensional.**

4.2 General Case

When Ω depends on τ and we take k to be nonzero [background line element $ds^2 = \Omega^2(\tau)(d\tau^2 - \tilde{\gamma}_{ij}dx^i dx^j)$], the general gauge invariants are

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \Omega\dot{\Omega}^{-1}\psi - B + \dot{E}, \qquad B_i - \dot{E}_i, \qquad E_{ij}. \tag{4.7}$$

Interestingly, they have no explicit dependence on k even though it is nonzero.

4.3 Perturbed Energy-Momentum Tensor

We take the background $T_{\mu\nu}$ to be of the perfect fluid form

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \tag{4.8}$$

with fluctuation

$$\delta T_{\mu\nu} = (\delta\rho + \delta p)U_\mu U_\nu + \delta p g_{\mu\nu} + (\rho + p)(\delta U_\mu U_\nu + U_\mu \delta U_\nu) + ph_{\mu\nu}. \tag{4.9}$$

Here $g^{\mu\nu}U_\mu U_\nu = -1$, $U^0 = \Omega^{-1}(\tau)$, $U_0 = -\Omega(\tau)$, $U^i = 0$, $U_i = 0$ for the background, while for the fluctuation we have

$$\delta g^{00}U_0U_0 + 2g^{00}U_0\delta U_0 = 0, \tag{4.10}$$

i.e.

$$\delta U_0 = -\frac{1}{2}(g^{00})^{-1}(-g^{00}g^{00}\delta g_{00})U_0 = -\Omega(\tau)\phi. \tag{4.11}$$

Thus δU_0 is not an independent degree of freedom. For the 3-vector we set $\delta U_i = V_i + \tilde{\nabla}_i V$, where now $\tilde{\gamma}^{ij}\tilde{\nabla}_j V_i = \tilde{\gamma}^{ij}[\partial_j V_i - \tilde{\Gamma}^k_{ij}V_k] = 0$. As constructed, in general we have 11 fluctuation variables, the six from the metric together with the three δU_i, and $\delta\rho$ and δp. But we only have ten fluctuation equations $-(1/8\pi G_N)\delta G_{\mu\nu} = \delta T_{\mu\nu}$. Thus to solve the theory when there is both a $\delta\rho$ and a δp we will need some constraint between δp and $\delta\rho$.

Proceeding as with the fluctuating metric, we find that for fluctuations around a background $ds^2 = \Omega^2(\tau)(d\tau^2 - \tilde{\gamma}_{ij}dx^i dx^j)$, the fluctuating $\delta T_{\mu\nu}$ gauge invariants are

$$\hat{V} = V - \Omega^2\dot{\Omega}^{-1}\psi, \qquad \hat{V}_i = V_i, \qquad \delta\hat{\rho} \;=\; \delta\rho - 3(\rho + p)\psi, \qquad \delta\hat{p} = \delta p + \Omega\dot{\Omega}^{-1}\dot{p}\psi. \tag{4.12}$$

Again there is no explicit dependence on k.

4.4 The Fluctuation Equations

For the background Einstein equations we have

$$
\begin{aligned}
G_{00} &= -3k - 3\dot{\Omega}^2\Omega^{-2}, \quad G_{0i} = 0, \quad G_{ij} = \tilde{\gamma}_{ij}\left[k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1}\right], \\
G_{00} + 8\pi G_N T_{00} &= -3k - 3\dot{\Omega}^2\Omega^{-2} + \Omega^2\rho = 0, \quad G_{ij} + 8\pi G_N T_{ij} = \tilde{\gamma}_{ij}\left[k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1} + \Omega^2 p\right] = 0, \\
\rho &= 3k\Omega^{-2} + 3\dot{\Omega}^2\Omega^{-4}, \quad p = -k\Omega^{-2} + \dot{\Omega}^2\Omega^{-4} - 2\ddot{\Omega}\Omega^{-3}, \quad p = -\rho - \frac{1}{3}\frac{\Omega}{\dot{\Omega}}\dot{\rho},
\end{aligned}
\tag{4.13}
$$

(after setting $8\pi G_N = 1$), with the last relation following from $\nabla_\nu T^{\mu\nu} = 0$, viz. conservation of the background energy-momentum tensor in the full 4-space. To solve these equations we would need an equation of state that would relate ρ and p.

For $\delta G_{\mu\nu}$ we have

$$
\begin{aligned}
\delta G_{00} &= -6k\phi - 6k\psi + 6\dot{\psi}\dot{\Omega}\Omega^{-1} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi, \\
\delta G_{0i} &= 3k\tilde{\nabla}_i B - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i B + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi + 2kB_i - k\dot{E}_i \\
&\quad - B_i\dot{\Omega}^2\Omega^{-2} + 2B_i\ddot{\Omega}\Omega^{-1} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i, \\
\delta G_{ij} &= -2\ddot{\psi}\tilde{\gamma}_{ij} + 2\dot{\Omega}^2\tilde{\gamma}_{ij}\phi\Omega^{-2} + 2\dot{\Omega}^2\tilde{\gamma}_{ij}\psi\Omega^{-2} - 2\dot{\phi}\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1} - 4\dot{\psi}\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1} - 4\ddot{\Omega}\tilde{\gamma}_{ij}\phi\Omega^{-1} \\
&\quad - 4\ddot{\Omega}\tilde{\gamma}_{ij}\psi\Omega^{-1} - 2\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\
&\quad - \tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\
&\quad + 2k\tilde{\nabla}_j\tilde{\nabla}_i E - 2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + 4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j \\
&\quad - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j + k\tilde{\nabla}_i E_j - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i E_j + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i E_j + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i \\
&\quad - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i + k\tilde{\nabla}_j E_i - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j E_i + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j E_i - \ddot{E}_{ij} - 2\dot{\Omega}^2 E_{ij}\Omega^{-2} \\
&\quad - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 4\ddot{\Omega}E_{ij}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}, \\
g^{\mu\nu}\delta G_{\mu\nu} &= 6\dot{\Omega}^2\phi\Omega^{-4} + 6\dot{\Omega}^2\psi\Omega^{-4} - 6\dot{\phi}\dot{\Omega}\Omega^{-3} - 18\dot{\psi}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\phi\Omega^{-3} - 12\ddot{\Omega}\psi\Omega^{-3} - 6\ddot{\psi}\Omega^{-2} + 6k\phi\Omega^{-2} \\
&\quad + 6k\psi\Omega^{-2} - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\
&\quad - 2\dot{\Omega}^2\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^a E + 4\ddot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a E + 2k\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a E - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi.
\end{aligned}
\tag{4.14}
$$

On using (4.13) for the background but without imposing any relation between the background ρ and p, we obtain evolution equations (again with $8\pi G_N = 1$) of the form

$$\Delta_{00} = 6\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) + \delta\hat{\rho}\Omega^2 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma = 0, \tag{4.15}$$

$$\begin{aligned}\Delta_{0i} = &-2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(\alpha - \dot{\gamma}) + 2k\tilde{\nabla}_i\gamma + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i\hat{V} \\ &+k(B_i - \dot{E}_i) + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i - \dot{E}_i) + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i = 0,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\Delta_{ij} = &\tilde{\gamma}_{ij}\big[2\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha} - \ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha - \dot{\gamma}) + \Omega^2\delta\hat{p} - \tilde{\nabla}_a\tilde{\nabla}^a(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma)\big] \\ &+\tilde{\nabla}_i\tilde{\nabla}_j(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma) + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(B_j - \dot{E}_j) + \tfrac{1}{2}\tilde{\nabla}_i(\dot{B}_j - \ddot{E}_j) + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j(B_i - \dot{E}_i) + \tfrac{1}{2}\tilde{\nabla}_j(\dot{B}_i - \ddot{E}_i) \\ &-\ddot{E}_{ij} - 2kE_{ij} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} = 0,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\tilde{\gamma}^{ij}\Delta_{ij} &= 6\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) - 6\dot{\Omega}\Omega^{-1}(\dot{\alpha} - \ddot{\gamma}) - 12\ddot{\Omega}\Omega^{-1}(\alpha - \dot{\gamma}) + 3\Omega^2\delta\hat{p} - 2\tilde{\nabla}_a\tilde{\nabla}^a(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma) = 0, \\ g^{\mu\nu}\Delta_{\mu\nu} &= 3\delta\hat{p} - \delta\hat{\rho} - 12\ddot{\Omega}\Omega^{-3}(\alpha - \dot{\gamma}) - 6\dot{\Omega}\Omega^{-3}(\dot{\alpha} - \ddot{\gamma}) - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a(\alpha + 3\dot{\Omega}\Omega^{-1}\gamma) = 0,\end{aligned} \tag{4.18}$$

where the gauge invariants are

$$\begin{aligned}&\alpha = \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \Omega\dot{\Omega}^{-1}\psi - B + \dot{E}, \qquad B_i - \dot{E}_i, \qquad E_{ij}, \\ &\delta\hat{\rho} = \delta\rho - 3(\rho + p)\psi, \quad \delta\hat{p} = \delta p + \frac{\Omega}{\dot{\Omega}}\dot{p}\psi\end{aligned} \tag{4.19}$$

These $\Delta_{\mu\nu} = 0$ equations are remarkably compact and are manifestly 4-dimensionally gauge invariant, just as required. As constructed, we have 11 fluctuation variables, the six from the metric together with $\delta\hat{\rho}$, $\delta\hat{p}$, $\hat{V}$ and the transverse 2-component $\hat{V}_i$. But we only have ten fluctuation equations $\Delta_{\mu\nu} = 0$. Thus to solve the theory when there is both a $\delta\rho$ and a δp we will need some constraint between δp and $\delta\rho$. Usually one sets $\delta p/\delta\rho = v^2$, where v is the velocity of sound.

4.5 Decomposition Theorem

To unravel these equations we introduce the decomposition theorem, an ansatz that claims that we can decouple the scalar, vector and tensor sectors into nine equations for the fluctuation components, viz.

$$\begin{aligned}
6\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})+\delta\hat{\rho}\Omega^2+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma=0,\\
-2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(\alpha-\dot{\gamma})+2k\tilde{\nabla}_i\gamma+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})\tilde{\nabla}_i\hat{V}=0,\\
+k(B_i-\dot{E}_i)+\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i-\dot{E}_i)+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})V_i=0,\\
\tilde{\gamma}_{ij}\big[2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})-2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma})-4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+\Omega^2\delta\hat{p}-\tilde{\nabla}_a\tilde{\nabla}^a(\alpha+2\dot{\Omega}\Omega^{-1}\gamma)\big]\\
+\tilde{\nabla}_i\tilde{\nabla}_j(\alpha+2\dot{\Omega}\Omega^{-1}\gamma)=0,\\
\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(B_j-\dot{E}_j)+\tfrac{1}{2}\tilde{\nabla}_i(\dot{B}_j-\ddot{E}_j)+\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j(B_i-\dot{E}_i)+\tfrac{1}{2}\tilde{\nabla}_j(\dot{B}_i-\ddot{E}_i)=0,\\
-\ddot{E}_{ij}-2kE_{ij}-2\dot{E}_{ij}\dot{\Omega}\Omega^{-1}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij}=0,
\end{aligned} \tag{4.20}$$

with the trace condition being of the firm

$$3\delta\hat{p}-\delta\hat{\rho}-12\ddot{\Omega}\Omega^{-3}(\alpha-\dot{\gamma})-6\dot{\Omega}\Omega^{-3}(\dot{\alpha}-\ddot{\gamma})-2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a(\alpha+3\dot{\Omega}\Omega^{-1}\gamma)=0. \tag{4.21}$$

With $\tilde{\gamma}_{ij}$ and $\tilde{\nabla}_i\tilde{\nabla}_j$ not being equal to each other, the fourth equation in (4.20) splits into two pieces

$$2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})-2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma})-4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+\Omega^2\delta\hat{p}=0,\quad \alpha+2\dot{\Omega}\Omega^{-1}\gamma=0. \tag{4.22}$$

and now we have ten fluctuation equations.

However, this decomposition theorem is not obvious, since if S is a scalar then $\tilde{\nabla}_i S$ is a vector, and thus we can not use angular momentum conservation to decouple the scalar, vector and tensor components. Specifically, we note that if we have a generic equation of the form

$$B_i + \partial_i B = C_i + \partial_i C, \tag{4.23}$$

where the B^i and C^i obey $\partial_i B^i = 0$, $\partial_i C^i = 0$, it does not follow that

$$B_i = C_i, \quad \partial_i B = \partial_i C. \tag{4.24}$$

since on applying ∂^i to (4.23) we obtain

$$\partial^i \partial_i (B - C) = 0. \tag{4.25}$$

Thus in Cartesian coordinates we can only obtain $B - C = a + b_i x^i$, where a and b_i are constants. To be able to set $B = C$ (and thus $B^i = C^i$) we need to set $a = 0$, $b^i = 0$. Thus we can do by requiring that $B - C$ vanishes at infinity, in consequence of which it would then vanish everywhere.. Thus to get a decomposition theorem in this case, we need a boundary condition. To see how this can work in the general case we need to decouple the fluctuation equations, with Δ_{00} being the only component of $\Delta_{\mu\nu}$ that is already decoupled, being pure scalar sector.

5 Decoupling the Fluctuation Equations

5.1 Some general tensor algebra relations

Starting from the general identities

$$\nabla_k\nabla_n T_{\ell m}-\nabla_n\nabla_k T_{\ell m}=T^s{}_m R_{\ell snk}+T_\ell{}^s R_{msnk},\quad \nabla_k\nabla_n A_m-\nabla_n\nabla_k A_m=A^s R_{msnk} \tag{5.1}$$

that hold for any rank two tensor or vector in any geometry, for the 3-space Robertson-Walker geometry where $\tilde{R}_{msnk}=k(\tilde{\gamma}_{sn}\tilde{\gamma}_{mk}-\tilde{\gamma}_{mn}\tilde{\gamma}_{sk})$ we obtain

$$\begin{aligned}&\tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a A_j-\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i A_j=2k\tilde{\gamma}_{ij}\tilde{\nabla}_a A^a-2k(\tilde{\nabla}_i A_j+\tilde{\nabla}_j A_i),\\&\tilde{\nabla}^j\tilde{\nabla}_a\tilde{\nabla}^a A_j=(\tilde{\nabla}_a\tilde{\nabla}^a+2k)\tilde{\nabla}^j A_j,\quad \tilde{\nabla}^j\tilde{\nabla}_i A_j=\tilde{\nabla}_i\tilde{\nabla}^j A_j+2kA_i\end{aligned} \tag{5.2}$$

for any 3-vector A_i in a maximally symmetric 3-geometry with 3-curvature k. Similarly, noting that for any scalar S in any geometry we have

$$\begin{aligned}&\nabla_a\nabla_b\nabla_i S=\nabla_a\nabla_i\nabla_b S=\nabla_i\nabla_a\nabla_b S+\nabla^s S R_{bsia},\\&\nabla_\ell\nabla_k\nabla_n\nabla_m S=\nabla_n\nabla_m\nabla_\ell\nabla_k S+\nabla_n[\nabla^s S R_{ksm\ell}]+\nabla^s\nabla_k S R_{msn\ell}+\nabla_m\nabla^s S R_{ksn\ell}+\nabla_\ell[\nabla^s S R_{msnk}],\end{aligned} \tag{5.3}$$

in a Robertson-Walker 3-geometry background we obtain

$$\begin{aligned}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i S&=\tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a S+2k\tilde{\nabla}_i S,\quad \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i\tilde{\nabla}_j S=\tilde{\nabla}_i\tilde{\nabla}_j\tilde{\nabla}_a\tilde{\nabla}^a S+6k(\tilde{\nabla}_i\tilde{\nabla}_j-\tfrac{1}{3}\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a)S,\\ \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i\tilde{\nabla}_j S&=\tilde{\nabla}_i\tilde{\nabla}_j\tilde{\nabla}_a\tilde{\nabla}^a S+6k\tilde{\nabla}_i\tilde{\nabla}_j S-2k\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a S.\end{aligned} \tag{5.4}$$

5.2 Scalar sector

Thus we find the pure scalar

$$\tilde{\nabla}^i\Delta_{0i}=\tilde{\nabla}_a\tilde{\nabla}^a\big[-2\dot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+2k\gamma+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})\hat{V}\big]=0, \tag{5.5}$$

5.3 Vector sector

and thus the pure vector sector

$$(\tilde{\nabla}_k\tilde{\nabla}^k-2k)\Delta_{0i}=(\tilde{\nabla}_k\tilde{\nabla}^k-2k)\left[k(B_i-\dot{E}_i)+\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i-\dot{E}_i)+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})V_i\right]=0. \tag{5.6}$$

Also we obtain

$$\epsilon^{ij\ell}\tilde{\nabla}_j\Delta_{0i} = \epsilon^{ij\ell}\tilde{\nabla}_j\left[k(B_i - \dot{E}_i) + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i - \dot{E}_i) + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i\right] = 0. \tag{5.7}$$

5.4 Scalar, vector and tensor sector

Now in any maximally symmetric space for any given E_{ij} that is transverse and traceless, it follows that the quantity $\tilde{\nabla}_a\tilde{\nabla}^a E_{ij}$ is transverse and traceless too. Thus given (5.2) we obtain

$$\begin{aligned}\tilde{\nabla}^j\Delta_{ij} &= \tilde{\nabla}_i[2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma}) + \Omega^2\delta\hat{p} + 2k(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma)] \\ &\quad +[\tilde{\nabla}_a\tilde{\nabla}^a + 2k][\tfrac{1}{2}(\dot{B}_i - \ddot{E}_i) + \dot{\Omega}\Omega^{-1}(B_i - \dot{E}_i)] = 0,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\tilde{\nabla}^i\tilde{\nabla}^j\Delta_{ij} &= \tilde{\nabla}_a\tilde{\nabla}^a[2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma}) + \Omega^2\delta\hat{p} \\ &\quad +2k(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma)] = 0.\end{aligned} \tag{5.9}$$

5.5 Scalar sector

Thus we obtain the pure scalar sector

$$3\tilde{\nabla}^i\tilde{\nabla}^j\Delta_{ij} - \tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\gamma}^{ij}\Delta_{ij}) = 2\tilde{\nabla}^2[\tilde{\nabla}^2 + 3k](\alpha + 2\dot{\Omega}\Omega^{-1}\gamma) = 0, \tag{5.10}$$

$$\tilde{\nabla}^i\tilde{\nabla}^j\Delta_{ij} + k\tilde{\gamma}^{ij}\Delta_{ij} = [\tilde{\nabla}^2 + 3k][2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma}) + \Omega^2\delta\hat{p}] = 0. \tag{5.11}$$

We now define $A = 2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma}) + \Omega^2\delta\hat{p}$ and $C = \alpha + 2\dot{\Omega}\Omega^{-1}\gamma$. And using (5.4) obtain

$$(\tilde{\nabla}_a\tilde{\nabla}^a + k)\tilde{\nabla}_i(A + 2kC) = \tilde{\nabla}_i(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)(A + 2kC), \tag{5.12}$$

and thus with (5.10) and (5.11) obtain

$$(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_a\tilde{\nabla}^a + k)\tilde{\nabla}_i(A + 2kC) = \tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\nabla}_b\tilde{\nabla}^b + 3k)(A + 2kC) = 0. \tag{5.13}$$

5.6 Vector sector

Consequently, on comparing with (5.8) we obtain

$$(\tilde{\nabla}_a\tilde{\nabla}^a-2k)(\tilde{\nabla}_b\tilde{\nabla}^b+k)\tilde{\nabla}^j\Delta_{ij}=(\tilde{\nabla}_a\tilde{\nabla}^a-2k)(\tilde{\nabla}_b\tilde{\nabla}^b+k)[\tilde{\nabla}_c\tilde{\nabla}^c+2k][\tfrac{1}{2}(\dot{B}_i-\ddot{E}_i)+\dot{\Omega}\Omega^{-1}(B_i-\dot{E}_i)]=0, \tag{5.14}$$

to give a relation that only involves $B_i-\dot{E}_i$.

5.7 Vector and tensor sector

To obtain a relation that involves E_{ij} we proceed as follows. We note that sector of Δ_{ij} that contains the above A and C can be written as

$$D_{ij}=\tilde{\gamma}_{ij}(A-\tilde{\nabla}_a\tilde{\nabla}^aC)+\tilde{\nabla}_i\tilde{\nabla}_jC. \tag{5.15}$$

We thus introduce

$$\begin{aligned}
A_{ij} &= D_{ij}-\frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{ab}D_{ab}=(\tilde{\nabla}_i\tilde{\nabla}_j-\tfrac{1}{3}\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a)C,\\
B_{ij} &= \Delta_{ij}-\frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{ab}\Delta_{ab}=(\tilde{\nabla}_i\tilde{\nabla}_j-\tfrac{1}{3}\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a)C\\
&+ \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(B_j-\dot{E}_j)+\tfrac{1}{2}\tilde{\nabla}_i(\dot{B}_j-\ddot{E}_j)+\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j(B_i-\dot{E}_i)+\tfrac{1}{2}\tilde{\nabla}_j(\dot{B}_i-\ddot{E}_i)\\
&- \ddot{E}_{ij}-2kE_{ij}-2\dot{E}_{ij}\dot{\Omega}\Omega^{-1}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij}=0,
\end{aligned} \tag{5.16}$$

with (5.16) defining A_{ij} and B_{ij}, and with A dropping out. Using (5.2) and the third relation in (5.4) we obtain

$$(\tilde{\nabla}_b\tilde{\nabla}^b-3k)A_{ij}=(\tilde{\nabla}_i\tilde{\nabla}_j-\tfrac{1}{3}\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a)(\tilde{\nabla}_b\tilde{\nabla}^b+3k)C, \tag{5.17}$$

and via (5.4) and (5.10) thus obtain

$$(\tilde{\nabla}_a\tilde{\nabla}^a-6k)(\tilde{\nabla}_b\tilde{\nabla}^b-3k)A_{ij}=(\tilde{\nabla}_i\tilde{\nabla}_j-\tfrac{1}{3}\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a)\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{\nabla}_c\tilde{\nabla}^c+3k)C=0. \tag{5.18}$$

Comparing with the structure of Δ_{ij} and $\tilde{\gamma}^{ij}\Delta_{ij}$, we thus obtain

$$(\tilde{\nabla}_a\tilde{\nabla}^a-6k)(\tilde{\nabla}_b\tilde{\nabla}^b-3k)[B_{ij}-A_{ij}]=(\tilde{\nabla}_a\tilde{\nabla}^a-6k)(\tilde{\nabla}_b\tilde{\nabla}^b-3k)$$

$$\times\big[\dot\Omega\Omega^{-1}\tilde\nabla_i(B_j-\dot E_j)+\tfrac{1}{2}\tilde\nabla_i(\dot B_j-\ddot E_j)+\dot\Omega\Omega^{-1}\tilde\nabla_j(B_i-\dot E_i)+\tfrac{1}{2}\tilde\nabla_j(\dot B_i-\ddot E_i)$$
$$-\ddot E_{ij}-2kE_{ij}-2\dot\Omega\Omega^{-1}\dot E_{ij}+\tilde\nabla_a\tilde\nabla^a E_{ij}\big]=0. \tag{5.19}$$

We now note that for any vector A_i that obeys $\tilde\nabla^i A_i=0$, through repeated use of the first relation in (5.2) we obtain

$$(\tilde\nabla_b\tilde\nabla^b-3k)(\tilde\nabla_iA_j+\tilde\nabla_jA_i)=\tilde\nabla_i(\tilde\nabla_b\tilde\nabla^b+k)A_j+\tilde\nabla_j(\tilde\nabla_b\tilde\nabla^b+k)A_i,$$
$$(\tilde\nabla_a\tilde\nabla^a-6k)(\tilde\nabla_b\tilde\nabla^b-3k)(\tilde\nabla_iA_j+\tilde\nabla_jA_i)=\tilde\nabla_i(\tilde\nabla_a\tilde\nabla^a-2k)(\tilde\nabla_b\tilde\nabla^b+k)A_j+\tilde\nabla_j(\tilde\nabla_a\tilde\nabla^a-2k)(\tilde\nabla_b\tilde\nabla^b+k)A_i. \tag{5.20}$$

On using the first relation in (5.2) again, it follows that

$$(\tilde\nabla_c\tilde\nabla^c-2k)(\tilde\nabla_a\tilde\nabla^a-6k)(\tilde\nabla_b\tilde\nabla^b-3k)(\tilde\nabla_iA_j+\tilde\nabla_jA_i)$$
$$=\tilde\nabla_i(\tilde\nabla_c\tilde\nabla^c+2k)(\tilde\nabla_a\tilde\nabla^a-2k)(\tilde\nabla_b\tilde\nabla^b+k)A_j+\tilde\nabla_j(\tilde\nabla_c\tilde\nabla^c+2k)(\tilde\nabla_a\tilde\nabla^a-2k)(\tilde\nabla_b\tilde\nabla^b+k)A_i. \tag{5.21}$$

5.8 Vector sector

On setting $A_i=\frac{1}{2}(\dot B_i-\ddot E_i)+\dot\Omega\Omega^{-1}(B_i-\dot E_i)$ (so that A_i is such that $\tilde\nabla^iA_i=0$), and recalling (5.14) we obtain

$$(\tilde\nabla_c\tilde\nabla^c-2k)(\tilde\nabla_a\tilde\nabla^a-6k)(\tilde\nabla_b\tilde\nabla^b-3k)$$
$$\times\Big[\tilde\nabla_i[\tfrac{1}{2}(\dot B_j-\ddot E_j)+\dot\Omega\Omega^{-1}(B_j-\dot E_j)]+\tilde\nabla_j[\tfrac{1}{2}(\dot B_i-\ddot E_i)+\dot\Omega\Omega^{-1}(B_i-\dot E_i)]\Big]=0. \tag{5.22}$$

5.9 Tensor sector

Thus finally from (5.19) we obtain

$$(\tilde\nabla_c\tilde\nabla^c-2k)(\tilde\nabla_a\tilde\nabla^a-6k)(\tilde\nabla_b\tilde\nabla^b-3k)\big[-\ddot E_{ij}-2kE_{ij}-2\dot\Omega\Omega^{-1}\dot E_{ij}+\tilde\nabla_a\tilde\nabla^aE_{ij}\big]=0. \tag{5.23}$$

Thus with ten independent fluctuation equations, four for the scalars [(4.15), (5.5), (5.10), (5.11)], two two-component equations for the vectors [(5.6), (5.14)], and one two-component equation for the tensor [(5.23)], we have succeeded in decomposing the fluctuation equations for the components, with the various components obeying derivative equations that are higher than second order.

5.10 The decoupled relations

The decoupled relations

$$
\begin{aligned}
\Delta_{00} &= 6\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})+\delta\hat{\rho}\Omega^2+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma=0,\\
\tilde{\nabla}^i\Delta_{0i} &= \tilde{\nabla}_a\tilde{\nabla}^a\big[-2\dot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+2k\gamma+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})\hat{V}\big]=0,\\
3\tilde{\nabla}^i\tilde{\nabla}^j\Delta_{ij}-\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\gamma}^{ij}\Delta_{ij}) &= 2\tilde{\nabla}^2[\tilde{\nabla}^2+3k](\alpha+2\dot{\Omega}\Omega^{-1}\gamma)=0,\\
\tilde{\nabla}^i\tilde{\nabla}^j\Delta_{ij}+k\tilde{\gamma}^{ij}\Delta_{ij} &= [\tilde{\nabla}^2+3k][2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})-2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma})-4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+\Omega^2\delta\hat{p}]=0.\\
(\tilde{\nabla}_k\tilde{\nabla}^k-2k)\Delta_{0i} &= (\tilde{\nabla}_k\tilde{\nabla}^k-2k)\left[k(B_i-\dot{E}_i)+\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i-\dot{E}_i)+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})V_i\right]=0,\\
(\tilde{\nabla}_a\tilde{\nabla}^a-2k)(\tilde{\nabla}_b\tilde{\nabla}^b+k)\tilde{\nabla}^j\Delta_{ij} &= (\tilde{\nabla}_a\tilde{\nabla}^a-2k)(\tilde{\nabla}_b\tilde{\nabla}^b+k)[\tilde{\nabla}_c\tilde{\nabla}^c+2k][\tfrac{1}{2}(\dot{B}_i-\ddot{E}_i)+\dot{\Omega}\Omega^{-1}(B_i-\dot{E}_i)]=0,\\
(\tilde{\nabla}_c\tilde{\nabla}^c-2k)(\tilde{\nabla}_a\tilde{\nabla}^a-6k)&(\tilde{\nabla}_b\tilde{\nabla}^b-3k)\big[-\ddot{E}_{ij}-2kE_{ij}-2\dot{\Omega}\Omega^{-1}\dot{E}_{ij}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij}\big]=0
\end{aligned}
\tag{5.24}
$$

are exact without approximation. They are all in the form of derivative operators acting on the functions required of the decomposition theorem.

Then with appropriate asymptotic boundary conditions we obtain

$$
\begin{aligned}
6\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})+\delta\hat{\rho}\Omega^2+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma&=0,\\
-2\dot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+2k\gamma+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})\hat{V}&=0,\\
\alpha+2\dot{\Omega}\Omega^{-1}\gamma&=0,\\
2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})-2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma})-4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+\Omega^2\delta\hat{p}&=0.\\
k(B_i-\dot{E}_i)+\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i-\dot{E}_i)+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})V_i&=0,\\
\tfrac{1}{2}(\dot{B}_i-\ddot{E}_i)+\dot{\Omega}\Omega^{-1}(B_i-\dot{E}_i)&=0,\\
-\ddot{E}_{ij}-2kE_{ij}-2\dot{\Omega}\Omega^{-1}\dot{E}_{ij}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij}&=0.
\end{aligned}
\tag{5.25}
$$

These boundary conditions are not new conditions since we already required asymptotic convergence in order to set up the SVT basis in (3.3) in the first place. Having now set up the formalism we look at some solutions.

6 Fluctuations Around de Sitter

For de Sitter the background fluid is a cosmological constant term $T_{\mu\nu} = \Lambda g_{\mu\nu}$. With a time-independent Hubble parameter H and $k = 0$ the comoving time expansion radius is given by $a(t) = e^{Ht}$. Thus the conformal time $\tau = -e^{-Ht}/H$ and $\Omega(\tau) = 1/H\tau$. The decomposition theorem gives us

$$\begin{aligned}
\tfrac{6}{\tau^2}(\alpha - \dot{\gamma}) - \tfrac{2}{\tau}\tilde{\nabla}_a\tilde{\nabla}^a\gamma &= 0, \\
\tfrac{2}{\tau}(\alpha - \dot{\gamma}) &= 0, \\
\alpha - \tfrac{2}{\tau}\gamma &= 0, \\
\tfrac{2}{\tau^2}(\alpha - \dot{\gamma}) + \tfrac{2}{\tau}(\dot{\alpha} - \ddot{\gamma}) + \tfrac{8}{\tau^2}(\alpha - \dot{\gamma}) &= 0. \\
\tfrac{1}{2}(B_i - \dot{E}_i) &= 0, \\
\tfrac{1}{2}(\dot{B}_i - \ddot{E}_i) - \tfrac{1}{\tau}(B_i - \dot{E}_i) &= 0, \\
-\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} &= 0.
\end{aligned} \tag{6.1}$$

In this solution we have

$$\alpha = 0, \qquad \gamma = 0, \qquad \dot{B}_i - \ddot{E}_i = 0. \tag{6.2}$$

Thus the only nontrivial modes are the tensor modes. And in a plane wave mode with momentum $\mathbf{k}$, E_{ij} is given as

$$E_{ij} = \epsilon_{ij}(\mathbf{k})\tau^2[a_1(\mathbf{k})j_1(k\tau) + b_1(\mathbf{k})y_1(k\tau)]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.3}$$

where $\mathbf{k}\cdot\mathbf{k} = k^2$, j_1 and y_1 are spherical Bessel functions, and $a_1(\mathbf{k})$ and $b_1(\mathbf{k})$ are spacetime independent constants. For E_{ij} to obey the transverse and traceless conditions $\delta^{ij}E_{ij} = 0$, $\tilde{\nabla}^j E_{ij} = 0$ the polarization tensor $\epsilon_{ij}(\mathbf{k})$ must obey $\delta^{ij}\epsilon_{ij} = 0$, $\mathbf{k}^j\epsilon_{ij}(\mathbf{k}) = 0$. Then, by taking a family of separation constants we can form a transverse-traceless wave packet

$$\begin{aligned}
E_{ij} &= \sum_{\mathbf{k}} \epsilon_{ij}(\mathbf{k})\tau^2[a_1(\mathbf{k})j_1(k\tau) + b_1(\mathbf{k})y_1(k\tau)]e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{k}} \epsilon_{ij}(\mathbf{k})\left[a_1(\mathbf{k})\left(\frac{\sin(k\tau)}{k^2} - \frac{\tau\cos(k\tau)}{k}\right) + b_1(\mathbf{k})\left(\frac{\cos(k\tau)}{k^2} + \frac{\tau\sin(k\tau)}{k}\right)\right],
\end{aligned} \tag{6.4}$$

and can choose the $a_1(\mathbf{k})$ and $b_1(\mathbf{k})$ coefficients to make the packet be as well-behaved at spatial infinity as desired. Finally, since the full fluctuation is given not by E_{ij} but by $2E_{ij}/H^2\tau^2$, then with $\tau = -e^{-Ht}/H$, through the $\cos(k\tau)/k^2$ term we find that at large comoving time E_{ij}/τ^2 behaves as e^{2Ht}, viz. the standard de Sitter inflation fluctuation exponential growth.

7 Conformal Gravity

Conformal gravity is a candidate alternate metric gravitational theory that has not only general coordinate invariance but also local conformal invariance, i.e., invariance under $g_{\mu\nu}(x) \to e^{2\alpha(x)} g_{\mu\nu}(x)$ for arbitrary spacetime dependent $\alpha(x)$. Under this transformation the conformal Weyl tensor, defined as

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) + \frac{1}{6}R^{\alpha}{}_{\alpha}\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right) \tag{7.1}$$

transforms as $C^{\lambda}{}_{\mu\nu\kappa} \to C^{\lambda}{}_{\mu\nu\kappa}$ with all derivatives of $\alpha(x)$ dropping out. In consequence the action

$$I_{\rm W} = -\alpha_g \int d^4x\, (-g)^{1/2} C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} \equiv -2\alpha_g \int d^4x\, (-g)^{1/2}\left[R_{\mu\kappa}R^{\mu\kappa} - \frac{1}{3}(R^{\alpha}{}_{\alpha})^2\right] \tag{7.2}$$

is locally conformal invariant. Not only that, it is the unique action that possesses this invariance. The attraction of this theory is that it forbids the presence of any cosmological constant term at the level of the Lagrangian. With the gravitational coupling constant α_g being dimensionless, quantum-mechanically the theory is power counting renormalizable. It is also unitary, and thus provides a consistent quantum gravity theory in four spacetime dimensions. No strings, no extra dimensions, no supersymmetry.

With the Weyl action $I_{\rm W}$ given in (7.2) being a fourth-order derivative function of the metric, functional variation with respect to the metric $g_{\mu\nu}(x)$ generates fourth-order derivative gravitational equations of motion of the form

$$-\frac{2}{(-g)^{1/2}}\frac{\delta I_{\rm W}}{\delta g_{\mu\nu}} = 4\alpha_g W^{\mu\nu} = 4\alpha_g\left[2\nabla_{\kappa}\nabla_{\lambda}C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda}C^{\mu\lambda\nu\kappa}\right] = 4\alpha_g\left[W^{\mu\nu}_{(2)} - \frac{1}{3}W^{\mu\nu}_{(1)}\right] = T^{\mu\nu}, \tag{7.3}$$

where the functions $W^{\mu\nu}_{(1)}$ and $W^{\mu\nu}_{(2)}$ (respectively associated with the $(R^{\alpha}{}_{\alpha})^2$ and $R_{\mu\kappa}R^{\mu\kappa}$ terms in (7.2)) are given by

$$\begin{aligned}
W^{\mu\nu}_{(1)} &= 2g^{\mu\nu}\nabla_{\beta}\nabla^{\beta}R^{\alpha}{}_{\alpha} - 2\nabla^{\nu}\nabla^{\mu}R^{\alpha}{}_{\alpha} - 2R^{\alpha}{}_{\alpha}R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}(R^{\alpha}{}_{\alpha})^2, \\
W^{\mu\nu}_{(2)} &= \frac{1}{2}g^{\mu\nu}\nabla_{\beta}\nabla^{\beta}R^{\alpha}{}_{\alpha} + \nabla_{\beta}\nabla^{\beta}R^{\mu\nu} - \nabla_{\beta}\nabla^{\nu}R^{\mu\beta} - \nabla_{\beta}\nabla^{\mu}R^{\nu\beta} - 2R^{\mu\beta}R^{\nu}{}_{\beta} + \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta},
\end{aligned} \tag{7.4}$$

and where $T^{\mu\nu}$ is the conformal invariant, and thus traceless, energy-momentum tensor associated with a conformal matter source. Here $W^{\mu\nu} = W^{\mu\nu}_{(2)} - (1/3)W^{\mu\nu}_{(1)}$ is known as the Bach tensor. In addition, the conformal Weyl tensor vanishes in geometries that are conformal to flat, this precisely being the case for the Robertson-Walker and de Sitter geometries that are of relevance to cosmology. Thus with the cosmological principle it follows that $T_{\mu\nu} = 0$, so that it allows for the creation of a universe from nothing, provided of course that $T_{\mu\nu}$ vanishes nontrivially, something we now show to be the case.

8 The Conformal Gravity Background Cosmology

Since particles can only acquire mass in a conformal invariant theory by symmetry breaking, we introduce a scalar field $S(x)$ for this purpose. We take the matter sector fields to be represented by fermions, with the conformally invariant matter sector action then being of the form

$$I_M = -\int d^4x(-g)^{1/2}\left[\frac{1}{2}\nabla_\mu S\nabla^\mu S - \frac{1}{12}S^2 R^\mu{}_\mu + \lambda S^4 + i\bar{\psi}\gamma^c V^\mu_c(x)[\partial_\mu + \Gamma_\mu(x)]\psi - hS\bar{\psi}\psi\right], \tag{8.1}$$

where h and λ are dimensionless coupling constants and the $V^\mu_c(x)$ are vierbeins. As such, the $I_{\rm M}$ action is the most general curved space matter action for the $\psi(x)$ and $S(x)$ fields that is invariant under both general coordinate transformations and the local conformal transformation $S(x) \to e^{-\alpha(x)}S(x)$, $\psi(x) \to e^{-3\alpha(x)/2}\psi(x)$, $\bar{\psi}(x) \to e^{-3\alpha(x)/2}\bar{\psi}(x)$, $V^a_\mu(x) \to e^{\alpha(x)}V^a_\mu(x)$, $g_{\mu\nu}(x) \to e^{2\alpha(x)}g_{\mu\nu}(x)$. Variation of this action with respect to $\psi(x)$ and $S(x)$ yields the equations of motion

$$i\gamma^c V^\mu_c(x)[\partial_\mu + \Gamma_\mu(x)]\psi - hS\psi = 0, \qquad \nabla_\mu\nabla^\mu S + \frac{1}{6}SR^\mu{}_\mu - 4\lambda S^3 + h\bar{\psi}\psi = 0. \tag{8.2}$$

We take the fermions to form a general background matter sector perfect fluid (labelled by m), and thus when the scalar field acquires a constant symmetry breaking vacuum expectation value S_0 the total background matter sector $T^{\mu\nu}$ is then of the form

$$T^{\mu\nu} = \frac{1}{c}\left[(\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu}\right] - \frac{1}{6}S_0^2\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^\alpha{}_\alpha\right) - g^{\mu\nu}\lambda S_0^4. \tag{8.3}$$

8.1 The Background Equations

Since $W_{\mu\nu}$ is zero in RW geometries, then so is $T_{\mu\nu}$. Thus it follows that

$$\frac{1}{6}S_0^2\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^\alpha{}_\alpha\right)=\frac{1}{c}\left[(\rho_m+p_m)U^\mu U^\nu+p_m g^{\mu\nu}\right]-g^{\mu\nu}\lambda S_0^4. \tag{8.4}$$

We thus recognize the conformal cosmological evolution equation given in (8.4) as being of the form as none other than the cosmological evolution equation of the standard theory, viz. (on setting $\Lambda=\lambda S_0^4$)

$$-\frac{c^3}{8\pi G_N}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^\alpha{}_\alpha\right)=\frac{1}{c}\left[(\rho_m+p_m)U^\mu U^\nu+p_m g^{\mu\nu}\right]-g^{\mu\nu}\Lambda, \tag{8.5}$$

save only for the fact that the standard G has been replaced by an effective, dynamically induced one given by

$$G_{\rm eff}=-\frac{3c^3}{4\pi S_0^2}, \tag{8.6}$$

viz. by an effective gravitational coupling that is expressly negative. Conformal cosmology is thus controlled by an effective gravitational coupling that is repulsive rather than attractive, and which becomes smaller the larger S_0 might be. With $G_{\rm eff}$ being negative, cosmological gravity is repulsive, and thus naturally leads to cosmic acceleration.

Despite the fact that the global cosmological $G_{\rm eff}$ is negative, local inhomogeneous gravity associated with a static source is not controlled by the global $G_{\rm eff}$ associated with a homogeneous comoving geometry and a vanishing Weyl tensor but by an induced local attractive G that is associated with an inhomogeneous geometry and a non-vanishing Weyl tensor. The static limit consists of both a $1/r$ potential and a linear r potential. Because the potential grows with r one cannot ignore material outside of any galaxy. Moreover the material furthest away has the biggest impact and is thus of cosmological strength. And not only that it leads to an additional universal linear potential $\gamma_0 r$ where γ_0 is fixed by the spatial 3-curvature of the Universe according to $\gamma_0=(-4k)^{1/2}$, a relation that requires that k expressly be negative. This then enables conformal gravity to fit galactic rotation curves without any dark matter and determine that $(-4k)^{1/2}=3.06\times10^{-30}{\rm cm}^{-1}$. **Thus the missing mass is the rest of the visible mass in the Universe, and it has been hiding in plain sight all along.**

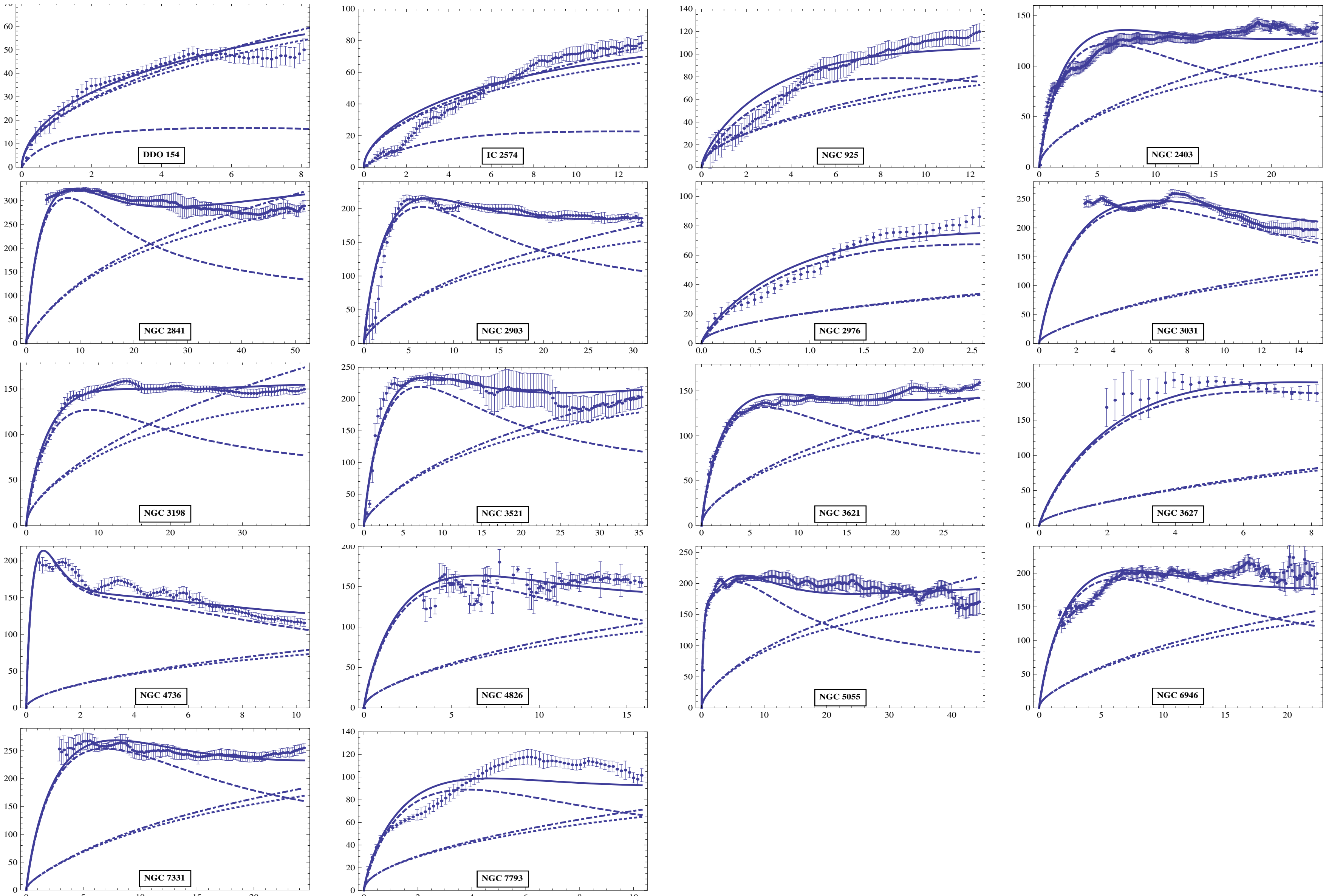


FIG. 1: Fitting to the rotational velocities (in km sec^{-1}) of the THINGS 18 galaxy sample

To be able to see how central the negative sign of $G_{\rm eff}$ is to cosmic acceleration we define

$$\bar{\Omega}_M(t)=\frac{8\pi G_{\rm eff}\rho_m(t)}{3c^2H^2(t)},\quad \bar{\Omega}_\Lambda(t)=\frac{8\pi G_{\rm eff}\Lambda}{3cH^2(t)},\quad \bar{\Omega}_k(t)=-\frac{kc^2}{\dot{a}^2(t)}, \tag{8.7}$$

where $H=\dot{a}/a$. And on introducing the deceleration parameter $q=-a\ddot{a}/\dot{a}^2$, from (8.4) we obtain

$$\dot{a}^2(t)+kc^2 \;=\; \dot{a}^2(t)\left(\bar{\Omega}_M(t)+\bar{\Omega}_\Lambda(t)\right),\quad \bar{\Omega}_M(t)+\bar{\Omega}_\Lambda(t)+\bar{\Omega}_k(t)=1,\qquad q(t)=\frac{1}{2}\left(1+\frac{3p_m}{\rho_m}\right)\bar{\Omega}_M(t)-\bar{\Omega}_\Lambda(t) \tag{8.8}$$

as the background evolution equations of conformal cosmology.

Given (8.8), and without needing to specify any matter sector equation of state and without even needing to solve the theory explicitly at all, we are still able to constrain $q(t)$. Specifically, we note that since Λ represents the free energy that is released in the phase transition that generated S_0 in the first place, Λ (and thus the scalar field coupling constant λ) is necessarily negative. Then with $G_{\rm eff}$ also being negative the quantity $\bar{\Omega}_\Lambda(t)$ is positive, i.e., the conformal theory needs a negative $G_{\rm eff}$ in order to obtain a positive $\bar{\Omega}_\Lambda(t)$. (In contrast, the standard model rationale for positive $\Omega_\Lambda=8\pi G_N\Lambda/3c^2H^2$ is that since the Newtonian G is positive Λ has to be taken to be positive too.) Since ρ_m and p_m are associated with ordinary matter they are both positive. Thus $\bar{\Omega}_M(t)$ is negative and $\bar{\Omega}_\Lambda(t)$ is positive. Thus since $G_{\rm eff}$ is negative it follows that $q(t)$ is automatically negative, being so in every epoch. **Consequently, conformal cosmology is automatically accelerating in every cosmological epoch without any adjustment or fine tuning of parameters ever being needed.**

If we take Λ to be much bigger than ρ_m the evolution equations admit of an exact comoving frame solution of the form

$$a(t)=(-k/\sigma)^{1/2}\sinh(\sigma^{1/2}ct), \tag{8.9}$$

where $\sigma=-2\lambda S_0^2=8\pi G_{\rm eff}\Lambda/3c$ is positive. With such an $a(t)$ we obtain

$$\bar{\Omega}_\Lambda(t)=\tanh^2(\sigma^{1/2}ct),\quad \bar{\Omega}_k(t)={\rm sech}^2(\sigma^{1/2}ct),\quad q(t)=-\tanh^2(\sigma^{1/2}ct), \tag{8.10}$$

As we see, no matter how big Λ might be, $\bar{\Omega}_\Lambda(t)$ has to lie between zero and one, i.e., because k is negative $\bar{\Omega}_\Lambda(t)$ approaches one from below. The cosmological constant problem is thus solved not by making Λ small but by making the amount by which it gravitates small (i.e., small $G_{\rm eff}$ and large S_0). Similarly, $q(t)$ has to lie between zero and minus one, with measured value $q_0=-0.37$.

The current value of the Hubble parameter is given by $H(t_0)=\sigma^{1/2}c\coth(\sigma^{1/2}ct_0)$. With $q_0=-0.37$ we obtain $\sigma^{1/2}ct_0=0.71$, and $t_0=1.16/H(t_0)=5\times10^{17}$ sec. Also $\sigma^{1/2}=0.50\times10^{-28}{\rm cm}^{-1}$. With $(-k)^{1/2}=1.53\times10^{-30}$ cm^{-1} we obtain $a(t_0)=2.36\times10^{-2}$. As we will see, this number is small enough to enable us to reliably do perturbation theory.

The lluminosity distance redshift relation of the form

$$d_L = -\frac{c}{H(t_0)}\frac{(1+z)^2}{q_0}\left[1-\left(1+q_0-\frac{q_0}{(1+z)^2}\right)^{1/2}\right], \tag{8.11}$$

where $q_0 = q(t_0)$ and $H(t_0)$ are the current era values of the deceleration parameter and the Hubble parameter.

Fitting the type 1A supernovae accelerating universe data with (8.11) gives a fit that is comparable in quality with that of the standard model $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ dark matter dark energy paradigm. In the conformal gravity fit q_0 is fitted to the value -0.37, i.e., quite non-trivially found to be right in the allowed $-1 \leq q_0 \leq 0$ range, with $\bar{\Omega}_\Lambda(t_0) = 0.37$, $\bar{\Omega}_k(t_0) = 0.63$. Since $\bar{\Omega}_M(t_0)$ is negligible no dark matter is needed, and since q_0 and $\bar{\Omega}_\Lambda(t_0) = -q_0$ fall right in the allowed region, no fine tuning is needed either. The ability of the conformal gravity theory to fit the accelerating universe data thus confirms that in conformal cosmology k is indeed negative. So now let us see what the fluctuations look like.

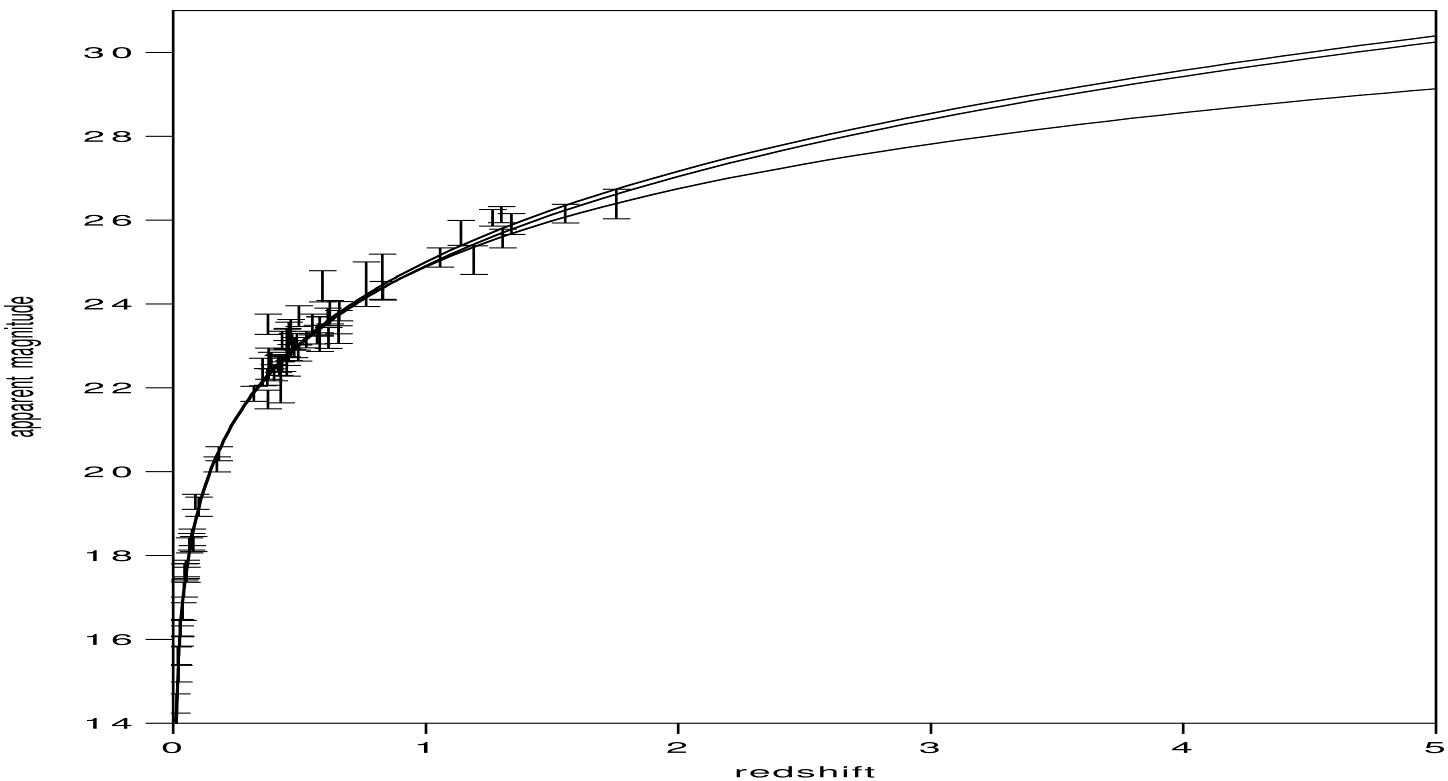


Figure 3: **Hubble plot expectations for $q_0 = -0.37$ (highest curve) and $q_0 = 0$ (middle curve) conformal gravity and for $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard gravity (lowest curve).**

9 Conformal Gravity Fluctuations

Taking the line element to be

$$ds^2 = -(g_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\tau)\left[d\tau^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2\right]$$
$$+\,\Omega^2(\tau)\left[2\phi d\tau^2 - 2(\tilde{\nabla}_i B + B_i)d\tau dx^i - [-2\psi\tilde{\gamma}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\right], \tag{9.1}$$

we need to solve $4\alpha_g\delta W_{\mu\nu} = \delta T_{\mu\nu}$ about a background in which both $W_{\mu\nu}$ and $T_{\mu\nu}$ vanish. $\delta T_{\mu\nu}$ is the same $\Delta_{\mu\nu}$ that we gave in (4.15) to (4.17) but with the repulsive $G_{\rm eff}$ replacing the attractive G_N.

It is convenient to define

$$\eta = -\frac{24\alpha_g}{S_0^2}, \quad R = -\frac{6(\rho_m + c\Lambda)}{S_0^2}, \quad P = -\frac{6(p_m - c\Lambda)}{S_0^2}, \quad \delta R = -\frac{6\delta\rho_m}{S_0^2}, \quad \delta P = -\frac{6\delta p_m}{S_0^2}. \tag{9.2}$$

The background and fluctuation equations then take the form

$$\eta W_{\mu\nu} = G_{\mu\nu} + \frac{1}{c}\left[(R+P)U_\mu U_\nu + P g_{\mu\nu}\right], \tag{9.3}$$

$$\eta\delta W_{\mu\nu} = \delta G_{\mu\nu} + \frac{1}{c}\left[(\delta R + \delta P)U_\mu U_\nu + \delta P g_{\mu\nu} + (R+P)(\delta U_\mu U_\nu + U_\mu \delta U_\nu) + P h_{\mu\nu}\right] = \Delta_{\mu\nu}, \tag{9.4}$$

The fluctuation $\delta W_{\mu\nu}$ in the Bach tensor $W_{\mu\nu}$ is of the form

$$\begin{aligned}
\delta W_{00} &= -\frac{2}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b\alpha,\\
\delta W_{0i} &= -\frac{2}{3\Omega^2}\tilde{\nabla}_i(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\dot{\alpha} + \frac{1}{2\Omega^2}(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_\tau^2 - 2k)(\tilde{\nabla}_c\tilde{\nabla}^c + 2k)(B_i - \dot{E}_i),\\
\delta W_{ij} &= -\frac{1}{3\Omega^2}\left[\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\nabla}_b\tilde{\nabla}^b + 2k - \partial_\tau^2)\alpha - \tilde{\nabla}_i\tilde{\nabla}_j(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_\tau^2)\alpha\right]\\
&\quad + \frac{1}{2\Omega^2}\left[\tilde{\nabla}_i(\tilde{\nabla}_a\tilde{\nabla}^a - 2k - \partial_\tau^2)(\dot{B}_j - \ddot{E}_j) + \tilde{\nabla}_j(\tilde{\nabla}_a\tilde{\nabla}^a - 2k - \partial_\tau^2)(\dot{B}_i - \ddot{E}_i)\right]\\
&\quad + \frac{1}{\Omega^2}\left[(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_\tau^2 - 2k)^2 + 4k\partial_\tau^2\right]E_{ij}.
\end{aligned} \tag{9.5}$$

With

$$
\begin{aligned}
\Omega^2 R &= 3k + 3\dot{\Omega}^2\Omega^{-2}, \quad \Omega^2 P = -k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1}, \quad \dot{R} + 3\dot{\Omega}(R+P)\Omega^{-1} = 0, \\
\alpha &= \phi + \psi + \dot{B} - \ddot{E}, \quad \gamma = -\dot{\Omega}^{-1}\Omega\psi + B - \dot{E}, \quad \hat{V} = V - \Omega^2\dot{\Omega}^{-1}\psi, \\
\delta\hat{R} &= \delta R - 12\dot{\Omega}^2\psi\Omega^{-4} + 6\ddot{\Omega}\psi\Omega^{-3} - 6k\psi\Omega^{-2} = \delta R + \dot{\Omega}^{-1}\dot{R}\psi\Omega = \delta R - 3(R+P)\psi, \\
\delta\hat{P} &= \delta P - 4\dot{\Omega}^2\psi\Omega^{-4} + 8\ddot{\Omega}\psi\Omega^{-3} + 2k\psi\Omega^{-2} - 2\dddot{\Omega}\dot{\Omega}^{-1}\psi\Omega^{-2} = \delta P + \dot{\Omega}^{-1}\dot{P}\psi\Omega,
\end{aligned} \tag{9.6}
$$

the full and exact conformal cosmological fluctuation equations are of the form

$$
\begin{aligned}
\eta\delta W_{00} &= -\frac{2\eta}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b\alpha \\
&= \Delta_{00} = 6\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) + \delta\hat{R}\Omega^2 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma,
\end{aligned} \tag{9.7}
$$

$$
\begin{aligned}
\eta\delta W_{0i} &= -\frac{2\eta}{3\Omega^2}\tilde{\nabla}_i(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\dot{\alpha} + \frac{\eta}{2\Omega^2}(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_\tau^2 - 2k)(\tilde{\nabla}_c\tilde{\nabla}^c + 2k)(B_i - \dot{E}_i) \\
&= \Delta_{0i} = -2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(\alpha - \dot{\gamma}) + 2k\tilde{\nabla}_i\gamma + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i\hat{V} \\
&\quad + k(B_i - \dot{E}_i) + \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a(B_i - \dot{E}_i) + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i,
\end{aligned} \tag{9.8}
$$

$$
\begin{aligned}
\eta\delta W_{ij} &= -\frac{\eta}{3\Omega^2}\left[\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\nabla}_b\tilde{\nabla}^b + 2k - \partial_\tau^2)\alpha - \tilde{\nabla}_i\tilde{\nabla}_j(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_\tau^2)\alpha\right] \\
&\quad + \frac{\eta}{2\Omega^2}\left[\tilde{\nabla}_i(\tilde{\nabla}_a\tilde{\nabla}^a - 2k - \partial_\tau^2)(\dot{B}_j - \ddot{E}_j) + \tilde{\nabla}_j(\tilde{\nabla}_a\tilde{\nabla}^a - 2k - \partial_\tau^2)(\dot{B}_i - \ddot{E}_i)\right] \\
&\quad + \frac{\eta}{\Omega^2}\left[(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_\tau^2 - 2k)^2 + 4k\partial_\tau^2\right]E_{ij} \\
&= \Delta_{ij} = \tilde{\gamma}_{ij}\big[2\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha} - \ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha - \dot{\gamma}) + \Omega^2\delta\hat{P} - \tilde{\nabla}_a\tilde{\nabla}^a(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma)\big] \\
&\quad + \tilde{\nabla}_i\tilde{\nabla}_j(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma) + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i(B_j - \dot{E}_j) + \frac{1}{2}\tilde{\nabla}_i(\dot{B}_j - \ddot{E}_j) + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j(B_i - \dot{E}_i) + \frac{1}{2}\tilde{\nabla}_j(\dot{B}_i - \ddot{E}_i) \\
&\quad - \ddot{E}_{ij} - 2kE_{ij} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}.
\end{aligned} \tag{9.9}
$$

9.1 The Decomposition Theorem

The Decomposition Theorem also holds in conforrnal gravity and yields

$$-\frac{2\eta}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a+3k)\tilde{\nabla}_b\tilde{\nabla}^b\alpha=6\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})+\delta\hat{R}\Omega^2+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma, \tag{9.10}$$

$$\begin{aligned}&\frac{1}{2}(\tilde{\nabla}_c\tilde{\nabla}^c+2k)\frac{\eta}{\Omega^2}(\tilde{\nabla}_b\tilde{\nabla}^b-\partial_\tau^2-2k)(B_i-\dot{E}_i)\\&=\frac{1}{2}(\tilde{\nabla}_c\tilde{\nabla}^c+2k)(B_i-\dot{E}_i)+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})V_i,\end{aligned} \tag{9.11}$$

$$\frac{\eta}{\Omega^2}\left[(\tilde{\nabla}_b\tilde{\nabla}^b-\partial_\tau^2-2k)^2+4k\partial_\tau^2\right]E_{ij}=-\ddot{E}_{ij}-2kE_{ij}-2\dot{E}_{ij}\dot{\Omega}\Omega^{-1}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij}. \tag{9.12}$$

$$\begin{aligned}&-\frac{\eta}{3\Omega^2}\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\nabla}_b\tilde{\nabla}^b+2k-\partial_\tau^2)\alpha\\&=2\dot{\Omega}^2\Omega^{-2}(\alpha-\dot{\gamma})-2\dot{\Omega}\Omega^{-1}(\dot{\alpha}-\ddot{\gamma})-4\ddot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+\Omega^2\delta\hat{P}-\tilde{\nabla}_a\tilde{\nabla}^a(\alpha+2\dot{\Omega}\Omega^{-1}\gamma),\end{aligned} \tag{9.13}$$

$$\frac{\eta}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a-3\partial_\tau^2)\alpha=\alpha+2\dot{\Omega}\Omega^{-1}\gamma, \tag{9.14}$$

$$-\frac{2\eta}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a+3k)\dot{\alpha}=-2\dot{\Omega}\Omega^{-1}(\alpha-\dot{\gamma})+2k\gamma+(-4\dot{\Omega}^2\Omega^{-3}+2\ddot{\Omega}\Omega^{-2}-2k\Omega^{-1})\hat{V}, \tag{9.15}$$

$$\frac{\eta}{2\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a-2k-\partial_\tau^2)(\dot{B}_i-\ddot{E}_i)=\dot{\Omega}\Omega^{-1}(B_i-\dot{E}_i)+\frac{1}{2}(\dot{B}_i-\ddot{E}_i), \tag{9.16}$$

Following some algebra we can manipulate these equations to obtain

$$\begin{aligned}
&\frac{d}{dp}\left(-3\dot{\Omega}\Omega^{-2}X\hat{V}+\Omega^2\delta\hat{R}\right)+\left(\tilde{\nabla}_b\tilde{\nabla}^b+3k-3\dot{\Omega}^2\Omega^{-2}\right)[\Omega^{-1}X\hat{V}]+\dot{\Omega}\Omega\delta\hat{R}=X(\nabla_b\nabla^b+3k)\left[\eta\Omega^{-2}\dot{\alpha}+\gamma\right],\\
&-\left(\frac{d}{dp}+2\dot{\Omega}\Omega^{-1}\right)(\Omega^{-1}X\hat{V})-\Omega^2\delta\hat{P}=X(\alpha-\dot{\gamma}),\\
&-3\dot{\Omega}\Omega^{-2}X\hat{V}+\Omega^2\delta\hat{R}=(\tilde{\nabla}_b\tilde{\nabla}^b+3k)\left[\frac{\eta}{\Omega^2}(\ddot{\alpha}-2\dot{\Omega}\Omega^{-1}\dot{\alpha}-\tilde{\nabla}_b\tilde{\nabla}^b\alpha)+\alpha\right],\\
&\gamma=\frac{\Omega}{2\dot{\Omega}}\left[\frac{\eta}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a-3\partial_\tau^2)\alpha-\alpha\right].
\end{aligned}\tag{9.17}$$

in the scalar sector, where $X=4\dot{\Omega}^2\Omega^{-2}-2\ddot{\Omega}\Omega^{-1}+2kc^2=-6\Omega^2c(\rho_m+p_m)/S_0^2$.

9.2 The Solution

So far everything is exact. We now specialize to the case where Λ is much bigger than ρ_m. Then in comoving time we have $a(t)=(-k/\sigma)^{1/2}\sinh(\sigma^{1/2}ct)$, so that in conformal time we have

$$\Omega(\tau)=\frac{S_0(k/2\Lambda)^{1/2}}{\sinh(-(-k)^{1/2}c\tau)}=-\frac{(-k/\sigma)^{1/2}}{\sinh((-k)^{1/2}c\tau)}.\tag{9.18}$$

With this $\Omega(\tau)$ we find that $X=0$. On dropping the matter sector $\delta\hat{R}$, with asymptotic boundedness we then find that the equations for α and γ simplify to

$$\begin{aligned}
&\frac{\eta}{\Omega^2}(\ddot{\alpha}-2\dot{\Omega}\Omega^{-1}\dot{\alpha}-\tilde{\nabla}_b\tilde{\nabla}^b\alpha)+\alpha=0,\\
&\gamma=\frac{\Omega}{2\dot{\Omega}}\left[\frac{\eta}{3\Omega^2}(\tilde{\nabla}_a\tilde{\nabla}^a-3\partial_\tau^2)\alpha-\alpha\right].
\end{aligned}\tag{9.19}$$

9.3 Separating the Variables

We introduce a dimensionless conformal time variable $\rho = (-k)^{1/2} c\tau$, and set $r = (-k)^{-1/2} \sinh\chi$ where χ is dimensionless. We set $\alpha = \alpha(\rho) S_\ell(\chi) Y_\ell^m(\theta,\phi)$ and introduce a separation constant $(-k)(\nu^2+1)$. Thus we obtain

$$\left(\tilde{\nabla}_a \tilde{\nabla}^a + (-k)(\nu^2+1)\right) S_\ell(\chi) Y_\ell^m(\theta,\phi) = 0, \tag{9.20}$$

$$\left[\frac{d^2}{d\chi^2} + 2\frac{\cosh\chi}{\sinh\chi}\frac{d}{d\chi} - \frac{\ell(\ell+1)}{\sinh^2\chi} + \nu^2 + 1\right] S_\ell(\chi) = 0, \tag{9.21}$$

$$\left(\frac{d^2}{d\rho^2} + 2\frac{\cosh\rho}{\sinh\rho}\frac{d}{d\rho} - \frac{K(K+1)}{\sinh^2\rho} + \nu^2 + 1\right) \alpha(\rho) = 0, \tag{9.22}$$

where $K = -1/2 \pm (1/4 - 1/\sigma\eta)^{1/2}$. Here ℓ is integer but K is not.
These are standard associated Legendre function equations, with solutions

$$\hat{S}_\ell = \frac{(-1)^{\ell+1}}{(2\pi)^{1/2}} \pi\nu^2(\nu^2+1^2)....(\nu^2+\ell^2) \frac{P^{-1/2-\ell}_{-1/2+i\nu}(\cosh\chi)}{\sinh^{1/2}\chi} = \sinh^\ell\chi \left(\frac{1}{\sinh\chi}\frac{d}{d\chi}\right)^{\ell+1} \cos(\nu\chi), \tag{9.23}$$

where ν is a continuous real variable that lies between zero and infinity, and

$$\alpha(\rho) = \frac{1}{\sinh^{1/2}\rho} P^{-1/2-K}_{-1/2+i\nu}(\cosh\rho) = \frac{1}{\sinh^{1/2}\rho}\frac{1}{\Gamma(3/2+K)} \coth^{-1/2-K}(\rho/2) F(1/2-i\nu, 1/2+i\nu; 3/2+K; -\sinh^2(\rho/2)),$$

$$\frac{1}{3K(K+1)} \sinh^2\rho(\nu^2+1+3\partial_\rho^2)\alpha(\rho) - \alpha(\rho) = -2\frac{\cosh\rho}{\sinh\rho}(-k)^{1/2}\gamma(\rho) \tag{9.24}$$

in conformal time. With $\xi = \sigma^{1/2} t$ in comoving time the solutions are

$$\alpha(\xi) = \sinh\xi P_K^{i\nu}(\cosh\xi) = \sinh\xi \frac{1}{\Gamma(1-i\nu)} \coth^{i\nu}(\xi/2) F(-K, K+1; 1-i\nu; -\sinh^2(\xi/2)),$$

$$2\gamma(\xi)(-k)^{1/2}\sinh\xi\cosh\xi = \frac{1}{K(K+1)}\left[K(K+1)\sinh^2\xi - \nu^2 + 1 + 2\sinh^2\xi - 2(K+1)(1+\sinh^2\xi)\right] P_K^{i\nu}(\cosh\xi)$$

$$+ \frac{K(K+1)}{3}(\nu^2+1)P_K^{i\nu}(\cosh\xi) + \frac{2}{K(K+1)}\cosh\xi(K+i\nu+1)P_{K+1}^{i\nu}(\cosh\xi) - \sinh^2\xi P_K^{i\nu}(\cosh\xi). \tag{9.25}$$

10 Growth of Structure

We had noted that currently $a(t_0) = 2.36 \times 10^{-2}$. With a current temperature $T_0 = 3^\circ K$ and an adiabatic expansion in which $a(t)$ behaves as $1/T$, at any earlier time we have $a(t) = a(t_0)T_0/T$. At last scattering, at which the CMB is produced, the temperature T_L is order $3000^\circ K$. So $a(t_L) = 2.36 \times 10^{-5}$. Now

$$a(t) = (-k/\sigma)^{1/2} \sinh(\sigma^{1/2} ct) = \Omega(\rho) = -\frac{(-k/\sigma)^{1/2}}{\sinh \rho}. \tag{10.1}$$

Thus in the early universe ρ is large (and negative). For large ρ the associated Legendre functions behave as

$$P^{-1/2-K}_{-1/2+i\nu}(\cosh \rho) \rightarrow \frac{1}{\cosh^{1/2} \rho}\left[\frac{\Gamma(i\nu)e^{i\nu\rho}}{(2\pi)^{1/2}\Gamma(i\nu + K + 1)} + \frac{\Gamma(-i\nu)e^{-i\nu\rho}}{(2\pi)^{1/2}\Gamma(-i\nu + K + 1)}\right]. \tag{10.2}$$

Now radial modes on the light cone obey $d\rho^2 - d\chi^2 = 0$. Thus light rays obey $\rho = -\chi$. Thus large ρ means large χ. With the conformal time behavior being of the form $\alpha(\rho) = P^{-1/2-K}_{-1/2+i\nu}(\cosh \rho)/\sinh^{1/2} \rho$, and with the χ behavior being of the form $\alpha(\chi) = P^{-1/2-\ell}_{-1/2+i\nu}(\cosh \chi)/\sinh^{1/2} \chi$, in the early universe light ray fluctuations behave as

$$\alpha(\rho, \chi) \sim \frac{1}{\sinh \rho \sinh \chi} = \frac{1}{\sinh^2 \rho} \sim \Omega^2(\rho) \sim \frac{1}{T^2} \tag{10.3}$$

Thus fluctuations grow as

$$\alpha(\rho_2, \chi_2) = \frac{T_1^2}{T_2^2}\, \alpha(\rho_1, \chi_1). \tag{10.4}$$

Thus in going from nucleosynthesis to last scattering the amplitude grows by a factor of $(10^9/10^3)^2 = 10^{12}$.
In going from nucleosynthesis to today the amplitude grows by a factor of $(10^9/3)^2 = 10^{17}$.
In going from $10^{23\circ}K$ to today the amplitude grows by a factor of $(10^{23}/3)^2 = 10^{45}$.
In a standard gravity de Sitter geometry the expansion radius grows as $a(t) = e^{Ht}$ and we saw that fluctuation grows as e^{2Ht}, i.e. as $a^2(t) \sim 1/T^2$. Thus conformal gravity gives the same growth rate as inflation. However, in conformal gravity at small t we have $a(t) \rightarrow (-k)^{1/2}ct$. Thus because $a(t_0)$ is small, at last scattering and earlier the contribution of Λ is negligible and the conformal gravity Universe is negative curvature dominated.

THUS WE CAN REPLACE STANDARD GRAVITY INFLATION BY NEGATIVE CURVATURE CONFORMAL GRAVITY.