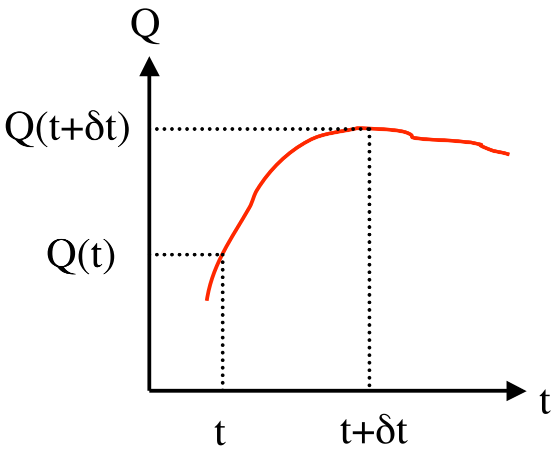
2.3 Relating Eulerian / Lagrangian descriptions

- Reminder: consider a function Q(t)

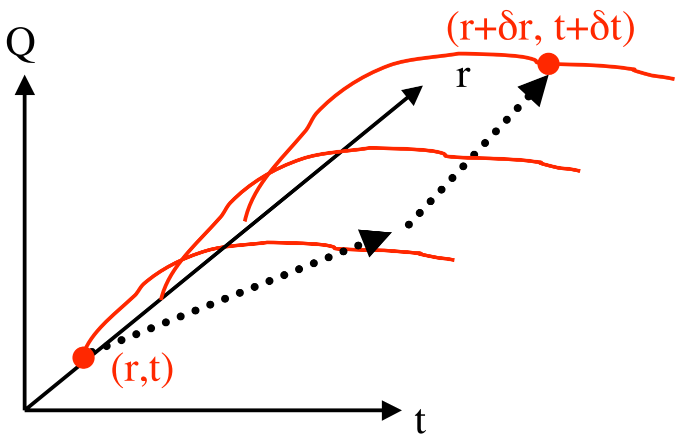
$$\frac{dQ}{dt} = \frac{Q(t+\delta t) - Q(t)}{\delta t}$$



- Now make Q a function of two variables:

$$\frac{dQ}{dt} = \frac{Q(\underline{r}+\delta\underline{r}, t+\delta t) - Q(\underline{r}, t)}{\delta t}$$

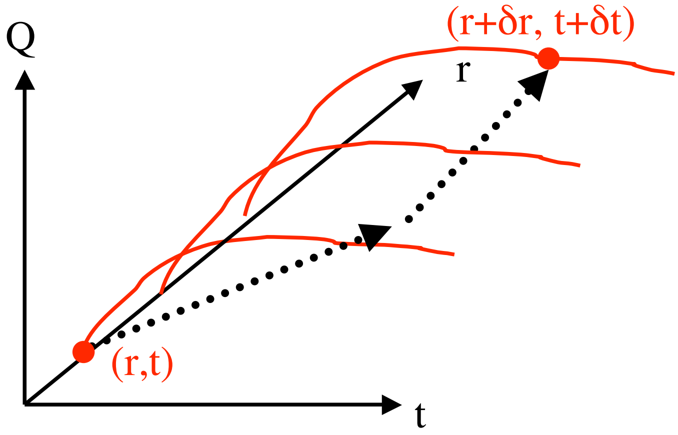
- r = position of fluid element at time t
- r+δr = position of same fluid element at time t+δt



•First of all, separate out the variation in r and t

$$Q(\underline{r}+\delta\underline{r},t+\delta t)-Q(\underline{r},t)$$
$$=\underbrace{Q(\underline{r},t+\delta t)-Q(\underline{r},t)}+\underbrace{Q(\underline{r}+\delta\underline{r},t+\delta t)-Q(\underline{r},t+\delta t)}$$

Variation in time t at fixed r   Variation in space r at fixed t+δt



• Then, write the numerator as an expansion in δr and δt

Remember that

$$\begin{cases} Q(t+\delta t)=Q(t)+\delta t\dfrac{\partial Q}{\partial t}+\ldots\ldots \\ Q(\underline{r}+\delta\underline{r})=Q(\underline{r})+\delta\underline{r}\cdot\underline{\nabla}Q+\ldots \end{cases}$$

So

$$Q(\underline{r}+\delta\underline{r},t+\delta t)-Q(\underline{r},t)$$
$$=\delta t\frac{\partial Q}{\partial t}+\ldots\ldots+\delta\underline{r}\cdot\underline{\nabla}Q+\ldots$$

- So we have:

$$Q(\underline{r}+\delta\underline{r},t+\delta t)-Q(\underline{r},t)$$

$$\approx \frac{\partial Q}{\partial t}\delta t+\delta\underline{r}\cdot\underline{\nabla Q}$$

Evaluated at t+δt

$$\approx \frac{\partial Q}{\partial t}\delta t+\delta\underline{r}\cdot\left[\underline{\nabla Q}+\delta t\frac{\partial}{\partial t}\underline{\nabla Q}\right]$$

Evaluated at t

- finally:

$$\frac{dQ}{dt}=\frac{Q(\underline{r}+\delta\underline{r},t+\delta t)-Q(\underline{r},t)}{\delta t}$$

$$\approx \frac{\frac{\partial Q}{\partial t}\delta t+\delta\underline{r}\cdot\left[\underline{\nabla Q}+\frac{\partial}{\partial t}\underline{\nabla Q}\delta t\right]}{\delta t}$$

$$\approx \frac{\partial Q}{\partial t}+\frac{\delta\underline{r}}{\delta t}\cdot\underline{\nabla Q}$$

But this is just the velocity!

- Hence if the flow velocity is u

$$\boxed{\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \underline{u} \cdot \underline{\nabla} Q} \qquad (2.1)$$

"Lagrangian" time derivative (what the thermometer on the boat measures)

"Eulerian" time derivative (what the riverbed thermometer measures)

"convective" derivative (due to the fact that the boat has moved in a finite time to a new location where the temperature is different)

NB: In a *steady state* the Eulerian time derivative would be zero, but the Lagrangian time derivative would only be zero *if Q was also uniform*.

2.4 Streamlines and streamfunctions

- We can write a 2D flow u(x,y) in terms of a scalar ψ (known as a *stream function*) such that

$$u_x = -\frac{\partial \psi}{\partial y}, u_y = \frac{\partial \psi}{\partial x}$$

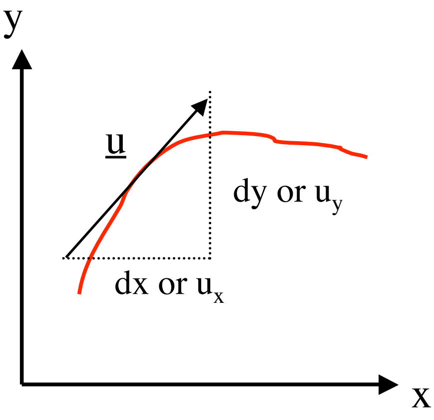
- Hence u is divergence-free

A *streamline* is defined as a curve that has u in the tangential direction,

$$\frac{dx}{u_x} = \frac{dy}{u_y}$$

$$\Rightarrow u_y dx - u_x dy = 0$$

$$\Rightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = d\psi = 0$$



Hence ψ is constant on a streamline.

Reminder

- If Q is a scalar:

$$\underline{u} \cdot \nabla Q = u_x \frac{\partial Q}{\partial x} + u_y \frac{\partial Q}{\partial y} + u_z \frac{\partial Q}{\partial z} \qquad \text{cartesians}$$

$$= u_r \frac{\partial Q}{\partial r} + \frac{u_\theta}{r} \frac{\partial Q}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial Q}{\partial \phi} \qquad \text{sphericals}$$

$$= u_R \frac{\partial Q}{\partial R} + u_z \frac{\partial Q}{\partial z} + \frac{u_\phi}{R} \frac{\partial Q}{\partial \phi} \qquad \text{cylindricals}$$

- If Q is a vector, then (u.grad)Q is also a vector, each component of which is (u.grad) acting on each component of Q.
- Hence in cartesians:

$$
\begin{aligned}
(\underline{u}\cdot\underline{\nabla})\underline{Q} &= \left(u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}+u_z\frac{\partial}{\partial z}\right)\left(Q_x, Q_y, Q_z\right) \\
&= \left[u_x\frac{\partial Q_x}{\partial x}+u_y\frac{\partial Q_x}{\partial y}+u_z\frac{\partial Q_x}{\partial z},\right. \\
&\quad u_x\frac{\partial Q_y}{\partial x}+u_y\frac{\partial Q_y}{\partial y}+u_z\frac{\partial Q_y}{\partial z}, \\
&\quad \left. u_x\frac{\partial Q_z}{\partial x}+u_y\frac{\partial Q_z}{\partial y}+u_z\frac{\partial Q_z}{\partial z}\right]
\end{aligned}
$$

Question 1

- The temperature variation in a river is

$$T(x,t) = e^x \sin t$$

- And the river flows with velocity

$$\underline{u}(x,t) = \left(xt^2, 0\right)$$

- Write down both the Lagrangian and Eulerian temperature derivatives.

Answer 1

- Did you get:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} = e^x \cos t + xt^2 e^x \sin t$$