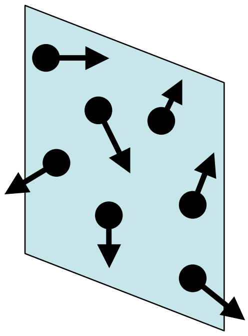
5. Conservation of momentum

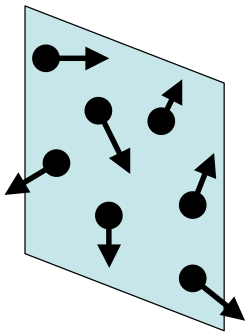
Rate of change of momentum = sum of forces



- 5.1 What are the forces acting on a parcel of fluid?

- For any surface within a fluid there is a momentum flux across it (from each side) *that has nothing to do with any bulk flow* but is a consequence of its thermal properties.

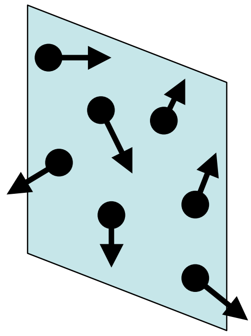
- *Microscopically* (in a perfect gas)
  - finite temperature imparts molecules with random motions
  - the pressure is the associated (one sided) momentum flux.



- Since these motions are isotropic, the momentum flux locally is:
  - independent of the orientation of the surface
  - always perp. to the surface (the parallel components cancel out).

- Quick check on units:

- Pressure is a force per unit area
- P~F/A
- Momentum flux is the rate of flow (rate of change) of momentum through unit area:
- (Momentum/s) / A
- And force is rate of change of momentum…



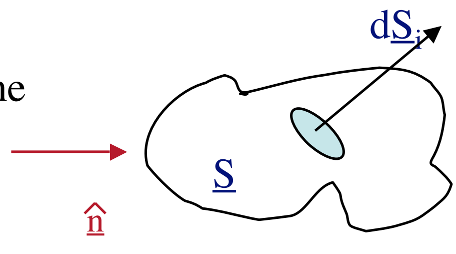
5.2 Deriving the equation

- Consider a lump of fluid subject to gravity and the inward pressure of the surrounding fluid

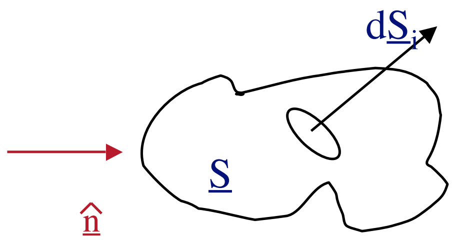
- Pressure force on $\underline{dS}$ = - p$\underline{dS}$
  - (minus because along inward normal)

- Component of inward pressure force along some direction $\hat{\underline{n}}$ is

$$-p\hat{\underline{n}}\cdot\underline{dS}$$



- Integrate over the whole surface



$$-\int_{s} p\hat{\underline{n}} \cdot d\underline{S} = -\int_{V} \underline{\nabla} \cdot (p\hat{\underline{n}})dV$$

Divergence theorem

- The total momentum in the volume V is:

$$\int_{V} \rho \underline{u} dV$$

- The rate of change is:

$$\frac{d}{dt}\int_{V} \rho \underline{u} dV$$

- The component along $\hat{\underline{n}}$ is:

$$\left(\frac{d}{dt}\int_{V} \rho \underline{u} dV\right) \cdot \hat{\underline{n}}$$

- Hence equation of motion in direction $\hat{\underline{n}}$ is
  (rate of change of momentum=sum of forces)

Lagrangian derivative

$$\left(\frac{d}{dt}\int_V \rho \underline{u}\, dV\right)\cdot \hat{\underline{n}} = -\int_V \underline{\nabla}\cdot\left(p\hat{\underline{n}}\right)dV + \int_V \rho \underline{g}\cdot\hat{\underline{n}}\, dV$$

Momentum contained in V — Pressure force — "mg"

- But note that $\underline{\nabla}\cdot\left(p\hat{\underline{n}}\right) = \hat{\underline{n}}\cdot\underline{\nabla}p + \cancelto{0}{p\underline{\nabla}\cdot\hat{\underline{n}}}$

- And (assuming lump is small) replace $\int dV$ by δV

- So that

(Mass of lump is conserved)

$$\frac{d}{dt}\left(\rho\underline{u}\delta V\right)\cdot\hat{\underline{n}} = \underline{u}\cdot\hat{\underline{n}}\cancelto{0}{\frac{d}{dt}\left(\rho\delta V\right)} + \rho\delta V\frac{d\underline{u}}{dt}\cdot\hat{\underline{n}}$$

- Hence momentum conservation reduces to

$$\rho \delta V \frac{d\underline{u}}{dt} \cdot \underline{\hat{n}} = \delta V \left(-\underline{\nabla} p + \rho \underline{g}\right) \cdot \underline{\hat{n}}$$

$$\Rightarrow$$

$$\delta V \left(\rho \frac{d\underline{u}}{dt} + \underline{\nabla} p - \rho \underline{g}\right) \cdot \underline{\hat{n}} = 0$$

- Since this is true for all δV and $\underline{\hat{n}}$

$$\boxed{\rho \frac{d\underline{u}}{dt} = -\underline{\nabla} p + \rho \underline{g}} \qquad (5.1)$$

- Lagrangian form: momentum of a fluid element changes in response to pressure and gravitational forces

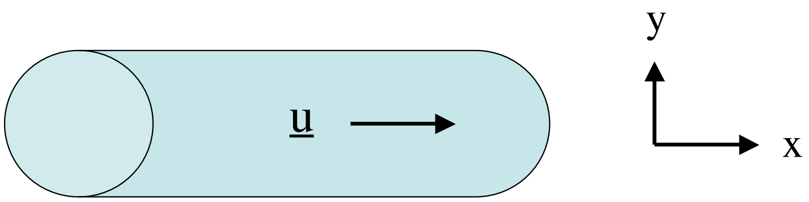
- Eulerian form:

$$\rho\frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla}p + \rho\underline{g} \qquad (5.2)$$

- The momentum contained in a fixed grid cell changes as a result of pressure and gravitational forces plus any imbalance in the momentum flux in and out of the cell.

Example

- Consider a flow $\underline{u}=u_x$ along a pipe in the absence of gravity



- (5.2) gives:

$$\rho\frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla}p$$

- The component along the pipe is

$$\rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} = -\frac{\partial p}{\partial x}$$

- If the fluid is incompressible, this gives

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u_x^2\right) = -\frac{\partial p}{\partial x}$$

Rate of change of momentum — "ram pressure" — thermal pressure

- Note: the thermal pressure is associated with random motions in the fluid which are isotropic. It is a scalar (acts the same way in any direction)

- The ram pressure is associated with bulk motions of the fluid. Only a surface whose normal has some component along the direction of flow feels the ram pressure.
  - Try putting your hand at the end of a hosepipe! Then rotate it till it's parallel to the flow…you only feel the ram pressure when the flow is "hitting" your hand.

Question 3

- Consider the velocity given in question 1:

$$\underline{u}(x,t) = \left(xt^2, 0\right)$$

- Determine (for a steady state) the density variation

$$\rho(x,y)$$

- And the pressure variation $p(x,y)$ in the absence of gravity.