6. The Energy Equation
6.1 Derviation

- From the second law of thermodynamics:

$$TdS \;=\; dQ \;=\; dU \;+\; pdV$$

Heat change   Internal energy change

- Write this equation for a small element of mass δm with

$$dQ = \delta m \, dq$$

$$dU = \delta m \, de$$

$$dV = \delta m \, d\left(\frac{1}{\rho}\right) \quad \longleftarrow \quad \text{Volume of unit mass}$$

- Where

$$e = \frac{p}{(\gamma - 1)\rho} \qquad \text{Internal energy per unit mass} \qquad (6.1)$$

$$\gamma = \frac{c_p}{c_v} \qquad \text{Ratio of specific heats}$$

- And s is the entropy per unit mass
- If L is the sum of sources and sinks of energy, we can divide by dt to get

$$\rho T \frac{ds}{dt} = \rho \left[\frac{de}{dt} + p \frac{d}{dt}\left(\frac{1}{\rho}\right) \right] = -L$$

- We can write this more simply using mass conservation:

$$\rho\frac{de}{dt} + p\rho\left(\frac{-1}{\rho^2}\right)\frac{d\rho}{dt} = -L$$

$$\Rightarrow \rho\frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = -L \qquad (6.2)$$

$$\Rightarrow \rho\frac{de}{dt} + p\underline{\nabla}\cdot\underline{u} = -L \qquad (6.3)$$

- Now take $\underline{u}$.[equation of motion (5.1)] - the "mechanical energy equation" to get

$$\rho\frac{d}{dt}\left(\frac{1}{2}u^2\right) = -\underline{u}\cdot\underline{\nabla}p + \underline{u}\cdot\rho\underline{g} \qquad (6.4)$$

- Add this to (6.3) to get

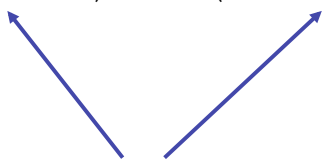$$\rho\frac{d}{dt}\left(\frac{1}{2}u^2 + e\right) = -L - p\underline{\nabla}\cdot\underline{u} - \underline{u}\cdot\underline{\nabla}p + \underline{u}\cdot\rho\underline{g}$$

$$= -L - \underline{\nabla}\cdot p\underline{u} + \underline{u}\cdot\rho\underline{g}$$

$$(6.5)$$

- Aside: for a scalar A:

$$\rho\frac{dA}{dt} = \frac{d(\rho A)}{dt} - A\frac{d\rho}{dt}$$

$$= \frac{d(\rho A)}{dt} + \rho A \underline{\nabla} \cdot \underline{u}$$

$$= \frac{\partial(\rho A)}{\partial t} + \underline{u} \cdot \underline{\nabla}(\rho A) + \rho A \underline{\nabla} \cdot \underline{u}$$

$$= \frac{\partial(\rho A)}{\partial t} + \underline{\nabla} \cdot (\rho A)\underline{u}$$

- And so writing g in terms of the gravitational potential ψ the energy equation (6.5) becomes:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \underline{\nabla} \cdot \left(\frac{1}{2}\rho u^2 + \rho e\right)\underline{u} = -L - \underline{\nabla} \cdot p\underline{u} - \underline{u} \cdot \rho\underline{\nabla}\psi$$

(6.6)

"ρA"

- But we can write

$$\underline{\nabla} \cdot (\rho \psi) \underline{u} = (\rho \psi) \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla}(\rho \psi)$$

$$= (\rho \psi) \underline{\nabla} \cdot \underline{u} + \psi \underline{u} \cdot \underline{\nabla} \rho + \rho \underline{u} \cdot \underline{\nabla} \psi$$

$$= \psi (\rho \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \rho) + \rho \underline{u} \cdot \underline{\nabla} \psi$$

$$= \psi \left(-\frac{\partial \rho}{\partial t} \right) + \rho \underline{u} \cdot \underline{\nabla} \psi$$

- And so in a steady state, the energy equation becomes:

$$\underline{\nabla} \cdot \left(\frac{1}{2} \rho u^2 + \rho e + p + \rho \psi \right) \underline{u} = -L \qquad (6.7)$$

- Where $\qquad \rho e + p = \dfrac{\gamma}{\gamma - 1} p \qquad$ is the *enthalpy*

Within some volume, the net effect (L) of the sources and sink of energy is equal (in a steady state) to the flux of energy through the surface of the volume.

6.2 Aside on equations of state

- In general, $p = p(\rho, T)$ and for an ideal gas

$$\boxed{p = nk_B T}$$ or $$\boxed{p = \frac{k_B}{m} \rho T}$$

n= total number of particles per unit volume

m= mean particle mass

- where $\rho = mn$
- For a fully-ionised H plasma

$$n \equiv n_p + n_e = 2n_e \qquad \rho \equiv n_p m_p + n_e m_e \approx n_e m_p$$

Barotropic Equations of State: p(ρ)

- Means that p can be written as a function of ρ only (e.g. for an ideal gas implies that ρ and T have some additional relation). E.g.:
- Isothermal $\Rightarrow T = $ constant

  so $p \propto \rho$

- For this to be a good approximation, require
  - Temperature for thermal equilibrium isn't very sensitive to the heating/cooling rate
  - In time-dependent problems, there is time for the system to reach this constant T thermal equilibrium

- Adiabatic $\qquad p = K\rho^{\gamma}$

- This is derived from the ideal gas laws on the assumption that there is no heat exchange with surroundings (i.e. no external heating/cooling) - changes in the internal energy result purely from pdV work. (Q4: use (6.2) to prove this)
- A fluid element behaves adiabatically if K is constant as the element's properties change. An isentropic fluid is one in which all the elements have the same value of K.

Answer to Q4

Start with the energy equation:

$$\rho\frac{\mathrm{d}e}{\mathrm{d}t} - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -L$$

$$\frac{\rho}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho}\right) - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -L$$

$$\frac{1}{\gamma-1}\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{(\gamma-1)p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) = -L$$

$$\frac{1}{\gamma-1}\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\gamma p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) = -L$$

$$\frac{\rho^{\gamma}}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^{\gamma}}\right) = -L$$

Hence if L=0, then following the motion, $p = K\rho^{\gamma}$