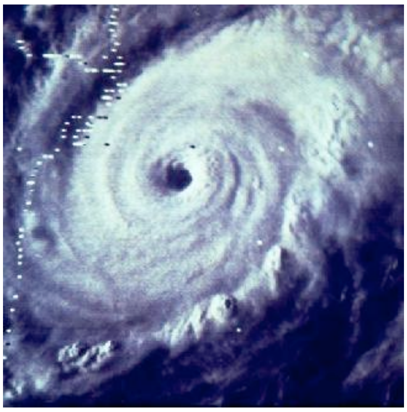
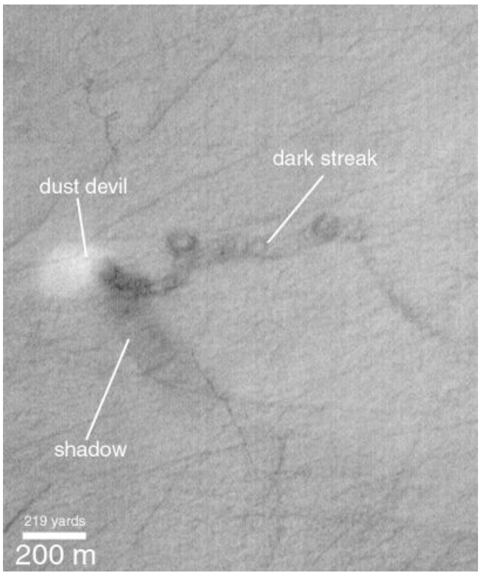
7. Vorticity and viscosity



Martian dust devils



7.1 What is vorticity?
(Angular momentum and all that..)

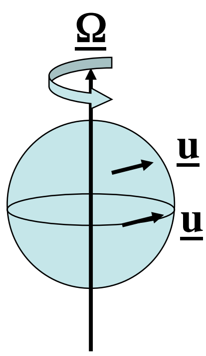
- Tendency for parcel of fluid to rotate about an axis through its centre of mass.
- Defined as $\underline{\omega} = \underline{\nabla} \times \underline{u}$
- A rough analogy: the Ferris wheel has rotational motion, but the passengers don't!

http://users.vnet.net/schulman/Columbian/ferris.html

Example 1

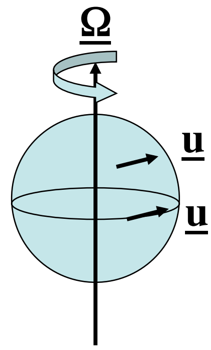
- A parcel of fluid rotates *as if it were a rigid body* with angular velocity Ω
- The velocity

$$\underline{u} = \underline{\Omega} \times \underline{r}$$



- The vorticity is the same everywhere:

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \underline{\nabla} \times \left(\underline{\Omega} \times \underline{r}\right) = 2\underline{\Omega}$$



- *cf*: Angular momentum per unit mass:

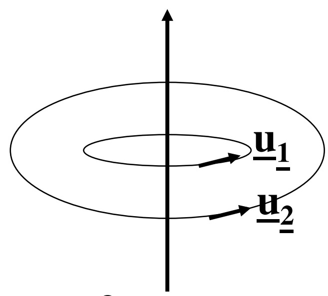
$$\underline{L} = \underline{r} \times \left(\underline{\Omega} \times \underline{r}\right) = r^2 \underline{\Omega} = \frac{r^2}{2} \underline{\omega}$$

- NB: *cylindrical* (r,ϕ,z)

$$u_\phi = \Omega r, \quad u_r = u_z = 0$$

$$\omega_z = 2\Omega, \quad \omega_r = \omega_\phi = 0$$

Example 2



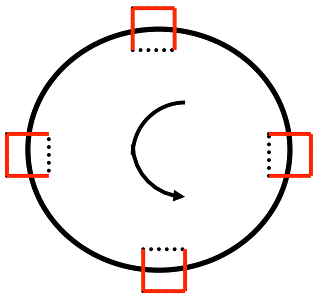
- *Every* fluid parcel moves in a circle, but with a different azimuthal velocity
- The velocity

$$u_\phi = \frac{K}{r}, \quad u_r = u_\theta = 0$$

- Gives a vorticity $\omega_\phi = \omega_r = 0$
- Zero everywhere except on the axis

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) = 0 \ \text{ for } \ r \neq 0$$

- So: circular motion doesn't necessarily imply vorticity!



Rigid body rotation. Each parcel of fluid changes its orientation as it moves

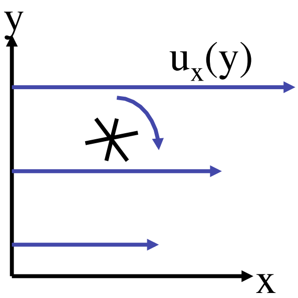
Circulation without rotation. Each parcel maintains the same orientation, even though it moves in a circle.

Example 3

- Shear flow $u = u_x(y), \quad u_y = u_z = 0$
- Vorticity

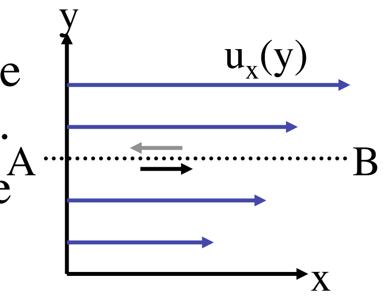
$$\omega_x = \omega_y = 0, \quad \omega_z = -\partial u / \partial y$$

- A paddle wheel put in this flow would rotate



7.2 What is viscosity?

- The faster fluid above plane AB drags the fluid below forwards (black arrow)
- The slower fluid below AB drags the fluid above backwards (grey arrow).
- Arrows show the equal and opposite forces acting on AB, drawn on the side of the fluid on which they act.
- Viscosity is responsible for this internal stress (force/unit area).

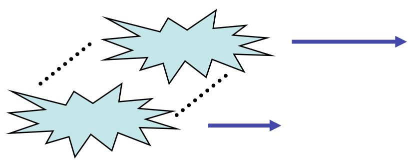


- For a Newtonian fluid* the force per unit area (*stress*) is

$$\tau = \mu \frac{\partial u}{\partial y}$$

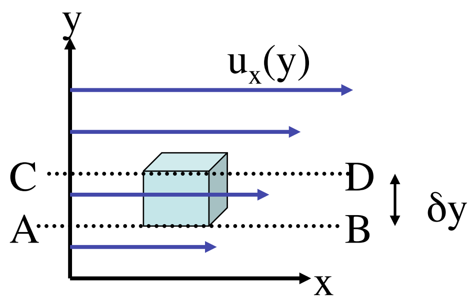
Coefficient of shear viscosity [kgm^{-1}s^{-1}]

- Hence, for a given flow u, the higher the viscosity, the greater the stress.
- Viscosity measures how "sticky" parcels of fluid are.



* Viscosity is independent of the velocity (may vary with p,T)

- Now, take a whole fluid parcel (not just a plane).



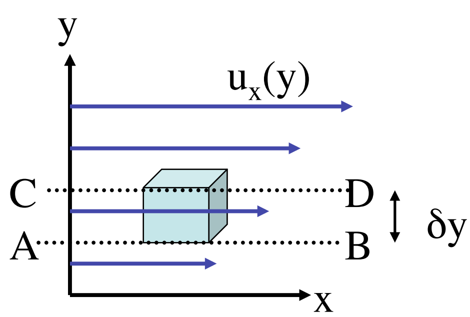
- Net viscous force is the difference of the viscous forces on each side.

- Viscous force/element of volume is

$$\left[\mu \frac{\partial u}{\partial y}\bigg|_{y+\delta y} - \mu \frac{\partial u}{\partial y}\bigg|_{y}\right]\delta x \delta z = \left[\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)\delta y\right]\delta x \delta z$$

$$= \mu \frac{\partial^2 u}{\partial y^2}\delta y \delta x \delta z \quad \text{(μ constant)}$$

i.e. $$F_v = \mu \frac{\partial^2 u}{\partial y^2}$$



7.3 Equation of Motion

•For an incompressible fluid, this generalises (using $\mu=\rho\nu$) to

$$F_v = \rho \nu \nabla^2 \underline{u}$$

kinematic viscosity [$m^2 s^{-1}$]

•The equation of motion (5.2) now becomes

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \underline{g} + \nu \nabla^2 \underline{u} \qquad (7.1)$$

7.4 Reynolds number

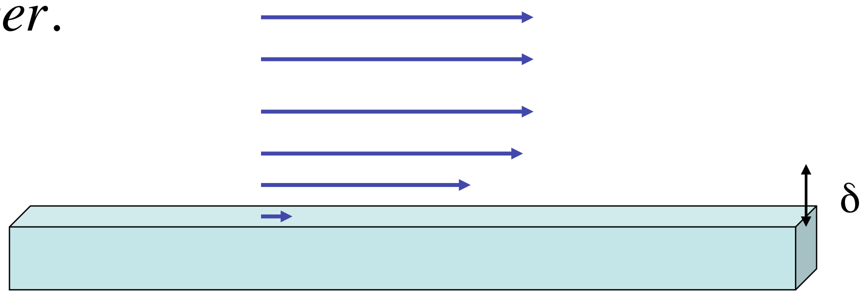
- Take the steady-state equation of motion with g=0

$$(\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u}$$

Dimensions: u^2/L $p/\rho L$ $\nu u/L^2$

- The ratio of the inertial term to the viscous term is the Reynolds number $R=uL/\nu$
  – R>>1 ->viscosity unimportant
  – R~1 (usually because L small) -> viscosity important

- Hence bees fly through a low-Reynolds number (sticky) fluid.
- Aeroplanes fly through a flow which has a high Reynolds number except in a narrow region around the aircraft the *boundary layer*.



Over a lengthscale δ, the flow slows down to zero at the boundary

7.5 Boundary layer thickness

- In the body of the flow, the inertial term has dimensions:

$$\left|(\underline{u}\cdot\underline{\nabla})\underline{u}\right| \sim \frac{u^2}{L}$$

- Within the boundary layer, however, the viscous term becomes important and it has dimensions:

$$\left|\nu\nabla^2\underline{u}\right| \sim \frac{\nu u}{\delta^2}$$

- For the viscous term to compete with the "large-scale" inertial term, these two must be equal, i.e.

$$\frac{u^2}{L} \sim \frac{\nu u}{\delta^2}$$

$$\Rightarrow \frac{\delta}{L} \sim \left(\frac{uL}{\nu}\right)^{-1/2} = \frac{1}{\sqrt{R}}$$

Low viscosity-> high R -> thin boundary layer

7.6 Bernoulli's equation

- Remember equation of motion without viscosity (5.2)

$$\rho\frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla}p + \rho\,\underline{g}$$

- Divide by density

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla}p - \underline{\nabla}\psi$$

- Now for a barotropic equation of state

$$\frac{1}{\rho}\underline{\nabla}p = \underline{\nabla}\int\frac{dp}{\rho}$$

- So..

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla}\left(\int \frac{dp}{\rho} + \psi\right)$$

- Use vector identity

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\nabla}\left(\frac{1}{2}u^2\right) - \underline{u}\times\underline{\nabla}\times\underline{u}$$

- To get

$$\frac{\partial \underline{u}}{\partial t} + -\underline{u}\times\underline{\omega} = -\underline{\nabla}\left(\frac{1}{2}u^2 + \int \frac{dp}{\rho} + \psi\right) \qquad (7.2)$$

- Remember: this is still just *rate of change of momentum=sum of forces*

1. Take the dot product with u̲ to see that in a steady state (remember $\underline{u}\cdot(\underline{u}\times\underline{\omega}) = 0$)

$$\underline{u}\cdot\underline{\nabla}\underbrace{\left(\frac{1}{2}u^2 + \int \frac{dp}{\rho} + \psi\right)}_{H} = 0 \qquad (7.3)$$

Bernoulli's constant H is constant along streamlines.

It may differ from one streamline to the next. Along each streamline the flow conserves the sum of its (kinetic + thermal + gravitational) energy.

This is Bernoulli's equation..just conservation of energy.

2. If the flow is steady and the vorticity ω=0 (curl-free or irrotational flow)

$$\underline{\nabla} H = 0$$

H is constant everywhere…ie the same on every streamline.

All the parcels of fluid not only conserve their own total mechanical energy H…they have the same H as all their neighbours.

- Example 1: for an *incompressible* fluid where the density is uniform

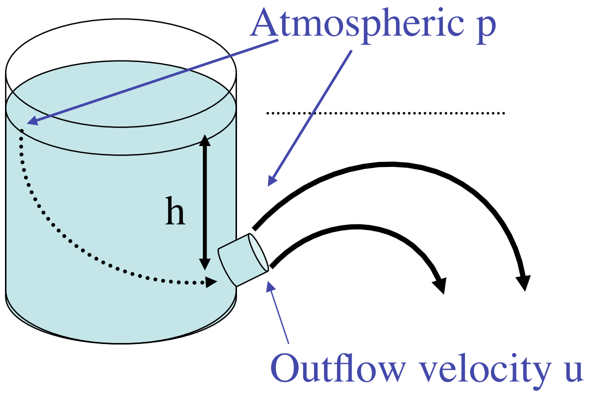
$$H = \frac{1}{2}u^2 + \frac{p}{\rho} + \psi$$

- A parcel of fluid following the dotted line has

$$\frac{p}{\rho} + \psi_{top} = \frac{1}{2}u^2 + \frac{p}{\rho} + \psi_{bottom}$$

$$\frac{1}{2}u^2 = \psi_{top} - \psi_{bottom} = gh$$

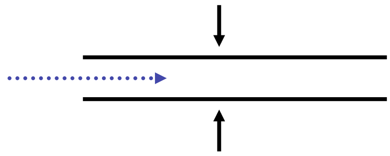
$$u = \sqrt{2gh}$$



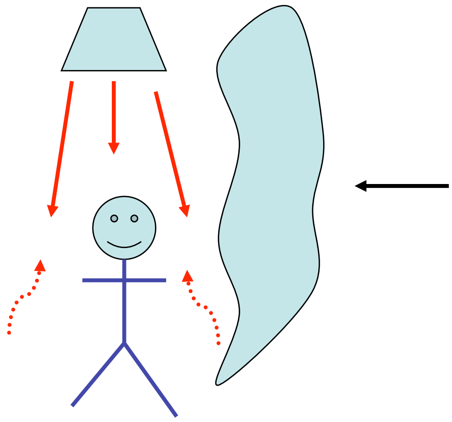
Why does the flow not rise to a height h?

- Example 2: blow between two pieces of paper …where the velocity is large, the pressure is small and the papers are pulled together.

$$H = \frac{1}{2}u^2 + \frac{p}{\rho}$$



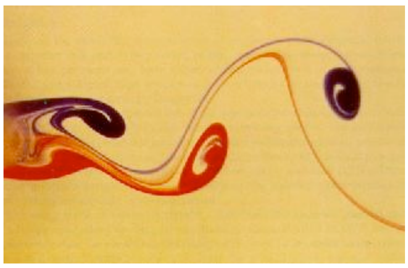
- Example 3: Why does the shower curtain cling to you? (Think where the warm air goes).



3. If we take the curl of both sides of (7.2), we get

$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times \left(\underline{u} \times \underline{\omega} \right) \qquad (7.4)$$

If the vorticity is zero initially, it has to stay zero. (*Helmholtz's equation*)



But this is clearly NOT true in a real fluid…why?

- Note if we add in viscosity, (7.4) becomes

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}} \qquad (7.5)$$

This term describes how vortex lines are pushed around by the flow..they can be stretched and twisted.

This term describes how vorticity is dissipated by viscosity)

- Question 5:

- Write down H for two equations of state:
  - adiabatic
  - isothermal

Answers to 5:

- For an adiabatic equation of state,

$$H = \frac{1}{2}u^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \psi$$

- For isothermal,

$$H = \frac{1}{2}u^2 + c_s^2 \ln \rho + \psi$$

- Aside:

$$\underline{\nabla}\left[\int \frac{dp}{\rho}\right] = \frac{\partial}{\partial \rho}\left[\int \frac{dp}{\rho}\right]\underline{\nabla}\rho$$

$$= \frac{\partial}{\partial \rho}\left[\int \frac{\partial p}{\partial \rho}\frac{d\rho}{\rho}\right]\underline{\nabla}\rho$$

$$= \frac{\partial p}{\partial \rho}\frac{1}{\rho}\underline{\nabla}\rho$$

$$= \frac{\underline{\nabla} p}{\rho}$$