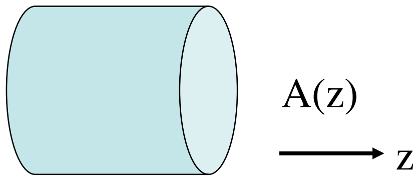
8. Example of flows: (de) Laval Nozzle

- Consider steady flow in a tube of variable cross-section, A(z).



- Mass conservation (4.1) ->

$$\underline{\nabla} \cdot (\rho \underline{u}) = 0$$

- or

$$\int_V \underline{\nabla} \cdot (\rho \underline{u}) dV = \int_s \rho \underline{u} \cdot d\underline{S}$$

- But there is no flow through the sides of the tube, so this is just

Mass loss rate

$$\rho u A = \text{constant} = \dot{M} \qquad (8.1)$$

- Steady momentum equation (no gravity)

$$(\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho} \underline{\nabla} p = -\frac{1}{\rho} \underline{\nabla} \rho \frac{dp}{d\rho} \qquad (8.2)$$

- But from (8.1)

$$\ln \rho + \ln u + \ln A = \ln \dot{M}$$

$$\therefore \frac{\nabla \rho}{\rho} = \nabla(\ln \rho) = -\nabla(\ln u) - \nabla(\ln A)$$

- And so (8.2) becomes

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = \left[\nabla(\ln u) + \nabla(\ln A)\right]\frac{dp}{d\rho}$$

$(\underline{u}\cdot\underline{u})\nabla \ln u$ (Assume irrotational) $\qquad c_s^2$ (see later)

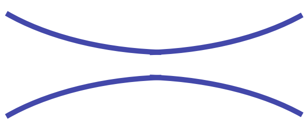
i.e.

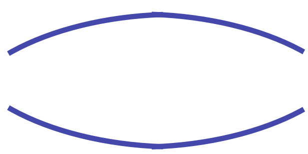
$$\left(u^2 - c_s^2\right)\nabla(\ln u) = c_s^2\nabla(\ln A)$$

- Or

$$\left(u^2 - c_s^2\right)\frac{\nabla u}{u} = c_s^2\frac{\nabla A}{A} \qquad (8.3)$$

- So.. a min or max in A:

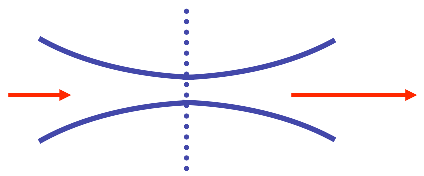




- Corresponds to either
  - a min or max in $\underline{u}$ or
  - $u=c_s$

- Conversely, gas can *only* make a sonic transition (from sub- to super-sonic or vice versa) at a max or min of the nozzle.

- In subsonic regime, $(u^2-c^2)<0$
  - if A gets smaller, u gets larger
- In supersonic regime, $(u^2-c^2)>0$
  - if A gets larger, u gets larger



- So a nozzle can be used to accelerate flow from subsonic -> supersonic

u increases monotonically

- Note also from (8.2)

$$u^2 \underline{\nabla} \ln u = -c_s^2 \underline{\nabla} \ln \rho$$

Subsonic: $u \ll c_s$ $\underline{\nabla} \ln u \gg \underline{\nabla} \ln \rho$

Nearly incompressible (so often a good assumption for everyday flows); acceleration important

Supersonic: $u \gg c_s$ $\underline{\nabla} \ln u \ll \underline{\nabla} \ln \rho$

Nearly constant u; pressure gradients not important in acceleration

Getting the velocity:

- Apply Bernoulli (7.3):

$$\frac{1}{2}u^2 + \int \frac{dp}{\rho} = const$$

- Assume isothermal

$$\frac{1}{2}u^2 + c_s^2 \ln \rho = const$$

- At a max/min of the nozzle, $A=A_m$ and $u=c_s$
- So, if M, c_s and A(z) are specified, (8.1) gives

$$\rho_{Am} = \frac{\dot{M}}{u_{Am} A_m} = \frac{\dot{M}}{c_s A_m}$$

- Wave Bernoulli at it…

$$\frac{1}{2}u^2 + c_s^2 \ln \rho = \frac{1}{2}c_s^2 + c_s^2 \ln \rho_{Am}$$

$$u^2 = c_s^2\left[1 + 2\ln\left(\rho_{Am}/\rho\right)\right]$$

- But, from (8.1)

$$\frac{\rho_{Am}}{\rho} = \frac{uA}{c_s A_m} \qquad (8.4)$$

- And so

$$u^2 = c_s^2\left[1 + 2\ln\left(uA/c_s A_m\right)\right] \qquad (8.5)$$

- Hence if we know A(z), get u(z) from (8.5) and use (8.1) to get ρ(z).