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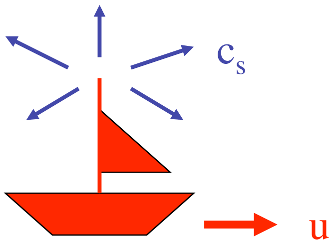
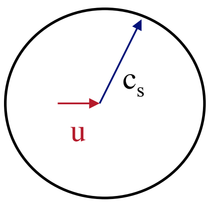
11. Shocks
11.1 General concepts

- Imagine sound waves released from a boat travelling at speed u.



- Disturbances (sound waves) propagate at speed c_s relative to fluid (moving at speed u).

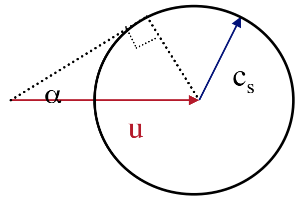
- Subsonic flow ($u<c_s$)
  – Resultant vector ($u+c_s$) sweeps out 4π steradians



- Supersonic flow ($u > c_s$)

  – Resultant vector is always to the right; it has a maximum angle α to the horizontal such that

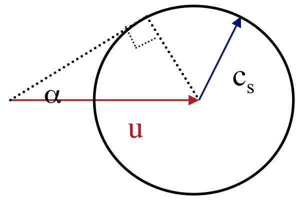
$$\sin\alpha = c_s / u$$



  – This delineates the "Mach cone" of directions in which disturbances can propagate.

- Mach number $M=u/c_s$

- Hence in a *supersonic* flow, information cannot travel upstream (to the left in this diagram).



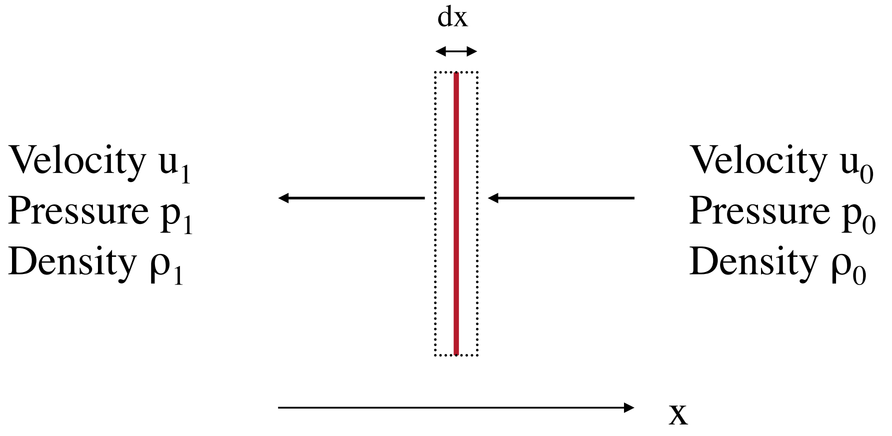
- The flow doesn't "know" about an obstacle in its path until it hits it!

- The properties of the flow have to change discontinuously in a shock.

- In a *subsonic flow*, the flow can adjust to the presence of an obstacle because its existence is communicated upstream in the flow.

11.2 Rankine-Hugoniot relations

- Consider a flow passing through a shock. In the frame of the shock:



- Mass conservation (4.1):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0$$

- Integrate over a layer of thickness dx around the shock:

$$\frac{\partial}{\partial t}\int \rho dx + \rho u_x\big|_{dx/2} - \rho u_x\big|_{-dx/2} = 0$$

- In the limit dx->0, ρu_x is the same on both sides of the shock
- i.e. no mass accumulates in the infinitely thin layer of the shock, so mass flux in=mass flux out

$$\boxed{\rho_0 u_0 = \rho_1 u_1} \qquad \text{(RH1)}$$

- Likewise, conservation of momentum (5.1) gives

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u} \cdot \underline{\nabla})\underline{u} = -\underline{\nabla} p$$

$$\Rightarrow$$

$$\rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} = -\frac{\partial p}{\partial x}$$

- Integrating gives

$$\boxed{p_0 + \rho_0 u_0^2 = p_1 + \rho_1 u_1^2} \qquad \text{(RH2)}$$

- Hence the sum of the thermal and ram pressures is constant.

- The shock represents a conversion of ram pressure to thermal pressure.

- Conservation of energy (6.7) gives (with L=0)

$$\underline{\nabla} \cdot \left[\left(\frac{1}{2} u^2 + e + \frac{p}{\rho} + \psi \right) \rho \underline{u} \right] = 0$$

- i.e.

$$\frac{\partial}{\partial x} \left[\left(\frac{1}{2} u^2 + e + \frac{p}{\rho} + \psi \right) \rho u_x \right] = 0$$

- so

$$\left[\frac{1}{2} u^2 + \psi + e + \frac{p}{\rho} \right] \rho u_x = \text{constant}$$

- hence

$$\boxed{\frac{1}{2}u_0^2 + e_0 + \frac{p_0}{\rho_0} = \frac{1}{2}u_1^2 + e_1 + \frac{p_1}{\rho_1}} \quad \text{(RH3)}$$

- The gravitational potential energy ψ (i.e the energy required to take unit mass to infinity) and ρu are continuous across the shock.

- The shock converts kinetic energy to enthalpy (ie it converts an ordered flow upstream into a hot (disordered) flow downstream).

11.3 Shock strength

- Eliminate velocities in RH equations by writing

$$\rho_0 u_0 = \rho_1 u_1 = j$$

- So that RH2 and RH3 become

$$p_1 + \frac{j^2}{\rho_1} = p_0 + \frac{j^2}{\rho_0}$$

$$\frac{1}{2}\frac{j^2}{\rho_1^2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}\frac{j^2}{\rho_0^2} + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0}$$

- Which can be re-arranged to eliminate j giving

$$\frac{\rho_1}{\rho_0} = \frac{(\gamma+1)p_1 + (\gamma-1)p_0}{(\gamma+1)p_0 + (\gamma-1)p_1} = \frac{u_0}{u_1} \qquad (11.1)$$

- Or, just using the first of the two,

$$p_1 - p_0 = \frac{j^2}{\rho_0}\left(1 - \frac{\rho_0}{\rho_1}\right) = \rho_0 u_0^2\left(1 - \frac{\rho_0}{\rho_1}\right) \qquad (11.2)$$

- Or

$$p_1 - p_0 = j(u_0 - u_1)$$

- i.e. the pressure must rise across the shock to balance the decrease in momentum flow rate.

- In the limit of a strong shock we neglect the upstream pressure i.e. $p_1 >> p_0$

$$\frac{u_0}{u_1} = \frac{\rho_1}{\rho_0} = \frac{(\gamma + 1)}{(\gamma - 1)} \qquad (11.3)$$

$$p_1 = \frac{2}{\gamma + 1} \rho_0 u_0^2 \qquad (11.4)$$

- Hence if we have $\gamma=5/3$:

$$\frac{u_0}{u_1} = \frac{\rho_1}{\rho_0} = 4 \qquad (11.5)$$

$$p_1 = \frac{3}{4} \rho_0 u_0^2 \qquad (11.6)$$

- Hence the commonly-used results that
  - the density increases by at most 4 through a shock (and the velocity falls by the same factor)
  - The pressure behind the shock is approximately the ram pressure of the upstream gas

- Hence for an *adiabatic* shock, you can't increase the compression ratio above $\gamma+1/\gamma-1$ by increasing the shock strength (*cf isothermal* shocks later)

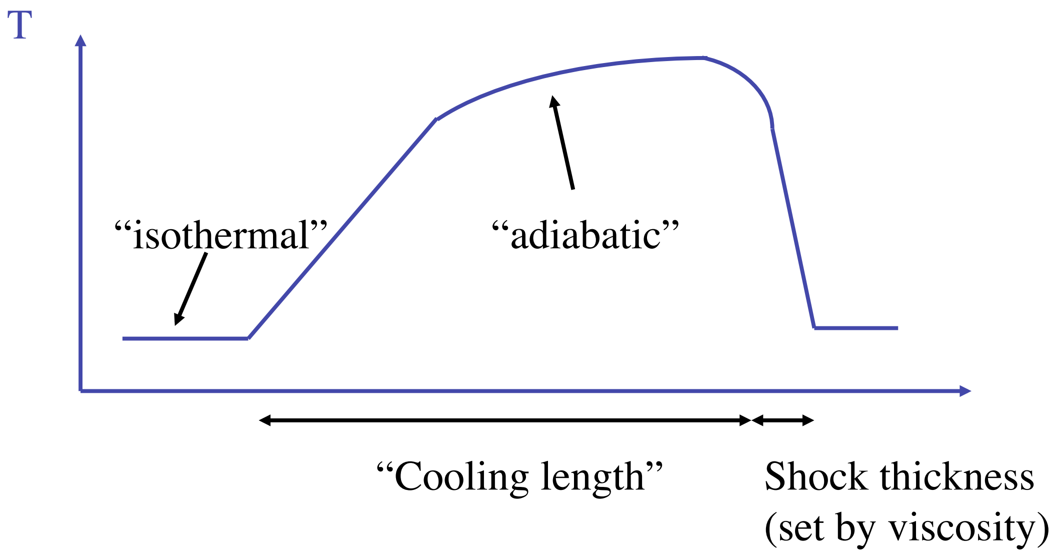
- As the shock strength is increased (higher M) p_1 goes up and stops ρ_1 increasing too much

11.4 Shock Thickness

- Set by viscosity (which allows the conversion of mechanical energy into heat).

- In practice, L is not zero so eventually the shocked gas can cool, e.g. back to its original temperature.

Shock structure



- Adiabatic shocks are therefore those where $l_{cool} > l_{scale}$ (where l_{scale} is the scale size of the system).

- Isothermal shocks have $l_{cool} << l_{scale}$

- Provided the flow downstream is steady, then (ρu) and ($p+\rho u^2$) are constant in the shocked flow.

- Therefore the isothermal portion of the flow also obeys RH1 + RH2, despite the adiabatic portion being sandwiched between it and the shock discontinuity.

11.5 Isothermal Shocks

- Replace RH3 with the condition that the flow returns to its original temperature, i.e. $T_1 = T_2$
- So RH1 and RH2 can be combined to give

$$(u_1 - u_0)c_s^2 = u_1 u_0 (u_1 - u_0)$$

using

$$c_s^2 = \frac{p_1}{\rho_1} = \frac{p_0}{\rho_0}$$

- Assuming

$$u_1 \neq u_0 \Rightarrow c_s^2 = u_1 u_0$$

- This gives

$$\frac{\rho_1}{\rho_0} = \frac{u_0}{u_1} = \left(\frac{u_0}{c_s}\right)^2 = M^2$$

- Hence, in an isothermal shock, the shock strength (or compression ratio) is the square of the Mach number of the pre-shocked flow.
- Hence it can be raised to an arbitrarily high value.