13. Water waves

- Why do waves break when they reach the shore?

- How can you tell if a storm is coming?

13. 1 How do water waves propagate?

Assume:

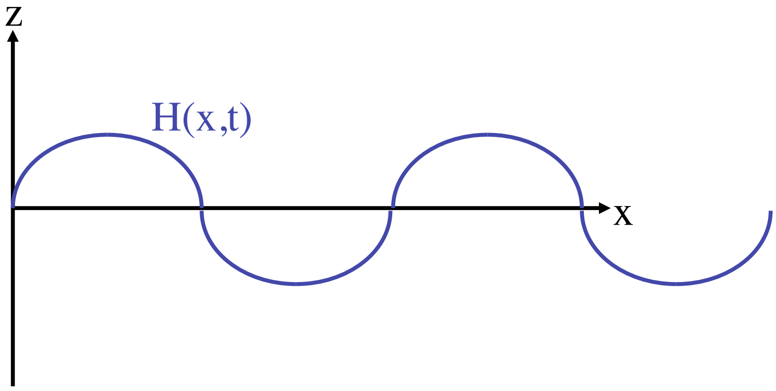
1. Incompressible, irrotational flow => Laplace's equation

$$\underline{\nabla} \times \underline{u} = 0, \qquad \underline{u} = \underline{\nabla}\Phi$$

$$\underline{\nabla} \cdot \underline{u} = 0, \qquad \Rightarrow \quad \underline{\nabla}^2 \Phi = 0 \qquad (13.1)$$

"potential" flow

2. Atmospheric (uniform) pressure at the water surface (=> p = p_0 and ρ constant too)



Surface boundary condition:

- Match pressures at the surface using Bernoulli's equation (7.2) for time-dependent but *ideal* (zero vorticity) flow:

$$\frac{\partial \underline{u}}{\partial t} = -\underline{\nabla}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \psi\right)$$

$$\underline{\nabla}\left(\frac{\partial \Phi}{\partial t} + \underbrace{\frac{1}{2}u^2}_{\text{Small => neglect}} + \underbrace{\frac{p}{\rho}}_{\text{Uniform near surface}} + \psi\right) = 0$$

- Hence at the water surface (z = H(x,t))

$$\frac{\partial \Phi}{\partial t} + \psi = F(t)$$

= g H(x,t)   -> 0 as no spatial dependence

- So

$$H(x,t) = \frac{\partial \Phi / \partial t}{-g} \qquad (13.2)$$

- Also, a parcel of fluid near the surface moves up and down *with the surface*

$$u_z = \frac{\partial \Phi}{\partial z} = \frac{\partial H}{\partial t}$$

- So differentiating (13.2) =>

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \qquad (13.3)$$

Method:

- Look for separable solutions of Laplace's equation (13.1):

$$\Phi = f(z)\phi(x)q(t)$$

- So

$$\nabla^2\Phi = 0 \quad \Rightarrow \quad \phi_{xx}f + \phi f_{zz} = 0$$

$$\frac{f_{zz}}{f} = -\frac{\phi_{xx}}{\phi} = \underset{\nwarrow}{k^2}$$

Positive to get f -> 0 below surface

- Choose

$$q(t) = e^{-i\omega t}$$

$$\phi(x) = e^{ikx}$$

- i.e.

$$\Phi = f(z)e^{i(kx-\omega t)}$$

$$f = Ae^{kz} + Be^{-kz}$$

- Choice for f(z) depends on whether deep or shallow water

13.2 Deep water waves

- Choose B = 0 so that

$$f \rightarrow 0 \quad \text{as} \quad z \rightarrow -\infty$$

$$\Phi = Ce^{kz}e^{i(kx-\omega t)} \qquad (13.4)$$

- Note that wave penetrates only to about z = -λ since here

$$e^{kz} = e^{z2\pi/\lambda} = e^{-2\pi} \approx 10^{-3}$$

- Boundary condition (13.3) now gives

$$-\omega^2 + gk = 0$$

- So $\omega = \sqrt{gk}$

- And the wave speed

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

- Waves are *dispersive* (long λ i.e. small k travel faster)
- First sign of Atlantic storm is long waves of small amplitude (swell) …. period ~ 30s, speed ~ 47 ms^{-1}, travelling ~ 4000 km in a day.
- Hence arrive before storm (takes several days to cross Atlantic)
- Followed by shorter waves that travel more slowly

13.3 Shallow water waves

- Still have

$$\Phi = f(z)e^{i(kx-\omega t)}$$

$$f = Ae^{kz} + Be^{-kz}$$

- But choose

$$f = C\cosh[k(z+h)]$$

- So that u_z ~ f’ -> 0 at z = -h

- So

$$\Phi = C\cosh\left[k\left(z+h\right)\right]e^{i(kx-\omega t)} \qquad (13.5)$$

- (13.3) now gives (at z = 0)

$$\left(i\omega\right)^2\cosh\left(kh\right) + gk\sinh\left(kh\right) = 0$$

- So

$$\omega = \sqrt{gk\tanh\left(kh\right)}$$

- And the wave speed

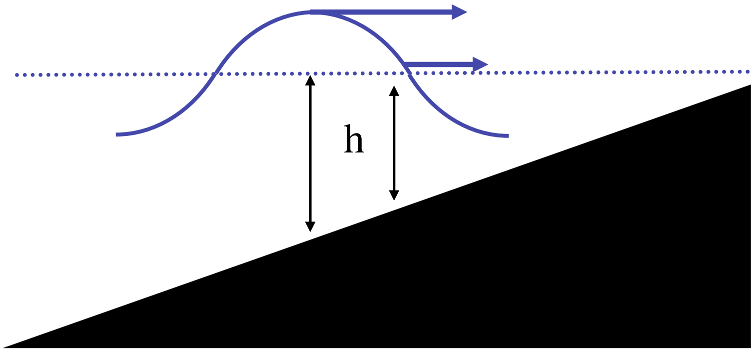
$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh\left(kh\right)}$$

- Limits:

Deep water	$h \to \infty,$	$c \to \sqrt{\frac{g}{k}}$	(as before)
Shallow water	$kh \to 0,$	$c \to \sqrt{gh}$	(indep. of k)

As h decreases, so does wave speed c

- Hence as waves approach shore:



- The back of the wave has a larger h -> larger speed so it catches up on the front of the wave.
- Wave steepens and breaks -> Surf!

13.4 Particle paths: deep water

- Write the particle position (x, z) as a displacement (X(t) , Z(t)) from a mean position (x_0, z_0).

$$x = x_0 + X(t)$$
$$z = z_0 + Z(t)$$

- Now from (13.4)

$$\Phi = Ce^{kz}e^{i(kx-\omega t)}$$

- So taking real parts

$$X'(t) = u_x(x_0, z_0) = \frac{\partial \Phi}{\partial x} = -kCe^{kz_0}\sin(kx_0 - \omega t)$$

$$Z'(t) = u_z(x_0, z_0) = \frac{\partial \Phi}{\partial z} = kCe^{kz_0}\cos(kx_0 - \omega t)$$

- Integrating gives

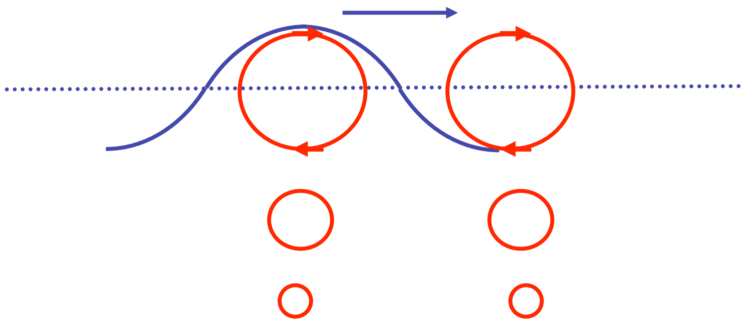
$$X(t) = -\frac{kC}{\omega}e^{kz_0}\cos(kx_0 - \omega t)$$

$$Z(t) = -\frac{kC}{\omega}e^{kz_0}\sin(kx_0 - \omega t)$$

- The particle paths are therefore circles, whose radius

$$R = \frac{kC}{\omega} e^{kz}$$

is the wave amplitude at z = 0 and decreases with z.



13.5 Particle paths: shallow water

- Once again

$$x = x_0 + X(t)$$
$$z = z_0 + Z(t)$$

- And with (13.5)

$$\Phi = C\cosh\left[k(z+h)\right]e^{i(kx-\omega t)}$$

- Obtain

$$X(t) = \frac{-kC}{\omega}\cosh\left[k\left(z_0 + h\right)\right]\cos\left(kx_0 - \omega t\right)$$
$$Z(t) = \frac{-kC}{\omega}\sinh\left[k\left(z_0 + h\right)\right]\sin\left(kx_0 - \omega t\right)$$

- Ellipses

$$\left\{\frac{x - x_0}{\cosh\left[k\left(z_0 + h\right)\right]}\right\}^2 + \left\{\frac{z - z_0}{\sinh\left[k\left(z_0 + h\right)\right]}\right\}^2 = \left(\frac{kC}{\omega}\right)^2$$

- h very large => circular motion
- h very small => almost horizontal motion