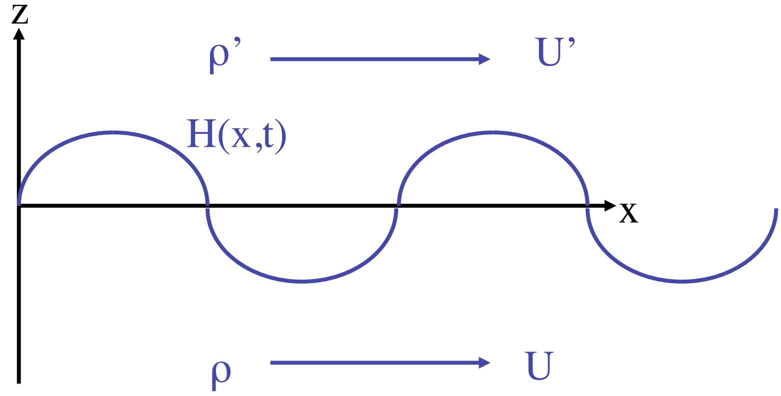
14. Interface instabilities
14.1 General solution

- Consider more general (but still incompressible) case: large-scale flow (U) and density (ρ) different on both sides



- Write the total velocity (the sum of the equilibrium + perturbed velocity) as

$$\underline{u} = \underline{\nabla}\Phi \qquad (14.1)$$

- Where

$$\Phi = Ux + \phi \qquad \text{for} \qquad z < 0$$
$$\Phi' = U'x + \phi' \qquad \text{for} \qquad z > 0$$

- ϕ is the term for the *perturbed* velocity. It satisfies

$$\nabla^2\phi = 0 \qquad \text{since} \qquad \underline{\nabla} \cdot \underline{u} = 0$$

- Consider a parcel of fluid just below the surface at z = 0. Its vertical velocity is

$$u_z = \frac{\partial \phi}{\partial z} = \frac{DH}{Dt} = \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial x}$$

- For a fluid parcel just above the surface,

$$u_z = \frac{\partial \phi'}{\partial z} = \frac{DH}{Dt} = \frac{\partial H}{\partial t} + U'\frac{\partial H}{\partial x}$$

- Look for solutions similar to those for ocean waves, where the perturbation dies away with distance from the interface.

$$H = Ae^{i(kx-\omega t)}$$

$$\phi = Ce^{kz}e^{i(kx-\omega t)} \quad \text{for} \quad z<0 \qquad (14.2)$$

$$\phi' = C'e^{-kz}e^{i(kx-\omega t)} \quad \text{for} \quad z>0$$

- How do we find the constants A, C and C' ?

- Put these into the expressions for u_z

$$\begin{aligned} kC &= iA(-\omega + Uk) \\ -kC' &= iA(-\omega + U'k) \end{aligned} \qquad (14.3)$$

- Now have two equations for 3 unknowns (A,C,C')…need another equation!

- Need to equate the pressures above and below the interface.

- Use Bernoulli's equation as before:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \psi = F(t)$$

- To get

$$\rho\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}u^2+gH\right)=\rho'\left(\frac{\partial\phi'}{\partial t}+\frac{1}{2}u'^2+gH\right)+K \qquad (14.4)$$

- Where u is the total velocity and

$$K=\rho F(t)-\rho' F'(t)$$

- But K can’t be a function of time as perturbations must vanish away from the boundary for *all* times.

- Get K by looking at the equilibrium:

$$u=U, u'=U'$$
$$\phi=\phi'=H=0$$

- Putting this in to (14.4) gives

$$K=\frac{1}{2}\rho U^2-\frac{1}{2}\rho' U'^2 \qquad (14.5)$$

- K is the difference in the K.E. of the flows above and below the interface (-> K = 0 when no flows).

- Get the total velocity by writing it as its equilibrium + perturbed part

$$u_x = U + \frac{\partial \phi}{\partial x}$$ ← δu_x

$$u_z = \frac{\partial \phi}{\partial z}$$ ← δu_z

- so

$$\underline{u} = \left(U + \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial z} \right)$$

$$u^2 = \underline{u} \cdot \underline{u} = U^2 + 2U \frac{\partial \phi}{\partial x} + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2$$

- Neglect terms in $(\delta u)^2$ and then substitute for u and K in (14.4):

$$\rho \left(\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} + gH \right) = \rho' \left(\frac{\partial \phi'}{\partial t} + U' \frac{\partial \phi'}{\partial x} + gH \right)$$

- Now (at z=0) substitute for φ,φ' and H from (14.2) (algebra!!):

$$\rho(-\omega + Uk)^2 + \rho'(-\omega + U'k)^2 - gk(\rho - \rho') = 0$$

Quadratic in ω

- More algebra!

$$\frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left\{ \frac{g}{k} \frac{(\rho - \rho')}{(\rho + \rho')} - \rho\rho' \frac{(U - U')^2}{(\rho + \rho')^2} \right\}^{1/2} \quad (14.6)$$

- This is a general dispersion relation. Valid for any densities or velocities.

- ω is real for real k => disturbance at interface travels as a wave
- waves are dispersive (speed depends on k).
- For air/water interface (e.g. ocean waves), neglect ρ' to get

$$\frac{\omega}{k} = \pm \left\{ \frac{g}{k} \right\}^{1/2}$$

as before

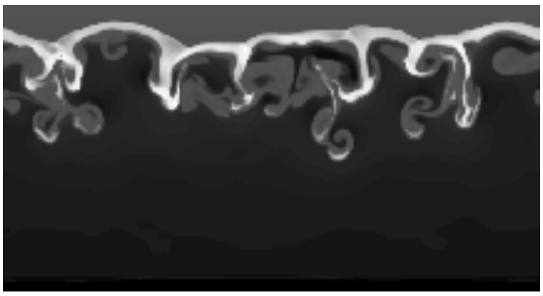
- NB: neglect surface tension, finite depth

14.2 Rayleigh-Taylor instability

- Consider fluids at rest (U = U' = 0) but the *heavier* fluid rests on top of the lighter one, (i.e. $\rho < \rho'$).

- This is an *unstable equilibrium*.

- From (14.6) , ω is imaginary, so from (14.2) the disturbance grows exponentially.

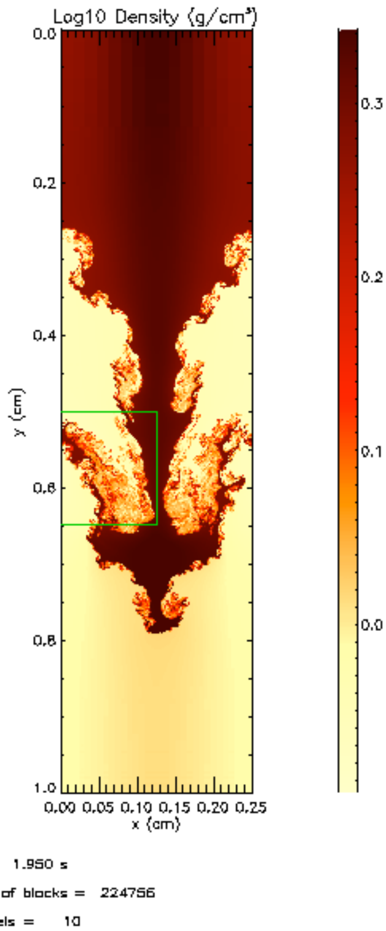


http://astron.berkeley.edu/~jrg/ay202/node142.html

- "fingers" of the denser fluid reach down into the less dense layer

- Eg dense sea water lying on top of less dense water

http://flash.uchicago.edu/~zingale/rt_gallery/rt_gallery.html



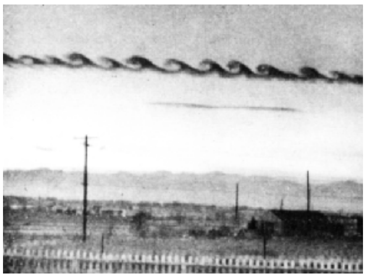
14.3 Kelvin-Helmholtz instability

- U and U' are *not* zero, but ($\rho > \rho'$) => lighter fluid rests on top of the heavier one => Rayleigh-Taylor stable.
- If the expression inside the square root in (14.6) is negative,

$$\rho\rho'(U - U')^2 > \frac{g}{k}\left(\rho^2 - \rho'^2\right)$$

- then ω has an imaginary part => instability.

- Eg: wind shear (not normally visible)
- Cloud over Denver … example of "clear air turbulence"



http://astron.berkeley.edu/~jrg/ay202/node142.html

http://www.efluids.com/efluids/gallery

- Or wind blowing over water forming waves

- “Kelvin cats eyes” formed between two fluids with a shear flow

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